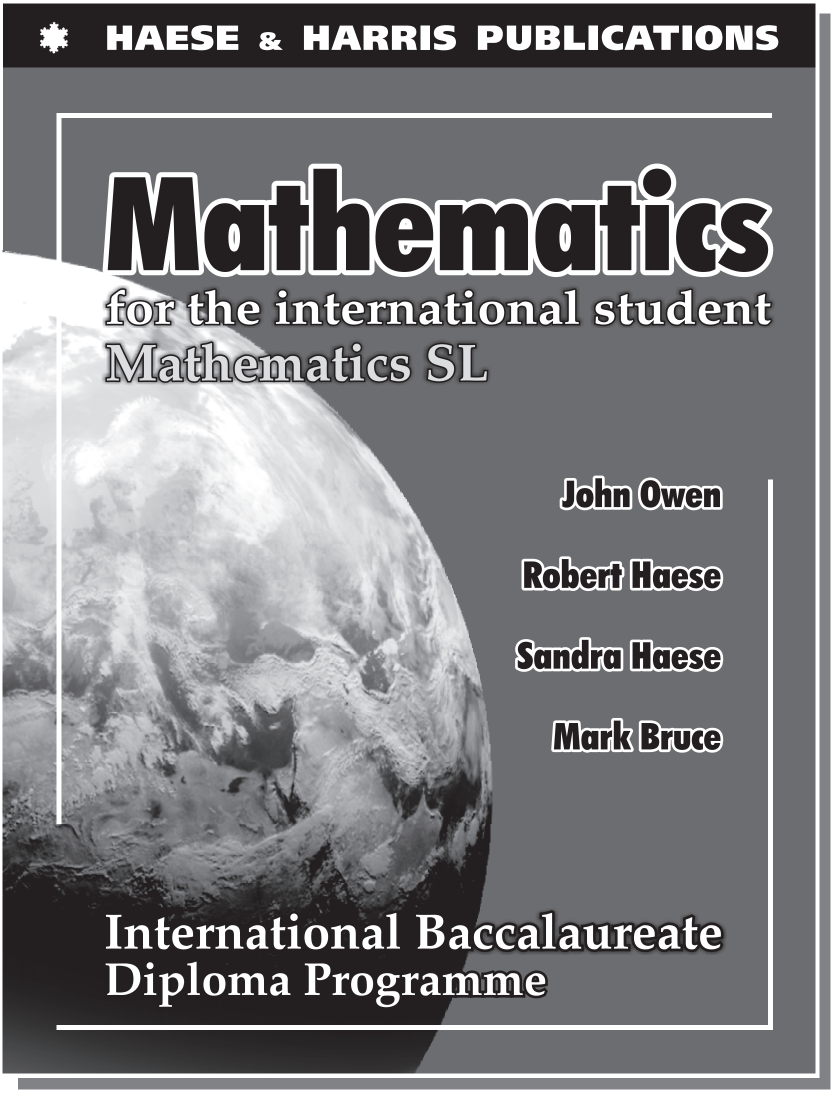
WORKED SOLUTIONS



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Haese & Harris Publications

MATHEMATICS FOR THE INTERNATIONAL STUDENT
Mathematics SL – WORKED SOLUTIONS
International Baccalaureate Diploma Programme

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FOREWORD

This book gives you fully worked solutions for every question in each chapter of the Haese & Harris Publications textbook **Mathematics SL** which is one of three textbooks in our series 'Mathematics for the International Student'. The other two textbooks are **Mathematics HL (Core)** and **Mathematical Studies SL**, and books of fully worked solutions are available for those textbooks also.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modeled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for **Mathematics SL** on our website. Please contact us if you have any additions to this list.

RLD PMH
RCH SHH

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TABLE OF CONTENTS

Background knowledge

EXERCISE A

1 **a** $\sqrt{3}\times\sqrt{5}$
$=\sqrt{3\times 5}$
$=\sqrt{15}$

b $(\sqrt{3})^2$
$=\sqrt{3}\times\sqrt{3}$
$=3$

c $2\sqrt{2}\times\sqrt{2}$
$=2(\sqrt{2}\times\sqrt{2})$
$=2\times 2$
$=4$

d $3\sqrt{2}\times 2\sqrt{2}$
$=(3\times 2)(\sqrt{2}\times\sqrt{2})$
$=6\times 2$
$=12$

e $3\sqrt{7}\times 2\sqrt{7}$
$=(3\times 2)(\sqrt{7}\times\sqrt{7})$
$=6\times 7$
$=42$

f $\dfrac{\sqrt{12}}{\sqrt{2}}$
$=\sqrt{\dfrac{12}{2}}$
$=\sqrt{6}$

g $\dfrac{\sqrt{12}}{\sqrt{6}}$
$=\sqrt{\dfrac{12}{6}}$
$=\sqrt{2}$

h $\dfrac{\sqrt{18}}{\sqrt{3}}$
$=\sqrt{\dfrac{18}{3}}$
$=\sqrt{6}$

2 **a** $2\sqrt{2}+3\sqrt{2}$
$=(2+3)\sqrt{2}$
$=5\sqrt{2}$

b $2\sqrt{2}-3\sqrt{2}$
$=(2-3)\sqrt{2}$
$=-\sqrt{2}$

c $5\sqrt{5}-3\sqrt{5}$
$=(5-3)\sqrt{5}$
$=2\sqrt{5}$

d $5\sqrt{5}+3\sqrt{5}$
$=(5+3)\sqrt{5}$
$=8\sqrt{5}$

e $3\sqrt{5}-5\sqrt{5}$
$=(3-5)\sqrt{5}$
$=-2\sqrt{5}$

f $7\sqrt{3}+2\sqrt{3}$
$=(7+2)\sqrt{3}$
$=9\sqrt{3}$

g $9\sqrt{6}-12\sqrt{6}$
$=(9-12)\sqrt{6}$
$=-3\sqrt{6}$

h $\sqrt{2}+\sqrt{2}+\sqrt{2}$
$=3\times\sqrt{2}$
$=3\sqrt{2}$

3 **a** $\sqrt{8}$
$=\sqrt{4\times 2}$
$=\sqrt{4}\times\sqrt{2}$
$=2\sqrt{2}$

b $\sqrt{12}$
$=\sqrt{4\times 3}$
$=\sqrt{4}\times\sqrt{3}$
$=2\sqrt{3}$

c $\sqrt{20}$
$=\sqrt{4\times 5}$
$=\sqrt{4}\times\sqrt{5}$
$=2\sqrt{5}$

d $\sqrt{32}$
$=\sqrt{16\times 2}$
$=\sqrt{16}\times\sqrt{2}$
$=4\sqrt{2}$

e $\sqrt{27}$
$=\sqrt{9\times 3}$
$=\sqrt{9}\times\sqrt{3}$
$=3\sqrt{3}$

f $\sqrt{45}$
$=\sqrt{9\times 5}$
$=\sqrt{9}\times\sqrt{5}$
$=3\sqrt{5}$

g $\sqrt{48}$
$=\sqrt{16\times 3}$
$=\sqrt{16}\times\sqrt{3}$
$=4\sqrt{3}$

h $\sqrt{54}$
$=\sqrt{9\times 6}$
$=\sqrt{9}\times\sqrt{6}$
$=3\sqrt{6}$

i $\sqrt{50}$
$=\sqrt{25\times 2}$
$=\sqrt{25}\times\sqrt{2}$
$=5\sqrt{2}$

j $\sqrt{80}$
$=\sqrt{16\times 5}$
$=\sqrt{16}\times\sqrt{5}$
$=4\sqrt{5}$

k $\sqrt{96}$
$=\sqrt{16\times 6}$
$=\sqrt{16}\times\sqrt{6}$
$=4\sqrt{6}$

l $\sqrt{108}$
$=\sqrt{36\times 3}$
$=\sqrt{36}\times\sqrt{3}$
$=6\sqrt{3}$

4 **a** $4\sqrt{3}-\sqrt{12}$
$=4\sqrt{3}-\sqrt{4\times 3}$
$=4\sqrt{3}-2\times\sqrt{3}$
$=4\sqrt{3}-2\sqrt{3}$
$=2\sqrt{3}$

b $3\sqrt{2}+\sqrt{50}$
$=3\sqrt{2}+\sqrt{25\times 2}$
$=3\sqrt{2}+5\times\sqrt{2}$
$=3\sqrt{2}+5\sqrt{2}$
$=8\sqrt{2}$

c $3\sqrt{6}+\sqrt{24}$
$=3\sqrt{6}+\sqrt{4\times 6}$
$=3\sqrt{6}+2\times\sqrt{6}$
$=3\sqrt{6}+2\sqrt{6}$
$=5\sqrt{6}$

d $2\sqrt{27}+2\sqrt{12}$
$=2\sqrt{9\times 3}+2\sqrt{4\times 3}$
$=6\sqrt{3}+4\sqrt{3}$
$=10\sqrt{3}$

e $\sqrt{75}-\sqrt{12}$
$=\sqrt{25\times 3}-\sqrt{4\times 3}$
$=5\sqrt{3}-2\sqrt{3}$
$=3\sqrt{3}$

f $\sqrt{2}+\sqrt{8}-\sqrt{32}$
$=\sqrt{2}+\sqrt{4\times 2}-\sqrt{16\times 2}$
$=\sqrt{2}+2\sqrt{2}-4\sqrt{2}$
$=-\sqrt{2}$

5 **a** $\frac{1}{\sqrt{2}}$
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2}}{2}$

b $\frac{6}{\sqrt{3}}$
$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{6\sqrt{3}}{3}$
$= 2\sqrt{3}$

c $\frac{7}{\sqrt{2}}$
$= \frac{7}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{7\sqrt{2}}{2}$

d $\frac{10}{\sqrt{5}}$
$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
$= \frac{10\sqrt{5}}{5}$
$= 2\sqrt{5}$

e $\frac{10}{\sqrt{2}}$
$= \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{10\sqrt{2}}{2}$
$= 5\sqrt{2}$

f $\frac{18}{\sqrt{6}}$
$= \frac{18}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$
$= \frac{18\sqrt{6}}{6}$
$= 3\sqrt{6}$

g $\frac{12}{\sqrt{3}}$
$= \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{12\sqrt{3}}{3}$
$= 4\sqrt{3}$

h $\frac{5}{\sqrt{7}}$
$= \frac{5}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$
$= \frac{5\sqrt{7}}{7}$

i $\frac{14}{\sqrt{7}}$
$= \frac{14}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$
$= \frac{14\sqrt{7}}{7}$
$= 2\sqrt{7}$

j $\frac{2\sqrt{3}}{\sqrt{2}}$
$= \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{2\sqrt{6}}{2}$
$= \sqrt{6}$

EXERCISE B

1 **a** 259
$= 2.59 \times 10^2$
$= 2.59 \times 10^2$

b $259\,000$
$= 2.590\,00 \times 10^5$
$= 2.59 \times 10^5$

c 2.59
$= 2.59 \times 1$
$= 2.59 \times 10^0$

d 0.259
$= 02.59 \div 10$
$= 2.59 \times 10^{-1}$

e $0.000\,259$
$= 00002.59 \div 10^4$
$= 2.59 \times 10^{-4}$

f 40.7
$= 4.07 \times 10$
$= 4.07 \times 10^1$

g 4070
$= 4.070 \times 10^3$
$= 4.07 \times 10^3$

h 0.0407
$= 004.07 \div 10^2$
$= 4.07 \times 10^{-2}$

i $407\,000$
$= 4.070\,00 \times 10^5$
$= 4.07 \times 10^5$

j $407\,000\,000$
$= 4.070\,000\,00 \times 10^8$
$= 4.07 \times 10^8$

k $0.000\,040\,7$
$= 000004.07 \div 10^5$
$= 4.07 \times 10^{-5}$

2 **a** $149\,500\,000\,000$ m
$= 1.49\,500\,000\,000 \times 10^{11}$
$= 1.495 \times 10^{11}$ m

b 0.0003 mm
$= 0003. \times 10^{-4}$
$= 3 \times 10^{-4}$ mm

c 0.001 mm
$= 001. \times 10^{-3}$
$= 1 \times 10^{-3}$ mm

d 15 million degrees
$= 15\,000\,000\ ^o\text{C}$
$= 1.500\,000\,0 \times 10^7\ ^o\text{C}$
$= 1.5 \times 10^7\ ^o\text{C}$

e $300\,000$ times
$= 3 \times 100\,000$
$= 3 \times 10^5$ times

3 **a** 4×10^3
$= 4 \times 1000$
$= 4000$

b 5×10^2
$= 5 \times 100$
$= 500$

c 2.1×10^3
$= 2.100 \times 10^3$
$= 2100$

d 7.8×10^4
$= 7.8000 \times 10^4$
$= 78\,000$

e 3.8×10^5
$= 3.800\,00 \times 10^5$
$= 380\,000$

f 8.6×10^1
$= 8.6 \times 10$
$= 86$

g 4.33×10^7
$= 4.330\,000\,0 \times 10^7$
$= 43\,300\,000$

h 6×10^7
$= 6 \times 10\,000\,000$
$= 60\,000\,000$

4 **a** 4×10^{-3}
$= 004. \div 10^3$
$= 0.004$

b 5×10^{-2}
$= 05. \div 10^2$
$= 0.05$

c 2.1×10^{-3}
$= 002.1 \div 10^3$
$= 0.0021$

d 7.8×10^{-4}
$= 0007.8 \div 10^4$
$= 0.000\,78$

e 3.8×10^{-5}
$= 00003.8 \div 10^5$
$= 0.000\,038$

f 8.6×10^{-1}
$= 8.6 \div 10^1$
$= 0.86$

g 4.33×10^{-7}
$= 0000004.33 \div 10^7$
$= 0.000\,000\,433$

h 6×10^{-7}
$= 0000006. \div 10^7$
$= 0.000\,000\,6$

5 **a** 9×10^{-7} m
$= 0000009. \div 10^7$
$= 0.000\,000\,9$ m

b 6.130×10^9 people
$= 6.130\,000\,000 \times 10^9$
$= 6\,130\,000\,000$ people

c 1×10^5 light years
$= 1 \times 100\,000$
$= 100\,000$ light years

d 1×10^{-5} mm
$= 00001. \div 10^5$
$= 0.000\,01$ mm

6 **a**
$$\begin{aligned}&(3.42 \times 10^5) \times (4.8 \times 10^4)\\ &= (3.42 \times 4.8) \times (10^5 \times 10^4)\\ &= 16.416 \times 10^9\\ &= 1.6416 \times 10^{10}\\ &= 1.64 \times 10^{10} \quad \text{(2 d.p.)}\end{aligned}$$

b
$$\begin{aligned}&(6.42 \times 10^{-2})^2\\ &= (6.42)^2 \times (10^{-2})^2\\ &= 41.2164 \times 10^{-4}\\ &= 4.121\,64 \times 10^{-3}\\ &= 4.12 \times 10^{-3} \quad \text{(2 d.p.)}\end{aligned}$$

c
$$\begin{aligned}&\frac{3.16 \times 10^{-10}}{6 \times 10^7}\\ &= \frac{3.16}{6} \times \frac{10^{-10}}{10^7}\\ &= 0.52\overline{6} \times 10^{-17}\\ &= 5.2\overline{6} \times 10^{-18}\\ &= 5.27 \times 10^{-18} \quad \text{(2 d.p.)}\end{aligned}$$

d
$$\begin{aligned}&(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})\\ &= \frac{9.8 \times 10^{-4}}{7.2 \times 10^{-6}}\\ &= \frac{9.8}{7.2} \times \frac{10^{-4}}{10^{-6}}\\ &= 1.36\overline{1} \times 10^2\\ &= 1.36 \times 10^2 \quad \text{(2 d.p.)}\end{aligned}$$

e
$$\begin{aligned}&\frac{1}{3.8 \times 10^5}\\ &= \frac{1}{3.8} \times 10^{-5}\\ &= 2.63 \times 10^{-6} \quad \text{(2 d.p.)}\end{aligned}$$

f
$$\begin{aligned}&(1.2 \times 10^3)^3\\ &= (1.2)^3 \times (10^3)^3\\ &= 1.728 \times 10^9\\ &= 1.73 \times 10^9 \quad \text{(2 d.p.)}\end{aligned}$$

7 **a** 1 day $= 24$ hours
i.e., missile travels 5400×24
$= 129\,600$
$= 1.296 \times 10^5$
$\doteqdot 1.30 \times 10^5$ km

b 1 week $= 7$ days
$= 7 \times 24$ hours
$= 168$ hours
i.e., missile travels 5400×168
$= 907\,200$
$= 9.072 \times 10^5$
$\doteqdot 9.07 \times 10^5$ km

c 2 years $= 2 \times 365.25$ days
$= 730.5$ days
$= 730.5 \times 24$ hours
$= 17\,532$ hours
i.e., missile travels $5400 \times 17\,532$
$= 94\,672\,800$
$= 9.467\,28 \times 10^7$
$\doteqdot 9.47 \times 10^7$ km

8 **a** distance = speed × time

time = 1 minute = 60 seconds

so, light travels $(3 \times 10^8) \times 60$
$= 180 \times 10^8$
$= 1.80 \times 10^{10}$ m

b distance = speed × time

time = 1 day = 24 hours
$= 24 \times 60 \times 60$ seconds
$= 86\,400$ seconds
$= 8.64 \times 10^4$ seconds

i.e., light travels $(3 \times 10^8) \times (8.64 \times 10^4)$
$= 3 \times 8.64 \times 10^{12}$
$= 25.92 \times 10^{12}$
$\doteqdot 2.59 \times 10^{13}$ m

c distance = speed × time

time = 1 year = 365.25 days
$= 365.25 \times 8.64 \times 10^4$ sec {from **b**}
$= 3155.76 \times 10^4$
$\doteqdot 3.16 \times 10^7$ sec

i.e., light travels $(3 \times 10^8) \times (3.156 \times 10^7)$
$= 3 \times 3.156 \times 10^{15}$
$= 9.468 \times 10^{15}$
$\doteqdot 9.47 \times 10^{15}$ m

EXERCISE C

1 **a** $\{x : x > 5\}$ reads 'the set of all x such that x is greater than 5'

b $\{x : x \leqslant 3\}$ reads 'the set of all x such that x is less than or equal to 3'

c $\{y : 0 < y < 6\}$ reads 'the set of all y such that y lies between 0 and 6'

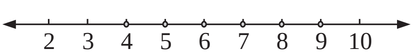
d $\{x : 2 \leqslant x \leqslant 4\}$ reads 'the set of all x such that x is greater than or equal to 2, but less than or equal to 4'

e $\{t : 1 < t < 5\}$ reads 'the set of all t such that t lies between 1 and 5'

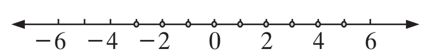
f $\{n : n < 2 \text{ or } n \geqslant 6\}$ reads 'the set of all n such that n is less than 2 or greater than or equal to 6'

2 **a** $\{x : x > 2\}$ **b** $\{x : 1 < x \leqslant 5\}$ **c** $\{x : x \leqslant -2 \text{ or } x \geqslant 3\}$
d $\{x : x \in Z, -1 \leqslant x \leqslant 3\}$ **e** $\{x : x \in Z, 0 \leqslant x \leqslant 5\}$ **f** $\{x : x < 0\}$

3 **a**

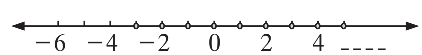


b



c

d



e

EXERCISE D

1 **a** $3x + 7x - 10$
$= 10x - 10$

b $3x + 7x - x$
$= 9x$

c $2x + 3x + 5y$
$= 5x + 5y$

d $8 - 6x - 2x$
$= 8 - 8x$

e $7ab + 5ba$
$= 7ab + 5ab$
$= 12ab$

f $3x^2 + 7x^3$
$= 3x^2 + 7x^3$
i.e., cannot be simplified

2 **a** $3(2x + 5) + 4(5 + 4x)$
$= 6x + 15 + 20 + 16x$
$= 22x + 35$

b $6 - 2(3x - 5)$
$= 6 - 6x + 10$
$= 16 - 6x$

c $5(2a-3b)-6(a-2b)$
$=10a-15b-6a+12b$
$=4a-3b$

d $3x(x^2-7x+3)-(1-2x-5x^2)$
$=3x^3-21x^2+9x-1+2x+5x^2$
$=3x^3-16x^2+11x-1$

3 **a** $2x(3x)^2$
$=2x\times 9x^2$
$=18x^3$

b $\dfrac{3a^2b^3}{9ab^4}$
$=\dfrac{\cancel{3}\times\cancel{a}\times a\times\cancel{b}\times\cancel{b}\times\cancel{b}}{\cancel{3}\times 3\times\cancel{a}\times\cancel{b}\times\cancel{b}\times\cancel{b}\times b}$
$=\dfrac{a}{3b}$

c $\sqrt{16x^4}$
$=\sqrt{16}\times\sqrt{x^4}$
$=4\times\sqrt{(x^2)^2}$
$=4x^2$

d $(2a^2)^3\times 3a^4$
$=2^3\times(a^2)^3\times 3a^4$
$=8\times a^6\times 3a^4$
$=24a^{10}$

EXERCISE E

1 **a** $2x+5=25$
$\therefore\ 2x=20$
$\therefore\ x=10$

b $3x-7>11$
$\therefore\ 3x>18$
$\therefore\ x>6$

c $5x+16=20$
$\therefore\ 5x=4$
$\therefore\ x=\frac{4}{5}$

d $\dfrac{x}{3}-7=10$
$\therefore\ \dfrac{x}{3}=17$
$\therefore\ x=51$

e $6x+11<4x-9$
$\therefore\ 2x<-20$
$\therefore\ x<-10$

f $\dfrac{3x-2}{5}=8$
$\therefore\ 3x-2=40$
$\therefore\ 3x=42$
$\therefore\ x=14$

g $1-2x\geqslant 19$
$\therefore\ -2x\geqslant 18$
$\therefore\ 2x\leqslant -18$
$\therefore\ x\leqslant -9$

h $\frac{1}{2}x+1=\frac{2}{3}x-2$
$\therefore\ \frac{3}{6}x-\frac{4}{6}x=-3$
$\therefore\ -\frac{1}{6}x=-3$
$\therefore\ x=18$

i $\dfrac{2}{3}-\dfrac{3x}{4}=\frac{1}{2}(2x-1)$

Multiplying each term by the LCD of 12 gives

$\therefore\ 8-9x=6(2x-1)$
$\therefore\ 8-9x=12x-6$
$\therefore\ 14=21x$ i.e., $x=\frac{2}{3}$

2 **a**
$x+2y=9$ ..... (1)
$x-y=3$ ..... (2)

Multiplying (2) by 2 gives

$x+2y=9$
$2x-2y=6$

$\therefore\ 3x=15$ {adding}
$\therefore\ x=5$

Substituting $x=5$ into (2) gives

$5-y=3$
$\therefore\ y=2$

$\therefore\ x=5$ and $y=2$

b
$2x+5y=28$ ..... (1)
$x-2y=2$ ..... (2)

Multiplying (2) by -2 gives

$2x+5y=28$
$-2x+4y=-4$

$\therefore\ 9y=24$ {adding}
$\therefore\ y=\frac{24}{9}=\frac{8}{3}$

Substituting $y=\frac{8}{3}$ into (2) gives

$x-2(\frac{8}{3})=2$ $\therefore\ x-\frac{16}{3}=2$ and so $x=\frac{22}{3}$

$\therefore\ x=\frac{22}{3}$ and $y=\frac{8}{3}$

c
$7x+2y=-4$ ..... (1)
$3x+4y=14$ ..... (2)

Multiplying (1) by -2 gives

$-14x-4y=8$
$3x+4y=14$

$\therefore\ -11x=22$ {adding}
$\therefore\ x=-2$

Substituting $x=-2$ into (2) gives

$3(-2)+4y=14$
$\therefore\ -6+4y=14$
$\therefore\ 4y=20$ and $\therefore\ y=5$

$\therefore\ x=-2$ and $y=5$

d
$5x-4y=27$ ..... (1)
$3x+2y=9$ ..... (2)

Multiplying (2) by 2 gives

$5x-4y=27$
$6x+4y=18$

$\therefore\ 11x=45$ {adding}
$\therefore\ x=\frac{45}{11}$

Substituting $x=\frac{45}{11}$ into (1) gives

$5(\frac{45}{11})-4y=27$ $\therefore\ \frac{225}{11}-27=4y$
$\therefore\ 4y=-\frac{72}{11}$ and $\therefore\ y=-\frac{18}{11}$

$\therefore\ x=\frac{45}{11}$ and $y=-\frac{18}{11}$

e
$$\begin{aligned} x+2y &= 5 \quad \text{..... (1)} \\ 2x+4y &= 1 \quad \text{..... (2)} \end{aligned}$$

Multiplying (1) by -2 gives

$$\begin{aligned} -2x-4y &= -10 \\ 2x+4y &= 1 \\ \hline \therefore\ 0 &= -9 \quad \{\text{adding}\} \end{aligned}$$

which is absurd

$\therefore$ there are no solutions

f
$$\begin{aligned} \frac{x}{2}+\frac{y}{3} &= 5 \quad \text{..... (1)} \\ \frac{x}{3}+\frac{y}{4} &= 1 \quad \text{..... (2)} \end{aligned}$$

Multiplying (1) by 18 and (2) by -24 gives

$$\begin{aligned} 9x+6y &= 90 \quad \text{..... (3)} \\ -8x-6y &= -24 \\ \hline \therefore\ x &= 66 \quad \{\text{adding}\} \end{aligned}$$

Substituting $x=66$ into (3) gives

$$\begin{aligned} 9\times 66+6y &= 90 \\ \therefore\ 6y &= 90-594=-504 \\ \therefore\ y &= -84 \end{aligned}$$

$\therefore$ $x=66$ and $y=-84$

EXERCISE F

1 **a** $5-(-11)$
$=5+11$
$=16$

b $|5|-|-11|$
$=5-11$
$=-6$

c $|5-(-11)|$
$=|5+11|$
$=|16|$
$=16$

d $\left|(-2)^2+11(-2)\right|$
$=|4-22|$
$=|-18|$
$=18$

e $|-6|-|-8|$
$=6-8$
$=-2$

f $|-6-(-8)|$
$=|-6+8|$
$=|2|$
$=2$

2 **a** $|a| = |-2|$
$= 2$

b $|b| = |3|$
$= 3$

c $|a|\,|b| = |-2|\,|3|$
$= 2\times 3$
$= 6$

d $|ab| = |-2\times 3|$
$= |-6|$
$= 6$

e $|a-b| = |-2-3|$
$= |-5|$
$= 5$

f $|a|-|b| = |-2|-|3|$
$= 2-3$
$= -1$

g $|a+b| = |-2+3|$
$= |1|$
$= 1$

h $|a|+|b| = |-2|+|3|$
$= 2+3$
$= 5$

i $|a|^2 = |-2|^2$
$= 2^2$
$= 4$

j $a^2 = (-2)^2 = 4$

k $\left|\dfrac{c}{a}\right| = \left|\dfrac{-4}{-2}\right| = |2| = 2$

l $\dfrac{|c|}{|a|} = \dfrac{|-4|}{|-2|} = \dfrac{4}{2} = 2$

3 **a** $|x| = 3$
$\therefore\ x = \pm 3$

b $|x| = -5$
but $|x| \geqslant 0$ for all x
(property of modulus)
$\therefore$ no solution

c $|x| = 0$
$\therefore\ x = 0$

d $|x-1| = 3$
$\therefore\ x-1 = \pm 3$
$\therefore\ x = 1\pm 3$
$\therefore\ x = -2$ or 4

e $|3-x| = 4$
$\therefore\ 3-x = \pm 4$
$\therefore\ -x = -3\pm 4$
$\therefore\ x = 3\mp 4$
$\therefore\ x = -1$ or 7

f $|x+5| = -1$
but $|x+5| \geqslant 0$ for all x
(property of modulus)
$\therefore$ no solution

g $|3x-2| = 1$
$\therefore\ 3x-2 = \pm 1$
$\therefore\ 3x = 2 \pm 1$
$\therefore\ 3x = 3$ or 1
$\therefore\ x = \frac{1}{3}$ or 1

h $|3-2x| = 3$
$\therefore\ 3-2x = \pm 3$
$\therefore\ 2x = 3 \mp 3$
$\therefore\ 2x = 0$ or 6
$\therefore\ x = 0$ or 3

i $|2-5x| = 12$
$\therefore\ 2-5x = \pm 12$
$\therefore\ 5x = 2 \mp 12$
$\therefore\ 5x = -10$ or 14
$\therefore\ x = -2$ or $\frac{14}{5}$

EXERCISE G

1 **a** $(2x+3)(x+1)$
$= 2x^2+2x+3x+3$
$= 2x^2+5x+3$

b $(3x+4)(x+2)$
$= 3x^2+6x+4x+8$
$= 3x^2+10x+8$

c $(5x-2)(2x+1)$
$= 10x^2+5x-4x-2$
$= 10x^2+x-2$

d $(x+2)(3x-5)$
$= 3x^2-5x+6x-10$
$= 3x^2+x-10$

e $(7-2x)(2+3x)$
$= 14+21x-4x-6x^2$
$= -6x^2+17x+14$

f $(1-3x)(5+2x)$
$= 5+2x-15x-6x^2$
$= -6x^2-13x+5$

g $(3x+4)(5x-3)$
$= 15x^2-9x+20x-12$
$= 15x^2+11x-12$

h $(1-3x)(2-5x)$
$= 2-5x-6x+15x^2$
$= 15x^2-11x+2$

i $(7-x)(3-2x)$
$= 21-14x-3x+2x^2$
$= 2x^2-17x+21$

j $(5-2x)(3-2x)$
$= 15-10x-6x+4x^2$
$= 4x^2-16x+15$

k $-(x+1)(x+2)$
$= -(x^2+2x+x+2)$
$= -(x^2+3x+2)$
$= -x^2-3x-2$

l $-2(x-1)(2x+3)$
$= -2(2x^2+3x-2x-3)$
$= -2(2x^2+x-3)$
$= -4x^2-2x+6$

2 **a** $(x+6)(x-6)$
$= x^2-6^2$
$= x^2-36$

b $(x+8)(x-8)$
$= x^2-8^2$
$= x^2-64$

c $(2x-1)(2x+1)$
$= (2x)^2-1^2$
$= 4x^2-1$

d $(3x-2)(3x+2)$
$= (3x)^2-2^2$
$= 9x^2-4$

e $(4x+5)(4x-5)$
$= (4x)^2-5^2$
$= 16x^2-25$

f $(5x-3)(5x+3)$
$= (5x)^2-3^2$
$= 25x^2-9$

g $(3-x)(3+x)$
$= 3^2-x^2$
$= 9-x^2$

h $(7-x)(7+x)$
$= 7^2-x^2$
$= 49-x^2$

i $(7+2x)(7-2x)$
$= 7^2-(2x)^2$
$= 49-4x^2$

j $(x+\sqrt{2})(x-\sqrt{2})$
$= x^2-(\sqrt{2})^2$
$= x^2-2$

k $(x+\sqrt{5})(x-\sqrt{5})$
$= x^2-(\sqrt{5})^2$
$= x^2-5$

l $(2x-\sqrt{3})(2x+\sqrt{3})$
$= (2x)^2-(\sqrt{3})^2$
$= 4x^2-3$

3 **a** $(x+5)^2$
$= x^2+2(x)(5)+5^2$
$= x^2+10x+25$

b $(x+7)^2$
$= x^2+2(x)(7)+7^2$
$= x^2+14x+49$

c $(x-2)^2$
$= x^2-2(x)(2)+2^2$
$= x^2-4x+4$

d $(x-6)^2$
$= x^2-2(x)(6)+6^2$
$= x^2-12x+36$

e $(3+x)^2$
$= 3^2+2(3)(x)+x^2$
$= x^2+6x+9$

f $(5+x)^2$
$= 5^2+2(5)(x)+x^2$
$= x^2+10x+25$

g $(11-x)^2$
$= 11^2-2(11)(x)+x^2$
$= x^2-22x+121$

h $(10-x)^2$
$= 10^2-2(10)(x)+x^2$
$= x^2-20x+100$

i $(2x+7)^2$
$= (2x)^2+2(2x)(7)+7^2$
$= 4x^2+28x+49$

j $(3x+2)^2$
$= (3x)^2+2(3x)(2)+2^2$
$= 9x^2+12x+4$

k $(5-2x)^2$
$= 5^2-2(5)(2x)+(2x)^2$
$= 4x^2-20x+25$

l $(7-3x)^2$
$= 7^2-2(7)(3x)+(3x)^2$
$= 9x^2-42x+49$

4 **a**
$$\begin{aligned} y &= 2(x+2)(x+3) \\ &= 2(x^2+3x+2x+6) \\ &= 2(x^2+5x+6) \\ &= 2x^2+10x+12 \end{aligned}$$

b
$$\begin{aligned} y &= 3(x-1)^2+4 \\ &= 3(x^2-2(x)(1)+1^2)+4 \\ &= 3(x^2-2x+1)+4 \\ &= 3x^2-6x+3+4 \\ &= 3x^2-6x+7 \end{aligned}$$

c
$$\begin{aligned} y &= -(x+1)(x-7) \\ &= -(x^2-7x+x-7) \\ &= -(x^2-6x-7) \\ &= -x^2+6x+7 \end{aligned}$$

d
$$\begin{aligned} y &= -(x+2)^2-11 \\ &= -(x^2+2(x)(2)+2^2)-11 \\ &= -(x^2+4x+4)-11 \\ &= -x^2-4x-4-11 \\ &= -x^2-4x-15 \end{aligned}$$

e
$$\begin{aligned} y &= 4(x-1)(x-5) \\ &= 4(x^2-5x-x+5) \\ &= 4(x^2-6x+5) \\ &= 4x^2-24x+20 \end{aligned}$$

f
$$\begin{aligned} y &= -\tfrac{1}{2}(x+4)^2-6 \\ &= -\tfrac{1}{2}(x^2+2(x)(4)+4^2)-6 \\ &= -\tfrac{1}{2}(x^2+8x+16)-6 \\ &= -\tfrac{1}{2}x^2-4x-8-6 \\ &= -\tfrac{1}{2}x^2-4x-14 \end{aligned}$$

g
$$\begin{aligned} y &= -5(x-1)(x-6) \\ &= -5(x^2-6x-x+6) \\ &= -5(x^2-7x+6) \\ &= -5x^2+35x-30 \end{aligned}$$

h
$$\begin{aligned} y &= \tfrac{1}{2}(x+2)^2-6 \\ &= \tfrac{1}{2}(x^2+2(x)(2)+2^2)-6 \\ &= \tfrac{1}{2}(x^2+4x+4)-6 \\ &= \tfrac{1}{2}x^2+2x+2-6 \\ &= \tfrac{1}{2}x^2+2x-4 \end{aligned}$$

i
$$\begin{aligned} y &= -\tfrac{5}{2}(x-4)^2 \\ &= -\tfrac{5}{2}(x^2-2(x)(4)+4^2) \\ &= -\tfrac{5}{2}(x^2-8x+16) \\ &= -\tfrac{5}{2}x^2+20x-40 \end{aligned}$$

5 **a**
$$\begin{aligned} &1+2(x+3)^2 \\ &= 1+2(x^2+2(x)(3)+3^2) \\ &= 1+2(x^2+6x+9) \\ &= 1+2x^2+12x+18 \\ &= 2x^2+12x+19 \end{aligned}$$

b
$$\begin{aligned} &2+3(x-2)(x+3) \\ &= 2+3(x^2+3x-2x-6) \\ &= 2+3(x^2+x-6) \\ &= 2+3x^2+3x-18 \\ &= 3x^2+3x-16 \end{aligned}$$

c
$$\begin{aligned} &3-(3-x)^2 \\ &= 3-(9-2(3)(x)+x^2) \\ &= 3-(x^2-6x+9) \\ &= 3-x^2+6x-9 \\ &= -x^2+6x-6 \end{aligned}$$

d
$$\begin{aligned} &5-(x+5)(x-4) \\ &= 5-(x^2-4x+5x-20) \\ &= 5-(x^2+x-20) \\ &= 5-x^2-x+20 \\ &= -x^2-x+25 \end{aligned}$$

e
$$\begin{aligned} &1+2(4-x)^2 \\ &= 1+2(4^2-2(4)(x)+x^2) \\ &= 1+2(x^2-8x+16) \\ &= 1+2x^2-16x+32 \\ &= 2x^2-16x+33 \end{aligned}$$

f
$$\begin{aligned} &x^2-3x-(x+2)(x-2) \\ &= x^2-3x-(x^2-2^2) \\ &= x^2-3x-x^2+4 \\ &= -3x+4 \end{aligned}$$

g
$$\begin{aligned} &(x+2)^2-(x+1)(x-4) \\ &= x^2+2(x)(2)+2^2-(x^2-4x+x-4) \\ &= x^2+4x+4-(x^2-3x-4) \\ &= x^2+4x+4-x^2+3x+4 \\ &= 7x+8 \end{aligned}$$

h
$$\begin{aligned} &(2x+3)^2+3(x+1)^2 \\ &= (2x)^2+2(2x)(3)+3^2+3(x^2+2(x)(1)+1^2) \\ &= 4x^2+12x+9+3(x^2+2x+1) \\ &= 4x^2+12x+9+3x^2+6x+3 \\ &= 7x^2+18x+12 \end{aligned}$$

i
$$\begin{aligned}&x^2+3x-2(x-4)^2\\&=x^2+3x-2(x^2-2(x)(4)+4^2)\\&=x^2+3x-2(x^2-8x+16)\\&=x^2+3x-2x^2+16x-32\\&=-x^2+19x-32\end{aligned}$$

j
$$\begin{aligned}&(3x-2)^2-2(x+1)^2\\&=(3x)^2-2(3x)(2)+2^2-2(x^2+2(x)(1)+1^2)\\&=9x^2-12x+4-2(x^2+2x+1)\\&=9x^2-12x+4-2x^2-4x-2\\&=7x^2-16x+2\end{aligned}$$

EXERCISE H

1 **a**
$$\begin{aligned}&3x^2+9x\\&=3x(x+3)\end{aligned}$$

b
$$\begin{aligned}&2x^2+7x\\&=x(2x+7)\end{aligned}$$

c
$$\begin{aligned}&4x^2-10x\\&=2x(2x-5)\end{aligned}$$

d
$$\begin{aligned}&6x^2-15x\\&=3x(2x-5)\end{aligned}$$

e
$$\begin{aligned}&9x^2-25\\&=(3x)^2-5^2\\&=(3x+5)(3x-5)\end{aligned}$$

f
$$\begin{aligned}&16x^2-1\\&=(4x)^2-1^2\\&=(4x+1)(4x-1)\end{aligned}$$

g
$$\begin{aligned}&2x^2-8\\&=2(x^2-4)\\&=2(x^2-2^2)\\&=2(x+2)(x-2)\end{aligned}$$

h
$$\begin{aligned}&3x^2-9\\&=3(x^2-3)\\&=3(x^2-(\sqrt{3})^2)\\&=3(x+\sqrt{3})(x-\sqrt{3})\end{aligned}$$

i
$$\begin{aligned}&4x^2-20\\&=4(x^2-5)\\&=4(x^2-(\sqrt{5})^2)\\&=4(x+\sqrt{5})(x-\sqrt{5})\end{aligned}$$

j
$$\begin{aligned}&x^2-8x+16\\&=x^2-2(x)(4)+4^2\\&=(x-4)^2\end{aligned}$$

k
$$\begin{aligned}&x^2-10x+25\\&=x^2-2(x)(5)+5^2\\&=(x-5)^2\end{aligned}$$

l
$$\begin{aligned}&2x^2-8x+8\\&=2(x^2-4x+4)\\&=2(x^2-2(x)(2)+2^2)\\&=2(x-2)^2\end{aligned}$$

m
$$\begin{aligned}&16x^2+40x+25\\&=(4x)^2+2(4x)(5)+5^2\\&=(4x+5)^2\end{aligned}$$

n
$$\begin{aligned}&9x^2+12x+4\\&=(3x)^2+2(3x)(2)+2^2\\&=(3x+2)^2\end{aligned}$$

o
$$\begin{aligned}&x^2-22x+121\\&=x^2-2(x)(11)+11^2\\&=(x-11)^2\end{aligned}$$

2 **a**
$$\begin{aligned}&x^2+9x+8\\&=(x+1)(x+8)\end{aligned}$$
{as sum $=9$, product $=8$}

b
$$\begin{aligned}&x^2+7x+12\\&=(x+3)(x+4)\end{aligned}$$
{as sum $=7$, product $=12$}

c
$$\begin{aligned}&x^2-7x-18\\&=(x-9)(x+2)\end{aligned}$$
{as sum $=-7$, product $=-18$}

d
$$\begin{aligned}&x^2+4x-21\\&=(x+7)(x-3)\end{aligned}$$
{as sum $=4$, product $=-21$}

e
$$\begin{aligned}&x^2-9x+18\\&=(x-6)(x-3)\end{aligned}$$
{as sum $=-9$, product $=18$}

f
$$\begin{aligned}&x^2+x-6\\&=(x+3)(x-2)\end{aligned}$$
{as sum $=1$, product $=-6$}

g
$$\begin{aligned}&-x^2+x+2\\&=-(x^2-x-2)\\&=-(x-2)(x+1)\end{aligned}$$
{as sum $=-1$, product $=-2$}

h
$$\begin{aligned}&3x^2-42x+99\\&=3(x^2-14x+33)\\&=3(x-3)(x-11)\end{aligned}$$
{as sum $=-14$, product $=33$}

i
$$\begin{aligned}&-2x^2-4x-2\\&=-2(x^2+2x+1)\\&=-2(x^2+2(x)(1)+1^2)\\&=-2(x+1)^2\end{aligned}$$

j
$$\begin{aligned}&2x^2+6x-20\\&=2(x^2+3x-10)\\&=2(x+5)(x-2)\end{aligned}$$
{as sum $=3$, product $=-10$}

k $2x^2-10x-48$
$=2(x^2-5x-24)$
$=2(x-8)(x+3)$
$\{$as sum $=-5$, product $=-24\}$

l $-2x^2+14x-12$
$=-2(x^2-7x+6)$
$=-2(x-6)(x-1)$
$\{$as sum $=-7$, product $=6\}$

m $-3x^2+6x-3$
$=-3(x^2-2x+1)$
$=-3(x^2-2(x)(1)+1^2)$
$=-3(x-1)^2$

n $-x^2-2x-1$
$=-(x^2+2x+1)$
$=-(x^2+2(x)(1)+1^2)$
$=-(x+1)^2$

o $-5x^2+10x+40$
$=-5(x^2-2x-8)$
$=-5(x-4)(x+2)$
$\{$as sum $=-2$, prod. $=-8\}$

3 **a** $2x^2+5x-12$ has $ac=2\times-12=-24$
$=2x^2+8x-3x-12$ Factors of -24 which add to 5 are 8 and -3.
$=2x(x+4)-3(x+4)$
$=(2x-3)(x+4)$

b $3x^2-5x-2$ has $ac=3\times-2=-6$
$=3x^2-6x+x-2$ Factors of -6 which add to -5 are -6 and 1.
$=3x(x-2)+(x-2)$
$=(3x+1)(x-2)$

c $7x^2-9x+2$ has $ac=7\times2=14$
$=7x^2-7x-2x+2$ Factors of 14 which add to -9 are -7 and -2.
$=7x(x-1)-2(x-1)$
$=(7x-2)(x-1)$

d $6x^2-x-2$ has $ac=6\times-2=-12$
$=6x^2+3x-4x-2$ Factors of -12 which add to -1 are 3 and -4.
$=3x(2x+1)-2(2x+1)$
$=(3x-2)(2x+1)$

e $4x^2-4x-3$ has $ac=4\times-3=-12$
$=4x^2+2x-6x-3$ Factors of -12 which add to -4 are -6 and 2.
$=2x(2x+1)-3(2x+1)$
$=(2x-3)(2x+1)$

f $10x^2-x-3$ has $ac=10\times-3=-30$
$=10x^2+5x-6x-3$ Factors of -30 which add to -1 are -6 and 5.
$=5x(2x+1)-3(2x+1)$
$=(5x-3)(2x+1)$

g $2x^2-11x-6$ has $ac=2\times-6=-12$
$=2x^2-12x+x-6$ Factors of -12 which add to -11 are -12 and 1.
$=2x(x-6)+(x-6)$
$=(2x+1)(x-6)$

h $3x^2-5x-28$ has $ac=3\times-28=-84$
$=3x^2-12x+7x-28$ Factors of -84 which add to -5 are -12 and 7.
$=3x(x-4)+7(x-4)$
$=(3x+7)(x-4)$

i $8x^2+2x-3$ has $ac = 8 \times -3 = -24$
$= 8x^2-4x+6x-3$ Factors of -24 which add to 2 are 6 and -4.
$= 4x(2x-1)+3(2x-1)$
$= (4x+3)(2x-1)$

j $10x^2-9x-9$ has $ac = 10 \times -9 = -90$
$= 10x^2-15x+6x-9$ Factors of -90 which add to -9 are -15 and 6.
$= 5x(2x-3)+3(2x-3)$
$= (5x+3)(2x-3)$

k $3x^2+23x-8$ has $ac = 3 \times -8 = -24$
$= 3x^2-x+24x-8$ Factors of -24 which add to 23 are 24 and -1.
$= x(3x-1)+8(3x-1)$
$= (x+8)(3x-1)$

l $6x^2+7x+2$ has $ac = 6 \times 2 = 12$
$= 6x^2+3x+4x+2$ Factors of 12 which add to 7 are 4 and 3.
$= 3x(2x+1)+2(2x+1)$
$= (3x+2)(2x+1)$

m $-4x^2-2x+6$
$= -2(2x^2+x-3)$ has $ac = 2 \times -3 = -6$
$= -2(2x^2-2x+3x-3)$ Factors of -6 which add to 1 are 3 and -2.
$= -2[2x(x-1)+3(x-1)]$
$= -2(2x+3)(x-1)$

n $12x^2-16x-3$ has $ac = 12 \times -3 = -36$
$= 12x^2-18x+2x-3$ Factors of -36 which add to -16 are -18 and 2.
$= 6x(2x-3)+(2x-3)$
$= (6x+1)(2x-3)$

o $-6x^2-9x+42$
$= -3(2x^2+3x-14)$ has $ac = 2 \times -14 = -28$
$= -3(2x^2-4x+7x-14)$ Factors of -28 which add to 3 are 7 and -4.
$= -3[2x(x-2)+7(x-2)]$
$= -3(2x+7)(x-2)$

p $21x-10-9x^2$
$= -(9x^2-21x+10)$ has $ac = 9 \times 10 = 90$
$= -(9x^2-6x-15x+10)$ Factors of 90 which add to -21 are -6 and -15.
$= -[3x(3x-2)-5(3x-2)]$
$= -(3x-5)(3x-2)$

q $8x^2-6x-27$ has $ac = 8 \times -27 = -216$
$= 8x^2+12x-18x-27$ Factors of -216 which add to -6 are -18 and 12.
$= 4x(2x+3)-9(2x+3)$
$= (4x-9)(2x+3)$

r $12x^2+13x+3$ has $ac = 12 \times 3 = 36$
$= 12x^2+4x+9x+3$ Factors of 36 which add to 13 are 9 and 4.
$= 4x(3x+1)+3(3x+1)$
$= (4x+3)(3x+1)$

s $12x^2+20x+3$ has $ac = 12\times 3 = 36$
$= 12x^2+2x+18x+3$ Factors of 36 which add to 20 are 2 and 18.
$= 2x(6x+1)+3(6x+1)$
$= (2x+3)(6x+1)$

t $15x^2-22x+8$ has $ac = 15\times 8 = 120$
$= 15x^2-10x-12x+8$ Factors of 120 which add to -22 are -10 and -12.
$= 5x(3x-2)-4(3x-2)$
$= (5x-4)(3x-2)$

u $14x^2-11x-15$ has $ac = 14\times -15 = -210$
$= 14x^2-21x+10x-15$ Factors of -210 which add to -11 are -21 and 10.
$= 7x(2x-3)+5(2x-3)$
$= (7x+5)(2x-3)$

4 **a** $3(x+4)+2(x+4)(x-1)$
$= (x+4)[3+2(x-1)]$
$= (x+4)(3+2x-2)$
$= (x+4)(2x+1)$

b $8(2-x)-3(x+1)(2-x)$
$= (2-x)[8-3(x+1)]$
$= (2-x)(8-3x-3)$
$= (2-x)(5-3x)$

c $6(x+2)^2+9(x+2)$
$= (x+2)[6(x+2)+9]$
$= (x+2)(6x+12+9)$
$= (x+2)(6x+21)$
$= (x+2)\times 3(2x+7)$
$= 3(x+2)(2x+7)$

d $4(x+5)+8(x+5)^2$
$= (x+5)[4+8(x+5)]$
$= (x+5)(4+8x+40)$
$= (x+5)(8x+44)$
$= (x+5)\times 4(2x+11)$
$= 4(x+5)(2x+11)$

e $(x+2)(x+3)-(x+3)(2-x)$
$= (x+3)[(x+2)-(2-x)]$
$= (x+3)(x+2-2+x)$
$= (x+3)(2x)$
$= 2x(x+3)$

f $(x+3)^2+2(x+3)-x(x+3)$
$= (x+3)[(x+3)+2-x]$
$= (x+3)(x+3+2-x)$
$= (x+3)(5)$
$= 5(x+3)$

g $5(x-2)-3(2-x)(x+7)$
$= 5(x-2)+3(x-2)(x+7)$
$= (x-2)[5+3(x+7)]$
$= (x-2)(5+3x+21)$
$= (x-2)(3x+26)$

h $3(1-x)+2(x+1)(x-1)$
$= -3(x-1)+2(x+1)(x-1)$
$= (x-1)[-3+2(x+1)]$
$= (x-1)(-3+2x+2)$
$= (x-1)(2x-1)$

5 **a** $(x+3)^2-16$
$= (x+3)^2-4^2$
$= (x+3+4)(x+3-4)$
$= (x+7)(x-1)$

b $4-(1-x)^2$
$= 2^2-(1-x)^2$
$= [2+(1-x)][2-(1-x)]$
$= (2+1-x)(2-1+x)$
$= (3-x)(x+1)$

c $(x+4)^2-(x-2)^2$
$= [(x+4)+(x-2)][(x+4)-(x-2)]$
$= (x+4+x-2)(x+4-x+2)$
$= (2x+2)(6)$
$= 2(x+1)(6)$
$= 12(x+1)$

d $16-4(x+2)^2$
$= 4[4-(x+2)^2]$
$= 4[2+(x+2)][2-(x+2)]$
$= 4(x+4)(-x)$
$= -4x(x+4)$

e $(2x+3)^2 - (x-1)^2$
$= [(2x+3)+(x-1)][(2x+3)-(x-1)]$
$= (2x+3+x-1)(2x+3-x+1)$
$= (3x+2)(x+4)$

f $(x+h)^2 - x^2$
$= [(x+h)+x][(x+h)-x]$
$= (x+h+x)(x+h-x)$
$= (2x+h)(h)$
$= h(2x+h)$

g $3x^2 - 3(x+2)^2$
$= 3[x^2 - (x+2)^2]$
$= 3[x+(x+2)][x-(x+2)]$
$= 3(x+x+2)(x-x-2)$
$= 3(2x+2)(-2)$
$= -6(2x+2)$
$= -12(x+1)$

h $5x^2 - 20(2-x)^2$
$= 5[x^2 - 4(2-x)^2]$
$= 5[x^2 - 2^2(2-x)^2]$
$= 5[x^2 - (2(2-x))^2]$
$= 5[x+2(2-x)][x-2(2-x)]$
$= 5(x+4-2x)(x-4+2x)$
$= 5(-x+4)(3x-4)$
$= -5(x-4)(3x-4)$

i $12x^2 - 27(3+x)^2$
$= 3[4x^2 - 9(3+x)^2]$
$= 3[(2x)^2 - 3^2(3+x)^2]$
$= 3[(2x)^2 - (3(3+x))^2]$
$= 3[2x+3(3+x)][2x-3(3+x)]$
$= 3[(2x+9+3x)(2x-9-3x)]$
$= 3(5x+9)(-x-9)$
$= -3(5x+9)(x+9)$

EXERCISE I

1 **a** $a + x = b$
$\therefore\ x = b - a$

b $ax = b$
$\therefore\ x = \dfrac{b}{a}$

c $2x + a = d$
$\therefore\ 2x = d - a$
$\therefore\ x = \dfrac{d-a}{2}$

d $c + x = t$
$\therefore\ x = t - c$

e $5x + 2y = 20$
$\therefore\ 5x = 20 - 2y$
$\therefore\ x = \dfrac{20-2y}{5}$

f $2x + 3y = 12$
$\therefore\ 2x = 12 - 3y$
$\therefore\ x = \dfrac{12-3y}{2}$

g $7x + 3y = d$
$\therefore\ 7x = d - 3y$
$\therefore\ x = \dfrac{d-3y}{7}$

h $ax + by = c$
$\therefore\ ax = c - by$
$\therefore\ x = \dfrac{c-by}{a}$

i $y = mx + c$
$\therefore\ mx = y - c$
$\therefore\ x = \dfrac{y-c}{m}$

2 **a** $az = \dfrac{b}{c}$
$\therefore\ z = \dfrac{b}{ac}$

b $\dfrac{a}{z} = d$
$\therefore\ \dfrac{z}{a} = \dfrac{1}{d}$
$\therefore\ z = \dfrac{a}{d}$

c $\dfrac{3}{d} = \dfrac{2}{z}$
$\therefore\ 3z = 2d$
$\therefore\ z = \dfrac{2d}{3}$

3 **a** $F = ma$
$\therefore\ a = \dfrac{F}{m}$

b $C = 2\pi r$
$\therefore\ r = \dfrac{C}{2\pi}$

c $V = ldh$
$\therefore\ d = \dfrac{V}{lh}$

d $A = \dfrac{b}{K}$
$\therefore\ KA = b$
$\therefore\ K = \dfrac{b}{A}$

4 **a**
$$\begin{aligned} A &= \pi r^2 \\ \therefore \frac{A}{\pi} &= r^2 \\ \therefore r &= \sqrt{\frac{A}{\pi}} \end{aligned}$$

b
$$\begin{aligned} N &= \frac{x^5}{a} \\ \therefore aN &= x^5 \\ \therefore x &= \sqrt[5]{aN} \end{aligned}$$

c
$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ \therefore \tfrac{3}{4}V &= \pi r^3 \\ \therefore \frac{3V}{4\pi} &= r^3 \\ \therefore r &= \sqrt[3]{\frac{3V}{4\pi}} \end{aligned}$$

d
$$\begin{aligned} D &= \frac{n}{x^3} \\ \therefore Dx^3 &= n \\ \therefore x^3 &= \frac{n}{D} \\ \therefore x &= \sqrt[3]{\frac{n}{D}} \end{aligned}$$

5 **a**
$$\begin{aligned} d &= \frac{\sqrt{a}}{n} \\ \therefore dn &= \sqrt{a} \\ \therefore a &= (dn)^2 \\ \therefore a &= d^2n^2 \end{aligned}$$

b
$$\begin{aligned} T &= \tfrac{1}{5}\sqrt{l} \\ \therefore 5T &= \sqrt{l} \\ \therefore l &= (5T)^2 \\ \therefore l &= 25T^2 \end{aligned}$$

c
$$\begin{aligned} c &= \sqrt{a^2-b^2} \\ \therefore c^2 &= a^2-b^2 \\ \therefore a^2 &= c^2+b^2 \\ \therefore a &= \pm\sqrt{b^2+c^2} \end{aligned}$$

d
$$\begin{aligned} T &= 2\pi\sqrt{\frac{l}{g}} \\ \therefore \frac{T}{2\pi} &= \sqrt{\frac{l}{g}} \\ \therefore \left(\frac{T}{2\pi}\right)^2 &= \frac{l}{g} \\ \therefore \frac{T^2}{4\pi^2} &= \frac{l}{g} \\ \therefore l &= \frac{gT^2}{4\pi^2} \end{aligned}$$

e
$$\begin{aligned} P &= 2(a+b) \\ \therefore \frac{P}{2} &= a+b \\ \therefore a &= \frac{P}{2}-b \end{aligned}$$

f
$$\begin{aligned} A &= \pi r^2 + 2\pi rh \\ \therefore A-\pi r^2 &= 2\pi rh \\ \therefore h &= \frac{A-\pi r^2}{2\pi r} \end{aligned}$$

g
$$\begin{aligned} I &= \frac{E}{R+r} \\ \therefore E &= I(R+r) \\ \therefore \frac{E}{I} &= R+r \\ \therefore r &= \frac{E}{I}-R \end{aligned}$$

h
$$\begin{aligned} A &= \frac{B}{p-q} \\ \therefore A(p-q) &= B \\ \therefore p-q &= \frac{B}{A} \\ \therefore q &= p-\frac{B}{A} \end{aligned}$$

6 **a**
$$\begin{aligned} k &= \frac{d^2}{2ab} \\ \therefore d^2 &= k(2ab) \\ \therefore \frac{d^2}{k} &= 2ab \\ \therefore a &= \frac{d^2}{2kb} \end{aligned}$$

b When $k=112$, $d=24$, $b=2$,
$$\begin{aligned} a &= \frac{24^2}{2\times 112\times 2} \\ &= \frac{576}{448} \\ &\doteqdot 1.29 \end{aligned}$$

7 **a**
$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ \therefore \tfrac{3}{4}V &= \pi r^3 \\ \therefore r^3 &= \frac{3V}{4\pi} \\ \therefore r &= \sqrt[3]{\frac{3V}{4\pi}} \end{aligned}$$

b When $V=40$,
$$\begin{aligned} r &= \sqrt[3]{\frac{3\times 40}{4\pi}} \\ &\doteqdot 2.122 \text{ cm} \end{aligned}$$

8 **a**
$$\begin{aligned} S &= \tfrac{1}{2}at^2 \\ \therefore t^2 &= \frac{2S}{a} \\ \therefore t &= \sqrt{\frac{2S}{a}} \end{aligned}$$

b $10 \text{ m} \equiv 10\times 100 \text{ cm} = 1000 \text{ cm}$
i.e., we need to find t when $a=8$, $S=1000$
$$t = \sqrt{\frac{2\times 1000}{8}}$$
$\therefore$ $t \doteqdot 15.81$ seconds

9 **a** $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ $\therefore$ $\frac{1}{f} - \frac{1}{u} = \frac{1}{v}$

$\therefore$ $\frac{u-f}{uf} = \frac{1}{v}$ and so $\therefore$ $v = \frac{uf}{u-f}$

b **i** When $u = 50$, $f = 8$

so $v = \frac{50 \times 8}{50 - 8}$

$\therefore$ $v \doteqdot 9.52$ cm

ii When $u = 30$, $f = 8$

so $v = \frac{30 \times 8}{30 - 8}$

$\therefore$ $v \doteqdot 10.9$ cm

10 **a**

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\therefore\ m\sqrt{1 - \left(\frac{v}{c}\right)^2} = m_0$$

$$\therefore\ \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{m_0}{m}$$

$$\therefore\ 1 - \left(\frac{v}{c}\right)^2 = \left(\frac{m_0}{m}\right)^2$$

$$\therefore\ \left(\frac{v}{c}\right)^2 = 1 - \frac{m_0^2}{m^2}$$

$$\therefore\ \frac{v}{c} = \sqrt{\frac{m^2 - m_0^2}{m^2}}$$

$$\therefore\ v = \frac{c}{m}\sqrt{m^2 - m_0^2}$$

b When $m = 3m_0$,

$$\begin{aligned} v &= \frac{c}{3m_0}\sqrt{(3m_0)^2 - m_0^2} \\ &= \frac{c}{3m_0}\sqrt{9m_0^2 - m_0^2} \\ &= \frac{c}{3m_0}\sqrt{8m_0^2} \\ &= \frac{c}{3m_0}\sqrt{8}m_0 \\ &= \tfrac{\sqrt{8}}{3}\,c \end{aligned}$$

c When $m = 30m_0$, $c = 3 \times 10^8$,

$$\begin{aligned} v &= \frac{3 \times 10^8}{30m_0}\sqrt{(30m_0)^2 - m_0^2} \\ &= \frac{30 \times 10^7}{30m_0}\sqrt{900m_0^2 - m_0^2} \\ &= \frac{10^7}{m_0}\sqrt{899m_0^2} \\ &= \frac{10^7}{m_0} \times \sqrt{899} \times m_0 \\ &= 10^7 \times \sqrt{899} \\ &\doteqdot 2.998 \times 10^8 \text{ m/s} \end{aligned}$$

EXERCISE J

1 **a**

$$\begin{aligned} &3 + \frac{x}{5} \\ &= 3 \times \frac{5}{5} + \frac{x}{5} \\ &= \frac{15}{5} + \frac{x}{5} \\ &= \frac{x+15}{5} \end{aligned}$$

b

$$\begin{aligned} &1 + \frac{3}{x} \\ &= 1 \times \frac{x}{x} + \frac{3}{x} \\ &= \frac{x}{x} + \frac{3}{x} \\ &= \frac{x+3}{x} \end{aligned}$$

c

$$\begin{aligned} &3 + \frac{x-2}{2} \\ &= 3 \times \frac{2}{2} + \frac{x-2}{2} \\ &= \frac{6}{2} + \frac{x-2}{2} \\ &= \frac{6+x-2}{2} \\ &= \frac{x+4}{2} \end{aligned}$$

d $3-\dfrac{x-2}{4}$
$=3\times\dfrac{4}{4}-\dfrac{x-2}{4}$
$=\dfrac{12}{4}-\dfrac{x-2}{4}$
$=\dfrac{12-x+2}{4}$
$=\dfrac{14-x}{4}$

e $\dfrac{2+x}{3}+\dfrac{x-4}{5}$
$=\dfrac{2+x}{3}\times\dfrac{5}{5}+\dfrac{x-4}{5}\times\dfrac{3}{3}$
$=\dfrac{5(2+x)}{15}+\dfrac{3(x-4)}{15}$
$=\dfrac{10+5x+3x-12}{15}$
$=\dfrac{8x-2}{15}$

f $\dfrac{2x+5}{4}-\dfrac{x-1}{6}$
$=\dfrac{2x+5}{4}\times\dfrac{3}{3}-\dfrac{x-1}{6}\times\dfrac{2}{2}$
$=\dfrac{3(2x+5)}{12}-\dfrac{2(x-1)}{12}$
$=\dfrac{6x+15-2x+2}{12}$
$=\dfrac{4x+17}{12}$

2 **a** $1+\dfrac{3}{x+2}$
$=1\times\dfrac{x+2}{x+2}+\dfrac{3}{x+2}$
$=\dfrac{x+2+3}{x+2}$
$=\dfrac{x+5}{x+2}$

b $-2+\dfrac{3}{x-4}$
$=-2\times\dfrac{x-4}{x-4}+\dfrac{3}{x-4}$
$=\dfrac{-2(x-4)+3}{x-4}$
$=\dfrac{-2x+8+3}{x-4}$
$=\dfrac{11-2x}{x-4}$

c $-3-\dfrac{2}{x-1}$
$=-3\times\dfrac{x-1}{x-1}-\dfrac{2}{x-1}$
$=\dfrac{-3(x-1)-2}{x-1}$
$=\dfrac{-3x+3-2}{x-1}$
$=\dfrac{1-3x}{x-1}$

d $\dfrac{2x-1}{x+1}+3$
$=\dfrac{2x-1}{x+1}+3\times\dfrac{x+1}{x+1}$
$=\dfrac{2x-1+3(x+1)}{x+1}$
$=\dfrac{2x-1+3x+3}{x+1}$
$=\dfrac{5x+2}{x+1}$

e $3-\dfrac{x}{x+1}$
$=3\times\dfrac{x+1}{x+1}-\dfrac{x}{x+1}$
$=\dfrac{3(x+1)-x}{x+1}$
$=\dfrac{3x+3-x}{x+1}$
$=\dfrac{2x+3}{x+1}$

f $-1+\dfrac{4}{1-x}$
$=-1\times\dfrac{1-x}{1-x}+\dfrac{4}{1-x}$
$=\dfrac{-(1-x)+4}{1-x}$
$=\dfrac{x-1+4}{1-x}$
$=\dfrac{x+3}{1-x}$

3 **a** $\dfrac{3x}{2x-5}+\dfrac{2x+5}{x-2}$
$=\dfrac{3x}{2x-5}\times\dfrac{x-2}{x-2}+\dfrac{2x+5}{x-2}\times\dfrac{2x-5}{2x-5}$
$=\dfrac{3x(x-2)+(2x+5)(2x-5)}{(2x-5)(x-2)}$
$=\dfrac{3x^2-6x+(4x^2-5^2)}{(2x-5)(x-2)}$
$=\dfrac{3x^2-6x+4x^2-25}{(2x-5)(x-2)}$
$=\dfrac{7x^2-6x-25}{(2x-5)(x-2)}$

b $\dfrac{1}{x-2}-\dfrac{1}{x-3}$
$=\dfrac{1}{x-2}\times\dfrac{x-3}{x-3}-\dfrac{1}{x-3}\times\dfrac{x-2}{x-2}$
$=\dfrac{(x-3)-(x-2)}{(x-2)(x-3)}$
$=\dfrac{x-3-x+2}{(x-2)(x-3)}$
$=-\dfrac{1}{(x-2)(x-3)}$

c $\dfrac{5x}{x-4}+\dfrac{3x-2}{x+4}$

$=\dfrac{5x}{x-4}\times\dfrac{x+4}{x+4}+\dfrac{3x-2}{x+4}\times\dfrac{x-4}{x-4}$

$=\dfrac{5x(x+4)+(3x-2)(x-4)}{(x+4)(x-4)}$

$=\dfrac{5x^2+20x+3x^2-12x-2x+8}{(x+4)(x-4)}$

$=\dfrac{8x^2+6x+8}{(x+4)(x-4)}$

d $\dfrac{2x+1}{x-3}-\dfrac{x+4}{2x+1}$

$=\dfrac{2x+1}{x-3}\times\dfrac{2x+1}{2x+1}-\dfrac{x+4}{2x+1}\times\dfrac{x-3}{x-3}$

$=\dfrac{(2x+1)^2-(x+4)(x-3)}{(x-3)(2x+1)}$

$=\dfrac{(2x)^2+2(2x)(1)+1^2-(x^2-3x+4x-12)}{(x-3)(2x+1)}$

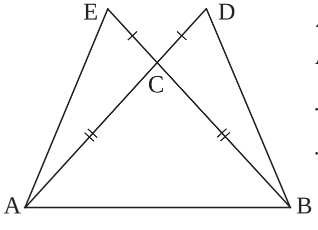
$=\dfrac{4x^2+4x+1-(x^2+x-12)}{(x-3)(2x+1)}$

$=\dfrac{4x^2+4x+1-x^2-x+12}{(x-3)(2x+1)}$

$=\dfrac{3x^2+3x+13}{(x-3)(2x+1)}$

EXERCISE K.1

1



$\measuredangle$ECA = $\measuredangle$DCB {opposite angles}

Also, EC = CD and AC = BC {given}

$\therefore$ Δs AEC and BDC are congruent (SAS)

$\therefore$ AE = BD

2 MP = NP {given}

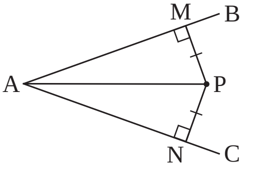
AP is common to both ΔAMP and ΔANP

$\therefore$ Δs AMP and ANP are congruent (RHS)

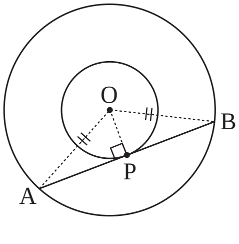
$\therefore$ $\measuredangle$MAP = $\measuredangle$NAP {corresponding angles}

i.e., AP bisects $\measuredangle$BAC

$\therefore$ P lies on the bisector of $\measuredangle$BAC



3



OA = OB {both radii of circle}

$\measuredangle$OPA = $\measuredangle$OPB = 90^o

{since AB is a tangent to inner circle}

OP is common to both ΔAOP and ΔBOP

$\therefore$ Δs AOP and BOP are congruent (RHS)

$\therefore$ AP = BP {corresponding sides}

$\therefore$ P is the midpoint of AB

EXERCISE K.2

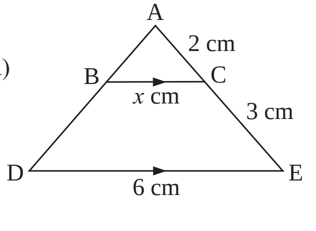
1 **a** $\measuredangle$ABC = $\measuredangle$ADE {corresponding angles}

$\measuredangle$ACB = $\measuredangle$AED {corresponding angles}

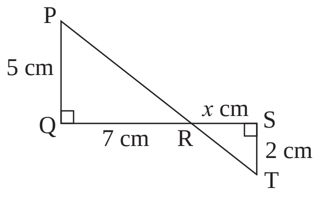
i.e., Δs ABC and ADE are equiangular (AAA) and hence similar.

$\therefore$ $\dfrac{x}{2}=\dfrac{6}{2+3}=\dfrac{6}{5}$

$\therefore$ $x=\frac{12}{5}=2.4$



b



$\measuredangle$PRQ $=$ $\measuredangle$TRS {opposite angles}

$\therefore$ Δs PQR and TSR are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{5}{7} = \frac{2}{x}$$

$$\therefore \quad 5x = 14$$

$$\therefore \quad x = \tfrac{14}{5} = 2.8$$

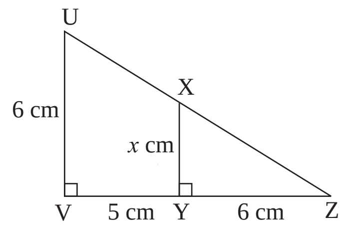
c $\measuredangle$VUZ $=$ $\measuredangle$YXZ {corresponding angles}

$\therefore$ Δs UVZ and XYZ are equiangular (AAA) and hence similar.

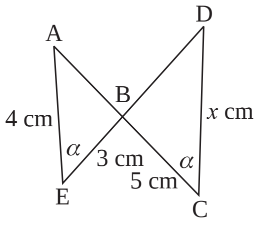
$$\therefore \quad \frac{x}{6} = \frac{6}{5+6}$$

$$\therefore \quad \frac{x}{6} = \frac{6}{11}$$

$$\therefore \quad x = \tfrac{36}{11} = 3\tfrac{3}{11}$$



d



$\measuredangle$ABE $=$ $\measuredangle$DBC {opposite angles}

$\therefore$ Δs ABE and DBC are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{5} = \frac{4}{3}$$

$$\therefore \quad x = \tfrac{20}{3} = 6\tfrac{2}{3}$$

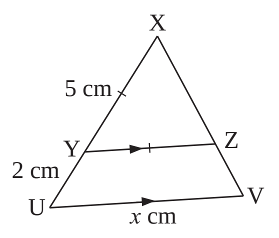
e $\measuredangle$XYZ $=$ $\measuredangle$XUV {corresponding angles}

$\measuredangle$XZY $=$ $\measuredangle$XVU {corresponding angles}

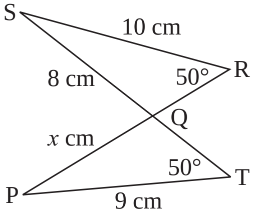
$\therefore$ Δs XYZ and XUV are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{2+5} = \frac{5}{5}$$

$$\therefore \quad \frac{x}{7} = 1 \quad \text{and so} \quad x = 7$$



f



$\measuredangle$SQR $=$ $\measuredangle$PQT {opposite angles}

$\therefore$ Δs SQR and PQT are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{9} = \frac{8}{10}$$

$$\therefore \quad x = \tfrac{72}{10} = 7.2$$

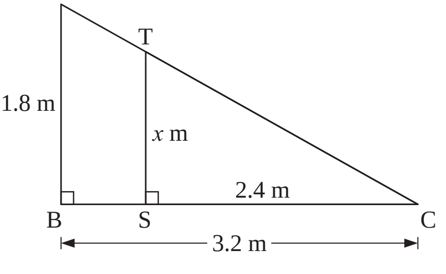
2 $\measuredangle$BAC $=$ $\measuredangle$STC {corresponding angles}

$\therefore$ Δs ABC and TSC are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{2.4} = \frac{1.8}{3.2}$$

$$\therefore \quad x = \frac{2.4 \times 1.8}{3.2} = 1.35$$

i.e., the son is 1.35 m tall



Chapter 1
FUNCTIONS

EXERCISE 1A

1 **a** (1, 3), (2, 4), (3, 5), (4, 6) is a function since no two ordered pairs have the same x-coordinate.

b (1, 3), (3, 2), (1, 7), (−1, 4) is not a function since two of the ordered pairs, (1, 3) and (1, 7), have the same x-coordinate of 1.

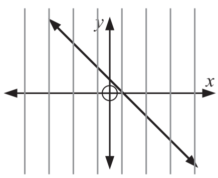
c (2, −1), (2, 0), (2, 3), (2, 11) is not a function since each ordered pair has the same x-coordinate of 2.

d (7, 6), (5, 6), (3, 6), (−4, 6) is a function since no two ordered pairs have the same x-coordinate.

e (0, 0), (1, 0), (3, 0), (5, 0) is a function since no two ordered pairs have the same x-coordinate.

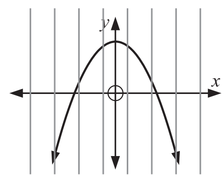
f (0, 0), (0, −2), (0, 2), (0, 4) is not a function since each ordered pair has the same x-coordinate of 0.

2 **a**



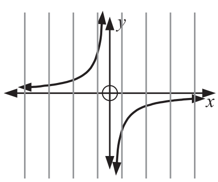
i.e., each line cuts the graph no more than once ∴ it is a function

b



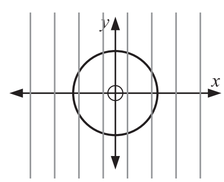
i.e., each line cuts the graph no more than once ∴ it is a function

c



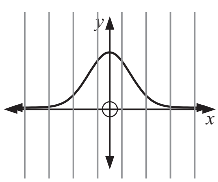
i.e., each line cuts the graph no more than once ∴ it is a function

d



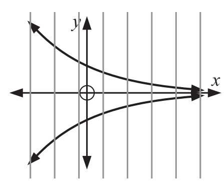
i.e., the lines cut the graph more than once ∴ it is not a function

e



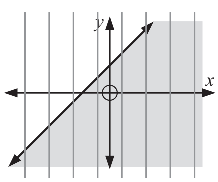
i.e., each line cuts the graph no more than once ∴ it is a function

f



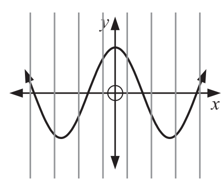
i.e., the lines cut the graph more than once ∴ it is not a function

g



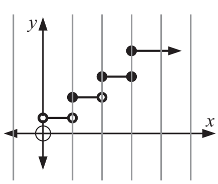
i.e., the lines cut the graph more than once ∴ it is not a function

h



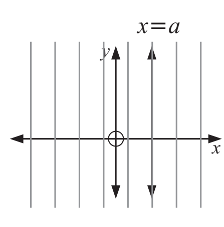
i.e., each line cuts the graph no more than once ∴ it is a function

i



i.e., one line cuts the graph more than once ∴ it is not a function

3 The graph of a straight line is not a function if the graph is a vertical line, i.e., $x = a$ for all a.
The vertical line through $x = a$ cuts the graph at every point ∴ it is not a function.



4 $x^2 + y^2 = 9$ is the equation of a circle, centre (0, 0) and radius 3.

Now
$$x^2 + y^2 = 9$$
$$\therefore\ y^2 = 9 - x^2$$
$$\therefore\ y = \pm\sqrt{9 - x^2}$$

Hence y has two real values for any value of x where $-3 < x < 3$.

EXERCISE 1B

1

a Domain is $\{x:\ x \geqslant -1\}$
Range is $\{y:\ y \leqslant 3\}$

b Domain is $\{x:\ -1 < x \leqslant 5\}$
Range is $\{y:\ 1 < y \leqslant 3\}$

c Domain is $\{x:\ x \neq 2\}$
Range is $\{y:\ y \neq -1\}$

d Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ 0 < y \leqslant 2\}$

e Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y \geqslant -1\}$

f Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y \leqslant 6\frac{1}{4}\}$ i.e., $\{y:\ y \leqslant \frac{25}{4}\}$

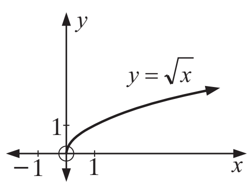
g Domain is $\{x:\ x \geqslant -4\}$
Range is $\{y:\ y \geqslant -3\}$

h Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y > -2\}$

i Domain is $\{x:\ x \neq \pm 2\}$ Range is $\{y:\ y \leqslant -1$ or $y > 0\}$

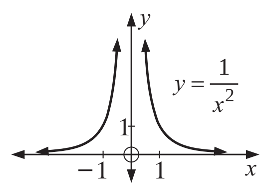
2

a



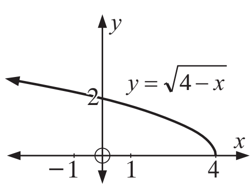
Domain is $\{x:\ x \geqslant 0\}$
Range is $\{y:\ y \geqslant 0\}$

b



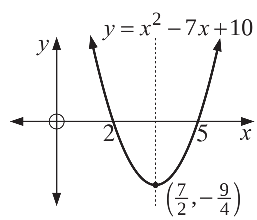
Domain is $\{x:\ x \neq 0\}$
Range is $\{y:\ y > 0\}$

c



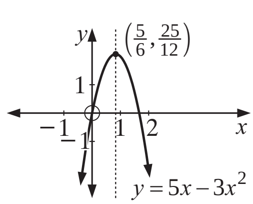
Domain is $\{x:\ x \leqslant 4\}$
Range is $\{y:\ y \geqslant 0\}$

d



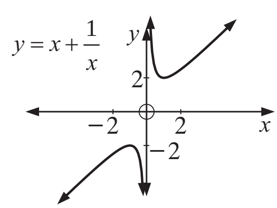
Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y \geqslant -2\frac{1}{4}\}$

e



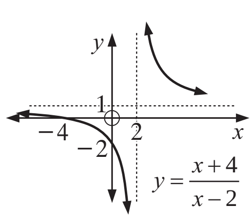
Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y \leqslant \frac{25}{12}\}$

f



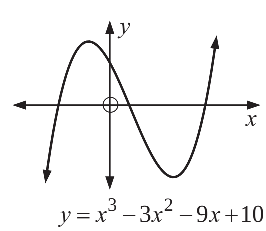
Domain is $\{x:\ x \neq 0\}$
Range is $\{y:\ y \leqslant -2$ or $y \geqslant 2\}$

g



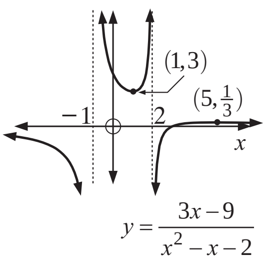
Domain is $\{x:\ x \neq 2\}$
Range is $\{y:\ y \neq 1\}$

h



Domain is $\{x:\ x \in \mathcal{R}\}$
Range is $\{y:\ y \in \mathcal{R}\}$

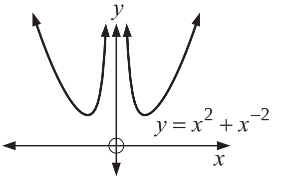
i



Domain is $\{x\colon\ x \neq -1,\ x \neq 2\}$

Range is $\{y\colon\ y \leqslant \frac{1}{3}$ or $y \geqslant 3\}$

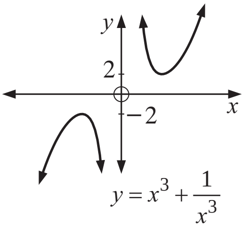
j



Domain is $\{x\colon\ x \neq 0\}$

Range is $\{y\colon\ y \geqslant 2\}$

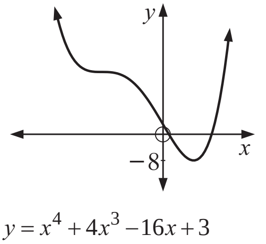
k



Domain is $\{x\colon\ x \neq 0\}$

Range is $\{y\colon\ y \leqslant -2$ or $y \geqslant 2\}$

l



Domain is $\{x\colon\ x \in \mathcal{R}\}$

Range is $\{y\colon\ y \geqslant -8\}$

EXERCISE 1C

1 **a** $f(0) = 3(0)+2 = 2$

b $f(2) = 3(2)+2 = 8$

c $f(-1) = 3(-1)+2 = -1$

d $f(-5) = 3(-5)+2 = -13$

e $f(-\frac{1}{3}) = 3(-\frac{1}{3})+2 = 1$

2 **a** $g(1) = 1 - \frac{4}{1} = 1-4 = -3$

b $g(4) = 4 - \frac{4}{4} = 4-1 = 3$

c $g(-1) = -1 - \frac{4}{-1} = -1+4 = 3$

d $g(-4) = -4 - \frac{4}{-4} = -4+1 = -3$

e $g(-\frac{1}{2}) = -\frac{1}{2} - \frac{4}{(-\frac{1}{2})} = -\frac{1}{2}+8 = 7\frac{1}{2}$

3 **a** $f(0) = 3(0) - 0^2 + 2 = 2$

b $f(3) = 3(3) - 3^2 + 2 = 9-9+2 = 2$

c $f(-3) = 3(-3) - (-3)^2 + 2 = -9-9+2 = -16$

d $f(-7) = 3(-7) - (-7)^2 + 2 = -21-49+2 = -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2 = \frac{9}{2} - \frac{9}{4} + 2 = \frac{17}{4}$

4 **a** $f(a) = 7-3a$

b $f(-a) = 7-3(-a) = 7+3a$

c $f(a+3) = 7-3(a+3) = 7-3a-9 = -3a-2$

d $f(b-1) = 7-3(b-1) = 7-3b+3 = 10-3b$

e $f(x+2) = 7-3(x+2) = 7-3x-6 = 1-3x$

5 **a**
$$\begin{aligned}&F(x+4)\\&=2(x+4)^2+3(x+4)-1\\&=2(x^2+8x+16)+3x+12-1\\&=2x^2+16x+32+3x+11\\&=2x^2+19x+43\end{aligned}$$

b
$$\begin{aligned}&F(2-x)\\&=2(2-x)^2+3(2-x)-1\\&=2(4-4x+x^2)+6-3x-1\\&=8-8x+2x^2+5-3x\\&=2x^2-11x+13\end{aligned}$$

c
$$\begin{aligned}&F(-x)\\&=2(-x)^2+3(-x)-1\\&=2x^2-3x-1\end{aligned}$$

d
$$\begin{aligned}&F(x^2)\\&=2(x^2)^2+3(x^2)-1\\&=2x^4+3x^2-1\end{aligned}$$

e
$$\begin{aligned}&F(x^2-1)\\&=2(x^2-1)^2+3(x^2-1)-1\\&=2(x^4-2x^2+1)+3x^2-3-1\\&=2x^4-4x^2+2+3x^2-4\\&=2x^4-x^2-2\end{aligned}$$

6 **a** **i** $G(2)=\dfrac{2(2)+3}{2-4}=\frac{7}{-2}=-\frac{7}{2}$

ii $G(0)=\dfrac{2(0)+3}{0-4}=\frac{3}{-4}=-\frac{3}{4}$

iii
$$\begin{aligned}G(-\tfrac{1}{2})&=\frac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4}\\&=\frac{-1+3}{-\frac{9}{2}}\\&=\frac{2}{(-\frac{9}{2})}\\&=-\tfrac{4}{9}\end{aligned}$$

b $G(x)=\dfrac{2x+3}{x-4}$ is undefined when $x-4=0$
i.e., when $x=4$

So, when $x=4$, $G(x)$ does not exist.

c
$$\begin{aligned}G(x+2)&=\frac{2(x+2)+3}{(x+2)-4}\\&=\frac{2x+4+3}{x+2-4}\\&=\frac{2x+7}{x-2}\end{aligned}$$

d
$$\begin{aligned}G(x)&=-3\\\text{i.e., }\frac{2x+3}{x-4}&=-3\\\therefore\ 2x+3&=-3(x-4)\\\therefore\ 2x+3&=-3x+12\\\therefore\ 5x&=9\quad\text{and so}\quad x=\tfrac{9}{5}\end{aligned}$$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 If $f(x)=2^x$, $\quad f(a)\,f(b)=2^a\times 2^b=2^{a+b}\quad$ and $\quad f(a+b)=2^{a+b}$

i.e., when $f(x)=2^x$, $f(a)\,f(b)=f(a+b)$.

9 **a**
$$\begin{aligned}\frac{f(x)-f(3)}{x-3}&=\frac{x^2-3^2}{x-3}\\&=\frac{x^2-9}{x-3}\\&=\frac{(x+3)(x-3)}{x-3}\\&=x+3,\quad x\neq 3\end{aligned}$$

b
$$\begin{aligned}\frac{f(2+h)-f(2)}{h}&=\frac{(2+h)^2-2^2}{h}\\&=\frac{4+4h+h^2-4}{h}\\&=\frac{h^2+4h}{h}\\&=\frac{h(h+4)}{h}\\&=h+4,\quad h\neq 0\end{aligned}$$

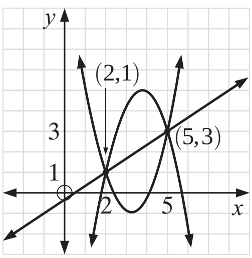
10 **a** $V(4) = 9650 - 860(4)$
$= 9650 - 3440$
$= 6210$

i.e., the value of the photocopier 4 years after purchase is \$6210.

b If $V(t) = 5780$,
then $9650 - 860t = 5780$
$\therefore\ 860t = 3870$
$\therefore\ t = 4.5$

i.e., the value of the photocopier $4\frac{1}{2}$ years after purchase is \$5780.

c Original purchase price is when $t = 0$,
i.e., $V(0) = 9650 - 860(0)$
$= 9650$ i.e., the original purchase price was \$9650.

11 

First sketch the linear function which passes through the two points (2, 1) and (5, 3).
Then sketch two quadratic functions which also pass through the two points.

12 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

i.e., $a(2) + b = 1$ and $a(-3) + b = 11$
$\therefore\ 2a + b = 1$ $\therefore\ -3a + b = 11$
$\therefore\ b = 1 - 2a$ ..... (1) $\therefore\ b = 11 + 3a$ ..... (2)

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$
$\therefore\ 5a = -10$
$\therefore\ a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$ i.e., $a = -2$, $b = 5$
Hence $f(x) = -2x + 5$

13 $f(x) = ax + \dfrac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

i.e., $a(1) + \dfrac{b}{1} = 1$ and $a(2) + \dfrac{b}{2} = 5$
$\therefore\ a + b = 1$
$\therefore\ a = 1 - b$ ..... (1)
$\therefore\ 2a + \dfrac{b}{2} = 5$ ..... (2)

Substituting (1) into (2) gives $2(1 - b) + \dfrac{b}{2} = 5$ $\therefore\ 2 - 2b + \dfrac{b}{2} = 5$
$\therefore\ -\dfrac{3b}{2} = 3$
$\therefore\ b = -2$

Substituting $b = -2$ into (1) gives $a = 1 - (-2) = 3$ i.e., $a = 3$, $b = -2$

14 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$

i.e., $a(0)^2 + b(0) + c = -4$
$\therefore\ c = -4$

Also, $a(1)^2 + b(1) + c = -2$ and $a(2)^2 + b(2) + c = 6$
$\therefore\ a + b + c = -2$ and $\therefore\ 4a + 2b + c = 6$

Substituting $c = -4$ into both equations gives

$a + b + (-4) = -2$ and $4a + 2b + (-4) = 6$

$\therefore\ a + b = 2$ $\therefore\ 4a + 2b = 10$ ..... (2)

$\therefore\ a = 2 - b$ ..... (1)

Substituting (1) into (2) gives $4(2-b) + 2b = 10$ $\therefore\ 8 - 4b + 2b = 10$

$\therefore\ -2b = 2$

$\therefore\ b = -1$

Now, substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$ i.e., $a = 3,\ b = -1,\ c = -4$

EXERCISE 1D

1 **a** $(f \circ g)(x)$
$= f(g(x))$
$= f(1-x)$
$= 2(1-x) + 3$
$= 2 - 2x + 3$
$= 5 - 2x$

b $(g \circ f)(x)$
$= g(f(x))$
$= g(2x+3)$
$= 1 - (2x+3)$
$= 1 - 2x - 3$
$= -2x - 2$

c $(f \circ g)(-3)$
$= f(g(-3))$
$= f(1-(-3))$
$= f(4)$
$= 2(4) + 3$
$= 8 + 3$
$= 11$

2 $(f \circ g)(x) = f(g(x))$
$= f(2-x)$
$= (2-x)^2$

$(g \circ f)(x) = g(f(x))$
$= g(x^2)$
$= 2 - x^2$

3 **a** $(f \circ g)(x)$
$= f(g(x))$
$= f(3-x)$
$= (3-x)^2 + 1$
$= 9 - 6x + x^2 + 1$
$= x^2 - 6x + 10$

b $(g \circ f)(x)$
$= g(f(x))$
$= g(x^2+1)$
$= 3 - (x^2+1)$
$= 3 - x^2 - 1$
$= -x^2 + 2$

c $(g \circ f)(x) = f(x)$
i.e., $-x^2 + 2 = f(x)$ {from **b**}
i.e., $-x^2 + 2 = x^2 + 1$
$\therefore\ 2x^2 = 1$
$\therefore\ x^2 = \frac{1}{2}$
$\therefore\ x = \pm\frac{1}{\sqrt{2}}$

4 **a** $ax + b = cx + d$ is true for all x {given}

Let $x = 0$,
$\therefore\ a(0) + b = c(0) + d$
$\therefore\ b = d$ ..... (1)

Let $x = 1$,
$\therefore\ a(1) + b = c(1) + d$
$\therefore\ a + b = c + d$
but $b = d$ (from (1))
$\therefore\ a + d = c + d$
$\therefore\ a = c$

b $(f \circ g)(x) = x$ for all x {given}
$\therefore\ f(g(x)) = x$
$\therefore\ f(ax+b) = x$
$\therefore\ 2(ax+b) + 3 = x$
$\therefore\ 2ax + 2b + 3 = x$

When $x = 0$,
$2a(0) + 2b + 3 = 0$
$\therefore\ 2b = -3$
$\therefore\ b = -\frac{3}{2}$

When $x = 1$, $2a(1) + 2b + 3 = 1$
$\therefore\ 2a + 2b = -2$
$\therefore\ 2a = -2 - 2(-\frac{3}{2}) = 1$
$\therefore\ a = \frac{1}{2}$

i.e., $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

c If $(g \circ f)(x) = x$
then $g(f(x)) = x$
i.e., $g(2x+3) = x$
i.e., $a(2x+3)+b = x$
$\therefore\ 2ax+3a+b = x$

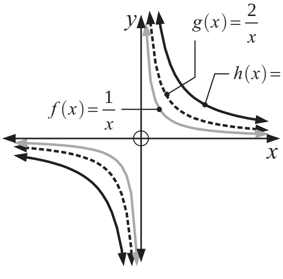
When $x = 0$,
$2a(0)+3a+b = 0$
$\therefore\ 3a = -b$
$\therefore\ a = -\frac{1}{3}b$ ..... (1)

When $x=1$,
$2a(1)+3a+b = 1$
$\therefore\ 2a+3a+b = 1$
$\therefore\ 5a+b = 1$
$\therefore\ 5(-\frac{1}{3}b)+b = 1$ (from (1))
$\therefore\ -\frac{2}{3}b = 1$
$\therefore\ b = -\frac{3}{2}$

Substituting $b = -\frac{3}{2}$ into (1) gives
$a = -\frac{1}{3}(-\frac{3}{2}) = \frac{1}{2}$
i.e., $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.
$\therefore$ the result in **b** is also true if $(g \circ f)(x) = x$ for all x.

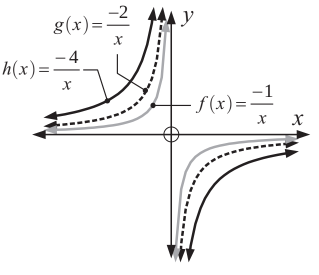
EXERCISE 1E

1



$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x- and y-axes.
The graphs all lie in the 1st and 3rd quadrants.
The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = \dfrac{1}{x}$ is closer to the axes than $g(x) = \dfrac{2}{x}$ for corresponding values of x, and $g(x) = \dfrac{2}{x}$ is closer to the axes than $h(x) = \dfrac{4}{x}$.

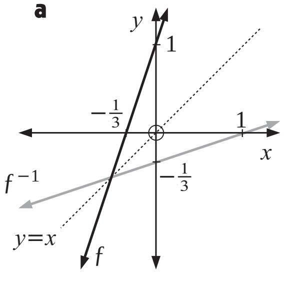
2



$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x- and y-axes.
The graphs all lie in the 2nd and 4th quadrants.
The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = -\dfrac{1}{x}$ is closer to the axes than $g(x) = -\dfrac{2}{x}$ for corresponding values of x, and $g(x) = -\dfrac{2}{x}$ is closer to the axes than $h(x) = -\dfrac{4}{x}$.

EXERCISE 1F

1 a



b $f(x)$ passes through (0, 1) and $(-\frac{1}{3}, 0)$
$\therefore\ f^{-1}(x)$ passes through (1, 0) and $(0, -\frac{1}{3})$

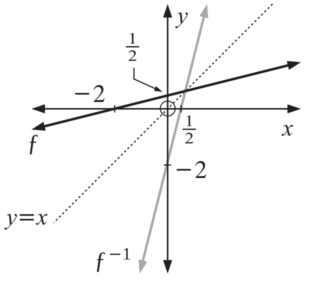
$f^{-1}(x)$ has slope $\dfrac{-\frac{1}{3}-0}{0-1} = \dfrac{-\frac{1}{3}}{-1} = \dfrac{1}{3}$

So, its equation is $\dfrac{y-0}{x-1} = \dfrac{1}{3}$

i.e., $y = \dfrac{x-1}{3}$ i.e., $f^{-1}(x) = \dfrac{x-1}{3}$

c f is $y = 3x+1$
so f^{-1} is $x = 3y+1$
$\therefore\ x - 1 = 3y$
$\therefore\ y = \dfrac{x-1}{3}$ i.e., $f^{-1}(x) = \dfrac{x-1}{3}$

2 a

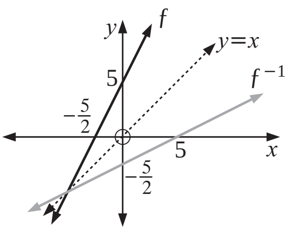


b $f(x)$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$
$\therefore\ f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$
$f^{-1}(x)$ has slope $\dfrac{-2-0}{0-\frac{1}{2}} = \dfrac{-2}{-\frac{1}{2}} = 4$
So, its equation is $\dfrac{y-0}{x-\frac{1}{2}} = 4$
i.e., $y = 4x-2$
i.e., $f^{-1}(x) = 4x-2$

c f is $y = \dfrac{x+2}{4}$
so f^{-1} is $x = \dfrac{y+2}{4}$
$\therefore\ 4x = y+2$
$\therefore\ y = 4x-2$
i.e., $f^{-1}(x) = 4x-2$

3 a i f is $y = 2x+5$
so f^{-1} is $x = 2y+5$
$\therefore\ x-5 = 2y$
$\therefore\ y = \dfrac{x-5}{2}$
i.e., $f^{-1}(x) = \dfrac{x-5}{2}$

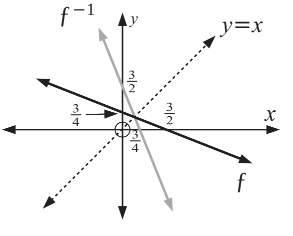
ii



$f(x)$ passes through $(0, 5)$ and $(-\frac{5}{2}, 0)$
$\therefore\ f^{-1}(x)$ passes through $(5, 0)$ and $(0, -\frac{5}{2})$

b i f is $y = \dfrac{3-2x}{4}$
so f^{-1} is $x = \dfrac{3-2y}{4}$
$\therefore\ 4x = 3-2y$
$\therefore\ 4x-3 = -2y$
$\therefore\ y = -2x+\frac{3}{2}$
i.e., $f^{-1}(x) = -2x+\frac{3}{2}$

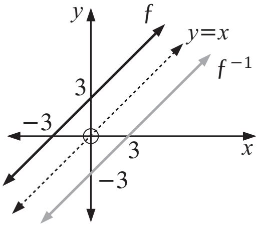
ii



$f(x)$ passes through $(0, \frac{3}{4})$ and $(\frac{3}{2}, 0)$
$\therefore\ f^{-1}(x)$ passes through $(\frac{3}{4}, 0)$ and $(0, \frac{3}{2})$

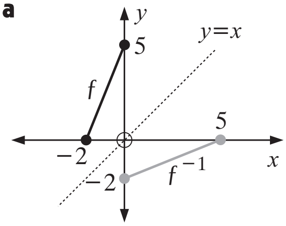
c i f is $y = x+3$
so f^{-1} is $x = y+3$
$\therefore\ y = x-3$
i.e., $f^{-1}(x) = x-3$

ii

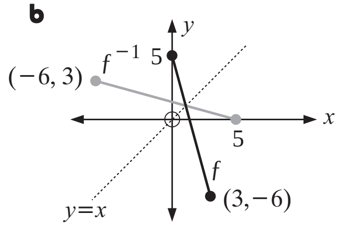


$f(x)$ passes through $(0, 3)$ and $(-3, 0)$
$\therefore\ f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$

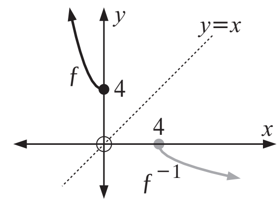
4 **a**



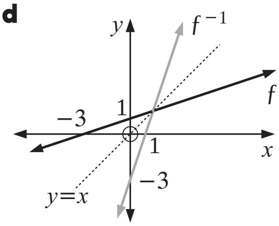
b



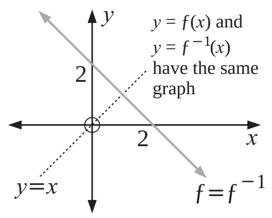
c



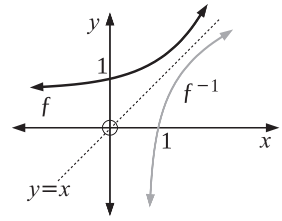
d



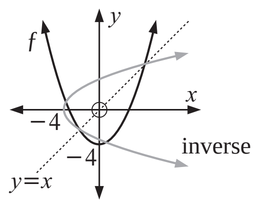
e



f



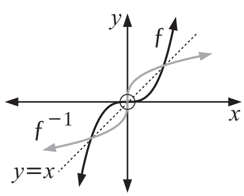
5 **a**



b Using the 'horizontal line test', f does not have an inverse function as a horizontal line through $y = x^2 - 4$ cuts it more than once.

c For $x \geqslant 0$, any horizontal line cuts it only once, i.e., f does have an inverse function for $x \geqslant 0$.

6



7 **a** If $y = f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the 'vertical line test', i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the 'horizontal line test'. Thus, the 'horizontal line test' is a valid test for the existence of an inverse function.

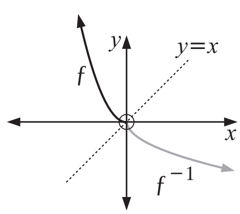
b **i** This graph satisfies the 'horizontal line test' and therefore has an inverse function.

ii, **iii** These graphs both fail the 'horizontal line test' so neither of these have inverse functions.

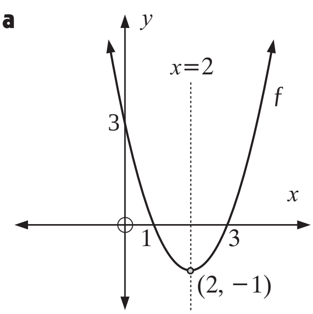
8 **a**

$$\begin{aligned} f \text{ is } \quad y &= x^2, \quad x \leqslant 0 \\ \text{so } f^{-1} \text{ is } \quad x &= y^2, \quad y \leqslant 0 \\ \therefore \quad y &= -\sqrt{x} \\ \text{i.e., } \quad f^{-1}(x) &= -\sqrt{x} \end{aligned}$$

b



9 **a**



$f : x \to x^2 - 4x + 3$ satisfies the 'vertical line test' so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b For $x \geqslant 2$, all horizontal lines cut the graph no more than once. Therefore f has an inverse function for $x \geqslant 2$.

c f is $y = x^2 - 4x + 3, \quad x \geqslant 2$

so f^{-1} is $x = y^2 - 4y + 3, \quad y \geqslant 2$

i.e., $x = (y-2)^2 - 4 + 3, \quad y \geqslant 2$ {completing the square}

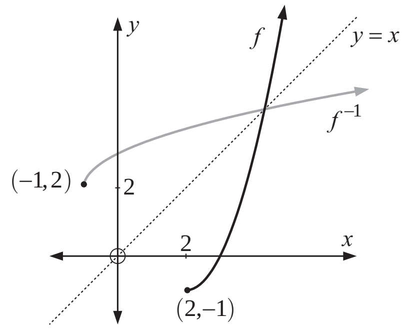
$= (y-2)^2 - 1, \quad y \geqslant 2$

$\therefore \quad x + 1 = (y-2)^2, \quad y \geqslant 2$

$\therefore \quad y - 2 = \sqrt{x+1}, \quad y \geqslant 2$

$\therefore \quad y = 2 + \sqrt{1+x}, \quad y \geqslant 2$

i.e., $f^{-1}(x) = 2 + \sqrt{1+x}$ as required



d **i** domain of f is $\{x\colon\ x \geqslant 2\}$, range is $\{y\colon\ y \geqslant -1\}$

ii domain of f^{-1} is $\{x\colon\ x \geqslant -1\}$, range is $\{y\colon\ y \geqslant 2\}$

10 **a** f is $y = \frac{1}{2}x - 1$ so f^{-1} is $x = \frac{1}{2}y - 1$

$\therefore \quad y = 2x + 2$

i.e., $f^{-1}(x) = 2x + 2$

b **i**
$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f(2x+2)\\ &= \tfrac{1}{2}(2x+2) - 1\\ &= x + 1 - 1\end{aligned}$$
i.e., $(f \circ f^{-1})(x) = x$

ii
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}(\tfrac{1}{2}x - 1)\\ &= 2(\tfrac{1}{2}x - 1) + 2\\ &= x - 2 + 2\\ &= x\end{aligned}$$

11 **a** f is $y = (x+1)^2 + 3, \quad x \geqslant -1$

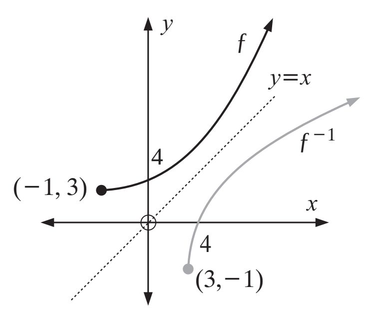
so f^{-1} is $x = (y+1)^2 + 3, \quad y \geqslant -1$

i.e., $x - 3 = (y+1)^2, \quad y \geqslant -1$

$\therefore \quad y + 1 = \sqrt{x-3}, \quad y \geqslant -1, \quad x \geqslant 3$

$\therefore \quad y = \sqrt{x-3} - 1, \quad y \geqslant -1, \quad x \geqslant 3$

b



c **i** Domain $\{x\colon\ x \geqslant -1\}$, Range $\{y\colon\ y \geqslant 3\}$

ii Domain $\{x\colon\ x \geqslant 3\}$, Range $\{y\colon\ y \geqslant -1\}$

12 **a** g is $y = \dfrac{8-x}{2}$

so g^{-1} is $x = \dfrac{8-y}{2}$

$\therefore \quad 2x = 8 - y$

$\therefore \quad y = 8 - 2x$

i.e., $g^{-1}(x) = 8 - 2x$

Now $g^{-1}(-1) = 8 - 2(-1) = 10$

b
$$\begin{aligned}(f \circ g^{-1})(x) &= 9\\ \therefore\quad f(g^{-1}(x)) &= 9\\ \therefore\quad f(8-2x) &= 9\\ \therefore\quad 2(8-2x) + 5 &= 9\\ \therefore\quad 16 - 4x + 5 &= 9\\ \therefore\quad -4x &= -12\\ \therefore\quad x &= 3\end{aligned}$$

13 **a** **i** f is $y = 5^x$

so, $f(2) = 5^2$

$= 25$

ii g is $y = \sqrt{x}$

so g^{-1} is $x = \sqrt{y}$

$\therefore \quad y = x^2$

i.e., $g^{-1}(x) = x^2, \quad x \geqslant 0$

$\therefore \quad g^{-1}(4) = 4^2$

$\therefore \quad g^{-1}(4) = 16$

b
$$\begin{aligned}(g^{-1}\circ f)(x) &= 25\\ \therefore\ g^{-1}(f(x)) &= 25\\ \therefore\ g^{-1}(5^x) &= 25\\ \therefore\ (5^x)^2 &= 25 \quad \{\text{as } g^{-1}(x)=x^2,\ x\geqslant 0\}\end{aligned}$$
and so
$$\begin{aligned}5^{2x} &= 5^2\\ \therefore\ 2x &= 2\\ \therefore\ x &= 1\end{aligned}$$

14 Show: $(f^{-1}\circ g^{-1})(x) = (g\circ f)^{-1}(x)$

$$\begin{aligned}f \text{ is } y &= 2x\\ \text{so } f^{-1} \text{ is } x &= 2y\\ \therefore\ y &= \frac{x}{2}\\ \text{i.e., } f^{-1}(x) &= \frac{x}{2}\end{aligned}$$

$$\begin{aligned}g \text{ is } y &= 4x-3\\ \text{so } g^{-1} \text{ is } x &= 4y-3\\ \therefore\ 4y &= x+3\\ \therefore\ y &= \frac{x+3}{4}\\ \text{i.e., } g^{-1}(x) &= \frac{x+3}{4}\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(2x)\\ &= 4(2x)-3\\ \text{i.e., } (g\circ f)(x) &= 8x-3\\ \text{i.e., } g\circ f \text{ is } y &= 8x-3\\ \text{so } (g\circ f)^{-1} \text{ is } x &= 8y-3\\ \therefore\ y &= \frac{x+3}{8}\\ \text{i.e., } (g\circ f)^{-1}(x) &= \frac{x+3}{8}\end{aligned}$$

$$\begin{aligned}\text{Now, } (f^{-1}\circ g^{-1})(x) &= f^{-1}(g^{-1}(x))\\ &= f^{-1}\left(\frac{x+3}{4}\right)\\ &= \frac{\left(\frac{x+3}{4}\right)}{2}\end{aligned}$$

$$\therefore\ (f^{-1}\circ g^{-1})(x) = \frac{x+3}{8} = (g\circ f)^{-1}(x) \quad \text{as required}$$

15 **a**
$$\begin{aligned}f \text{ is } y &= 2x\\ \text{so } f^{-1} \text{ is } x &= 2y\\ \therefore\ y &= \frac{x}{2}\\ \text{i.e., } f^{-1}(x) &= \frac{x}{2} \neq 2x\\ \text{So, } f^{-1}(x) &\neq f(x)\end{aligned}$$

b
$$\begin{aligned}f \text{ is } y &= x\\ \text{so } f^{-1} \text{ is } x &= y\\ \therefore\ y &= x\\ \text{i.e., } f^{-1}(x) &= x\\ \text{So, } f^{-1}(x) &= f(x)\end{aligned}$$

c
$$\begin{aligned}f \text{ is } y &= -x\\ \text{so } f^{-1} \text{ is } x &= -y\\ \therefore\ y &= -x\\ \text{i.e., } f^{-1}(x) &= -x\\ \text{So, } f^{-1}(x) &= f(x)\end{aligned}$$

d
$$\begin{aligned}f \text{ is } y &= \frac{1}{x}\\ \text{so } f^{-1} \text{ is } x &= \frac{1}{y}\\ \therefore\ y &= \frac{1}{x}\\ \text{i.e., } f^{-1}(x) &= \frac{1}{x}\\ \text{So, } f^{-1}(x) &= f(x)\end{aligned}$$

e
$$\begin{aligned}f \text{ is } y &= -\frac{6}{x}\\ \text{so } f^{-1} \text{ is } x &= -\frac{6}{y}\\ \therefore\ y &= -\frac{6}{x}\\ \text{i.e., } f^{-1}(x) &= -\frac{6}{x}\\ \text{So, } f^{-1}(x) &= f(x)\end{aligned}$$

i.e., $f^{-1}(x) = f(x)$ is true for parts **b**, **c**, **d** and **e**.

EXERCISE 1G

1 f is $y = 3x+1$

so f^{-1} is $x = 3y+1$

$\therefore\ y = \dfrac{x-1}{3}$ i.e., $f^{-1}(x) = \dfrac{x-1}{3}$

$$\begin{aligned}(f\circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f\left(\frac{x-1}{3}\right)\\ &= 3\left(\frac{x-1}{3}\right)+1\\ &= x-1+1\end{aligned}$$

i.e., $(f\circ f^{-1})(x) = x$

$$\begin{aligned}(f^{-1}\circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}(3x+1)\\ &= \frac{3x+1-1}{3}\\ &= \frac{3x}{3}\end{aligned}$$

i.e., $(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)$ as required

2 f is $y = \dfrac{x+3}{4}$ so f^{-1} is $x = \dfrac{y+3}{4}$

$\therefore\ 4x = y+3$

$\therefore\ y = 4x-3$

i.e., $f^{-1}(x) = 4x-3$

$$\begin{aligned}(f\circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f(4x-3)\\ &= \frac{4x-3+3}{4}\\ &= \frac{4x}{4}\end{aligned}$$

i.e., $(f\circ f^{-1})(x) = x$

$$\begin{aligned}(f^{-1}\circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}\left(\frac{x+3}{4}\right)\\ &= 4\left(\frac{x+3}{4}\right)-3\\ &= x+3-3\end{aligned}$$

i.e., $(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)$ as required

3 f is $y = \sqrt{x}$ for $x \geqslant 0$ so f^{-1} is $x = \sqrt{y}$

$\therefore\ y = x^2$

i.e., $f^{-1}(x) = x^2$ for $x \geqslant 0$

$$\begin{aligned}(f\circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f(x^2)\\ &= \sqrt{x^2}\end{aligned}$$

i.e., $(f\circ f^{-1})(x) = x$

$$\begin{aligned}(f^{-1}\circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}(\sqrt{x})\\ &= (\sqrt{x})^2\end{aligned}$$

i.e., $(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)$ as required

4 **a** $f(x)$ passes through $\mathrm{A}(x, f(x))$, so $f^{-1}(x)$ passes through $\mathrm{B}(f(x), x)$

b Substitute the coordinates of $\mathrm{B}(f(x), x)$ into $y = f^{-1}(x)$:

$$\begin{aligned}\text{i.e., } x &= f^{-1}(f(x))\\ &= (f^{-1}\circ f)(x)\end{aligned}$$

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,
so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

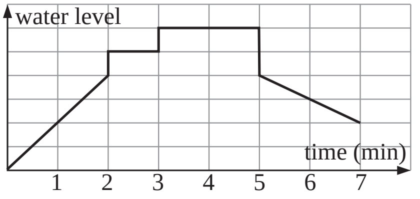
Substitute the coordinates of $\mathrm{A}(f^{-1}(x), x)$ into $y = f(x)$:

i.e., $x = f(f^{-1}(x))$

i.e., $f(f^{-1}(x)) = x$ as required

REVIEW SET 1A

1



2 **a** $\begin{aligned} f(x) &= 2x - x^2 \\ f(2) &= 2(2) - 2^2 \\ &= 0 \end{aligned}$

b $\begin{aligned} f(-3) &= 2(-3) - (-3)^2 \\ &= -6 - 9 \\ &= -15 \end{aligned}$

c $\begin{aligned} f(-\tfrac{1}{2}) &= 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\ &= -1 - \tfrac{1}{4} \\ &= -\tfrac{5}{4} \end{aligned}$

3 **a** **i** range is $\{y\colon\ y \geqslant -5\}$, domain is $\{x\colon\ x \in \mathcal{R}\}$

ii x-intercepts are -1 and 5; y-intercept is $-\frac{25}{9}$

iii The graph passes the 'vertical line test' so is therefore a function.

b **i** range is $\{y\colon\ y = 1 \text{ or } -3\}$, domain is $\{x\colon\ x \in \mathcal{R}\}$

ii there are no x-intercepts; y-intercept is 1

iii The graph passes the 'vertical line test' so is therefore a function.

4 **a** domain is $\{x\colon\ x \geqslant -2\}$, range is $\{y\colon\ 1 \leqslant y < 3\}$

b domain is $\{x\colon\ x \in \mathcal{R}\}$, range is $\{y\colon\ y = -1,\ 1 \text{ or } 2\}$

5 **a** $\begin{aligned} h(x) &= 7 - 3x \\ h(2x-1) &= 7 - 3(2x-1) \\ &= 7 - 6x + 3 \\ &= 10 - 6x \end{aligned}$

b $\begin{aligned} h(2x-1) &= -2 \\ \therefore\ 7 - 3(2x-1) &= -2 \\ \therefore\ 7 - 6x + 3 &= -2 \\ \therefore\ -6x &= -12 \\ \therefore\ x &= 2 \end{aligned}$

6 $f(x) = ax + b$, where $f(1) = 7$ and $f(3) = -5$

When $f(1) = 7$,
$$\begin{aligned} 7 &= a(1) + b \\ \therefore\ 7 &= a + b \\ \therefore\ a &= 7 - b \quad ..... (1) \end{aligned}$$

When $f(3) = -5$,
$$\begin{aligned} -5 &= a(3) + b \\ \therefore\ -5 &= 3a + b \\ \therefore\ -5 &= 3(7-b) + b \quad \{\text{using (1)}\} \\ \therefore\ -5 &= 21 - 3b + b \\ \therefore\ 2b &= 26 \quad \text{and so} \quad b = 13 \end{aligned}$$
i.e., $a = 7 - 13 = -6$

i.e., $a = -6$ and $b = 13$

7 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$

When $f(0) = 5$,
$$\begin{aligned} 5 &= a(0)^2 + b(0) + c \\ \therefore\ 5 &= c \\ \therefore\ c &= 5 \quad ..... (1) \end{aligned}$$

When $f(-2) = 21$,
$$\begin{aligned} 21 &= a(-2)^2 + b(-2) + c \\ &= 4a - 2b + c \\ &= 4a - 2b + 5 \quad \{\text{using (1)}\} \\ \therefore\ 4a - 2b &= 16 \\ \therefore\ 2a - b &= 8 \quad \text{and so} \quad b = 2a - 8 \quad ..... (2) \end{aligned}$$

When $f(3) = -4$,
$$\begin{aligned} -4 &= a(3)^2 + b(3) + c \\ \therefore\ -4 &= 9a + 3b + c \\ \therefore\ -4 &= 9a + 3b + 5 \quad \{\text{using (1)}\} \end{aligned}$$

$\therefore \quad -4 = 9a + 3(2a - 8) + 5 \quad \{\text{using (2)}\}$

$\therefore \quad -9 = 9a + 6a - 24$

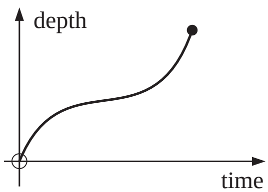
$\therefore \quad 15 = 15a$

$\therefore \quad a = 1$

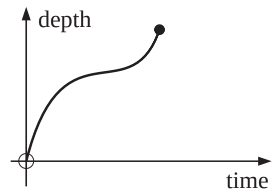
Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$

i.e., $a = 1, \quad b = -6, \quad c = 5$

8 **a**



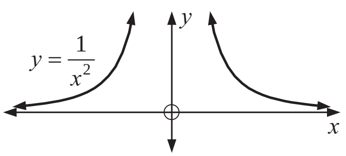
b



9 **a** $f(x) = \dfrac{1}{x^2}$ is meaningless when $x^2 = 0$

i.e., when $x = 0$

b



c domain of $f(x)$ is $\{x\colon\ x \neq 0\}$

range of $f(x)$ is $\{y\colon\ y > 0\}$

10 **a**
$$\begin{aligned} f(g(x)) &= f(x^2 + 2) \\ &= 2(x^2 + 2) - 3 \\ &= 2x^2 + 4 - 3 \\ &= 2x^2 + 1 \end{aligned}$$

b
$$\begin{aligned} g(f(x)) &= g(2x - 3) \\ &= (2x - 3)^2 + 2 \\ &= 4x^2 - 12x + 9 + 2 \\ &= 4x^2 - 12x + 11 \end{aligned}$$

11 **a**
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= 1 - 2\sqrt{x} \end{aligned}$$

b
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(1 - 2x) \\ &= \sqrt{1 - 2x} \end{aligned}$$

12 **a**
$$\begin{aligned} f(g(x)) &= \sqrt{1 - x^2} \\ &= f(1 - x^2) \end{aligned}$$

i.e., $f(x) = \sqrt{x}$,

$g(x) = 1 - x^2$

b
$$g(f(x)) = \left(\frac{x-2}{x+1}\right)^2 = g\left(\frac{x-2}{x+1}\right)$$

i.e., $g(x) = x^2$,

$f(x) = \dfrac{x-2}{x+1}$

REVIEW SET 1B

1 $f(x) = 5 - 2x$

a $f(0)$
$= 5 - 2(0)$
$= 5$

b $f(5)$
$= 5 - 2(5)$
$= -5$

c $f(-3)$
$= 5 - 2(-3)$
$= 11$

d $f(\frac{1}{2})$
$= 5 - 2(\frac{1}{2})$
$= 4$

2 $g(x) = x^2 - 3x$

a
$$\begin{aligned} g(x + 1) &= (x + 1)^2 - 3(x + 1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2 \end{aligned}$$

b
$$\begin{aligned} g(x^2 - 2) &= (x^2 - 2)^2 - 3(x^2 - 2) \\ &= x^4 - 4x^2 + 4 - 3x^2 + 6 \\ &= x^4 - 7x^2 + 10 \end{aligned}$$

3 **a**

$$\begin{aligned} f(x) &= 7-4x \\ \text{i.e.,}\quad y &= 7-4x \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 7-4y \\ \therefore\quad y &= \frac{7-x}{4} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{7-x}{4} \end{aligned}$$

b

$$\begin{aligned} f(x) &= \frac{3+2x}{5} \\ \text{i.e.,}\quad y &= \frac{3+2x}{5} \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= \frac{3+2y}{5} \\ \therefore\quad 5x &= 3+2y \\ \therefore\quad y &= \frac{5x-3}{2} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{5x-3}{2} \end{aligned}$$

4 **a** $y=(x-1)(x-5)$

i.e., x-intercepts are $x=1$ and 5

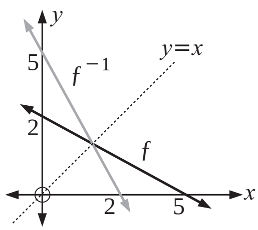
$\therefore$ vertex is at $x=3$, $\quad y = (3-1)(3-5) = 2\times(-2) = -4$

i.e., vertex is at $(3, -4)$

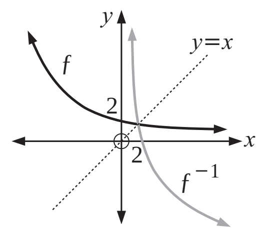
domain is $\{x\colon\ x\in\mathcal{R}\}$, range is $\{y\colon\ y\geqslant -4\}$

b From the graph, domain is $\{x\colon\ x\neq 0,\ 2\}$, range is $\{y\colon\ y\leqslant -1\ \text{ or }\ y>0\}$

5 **a**



b



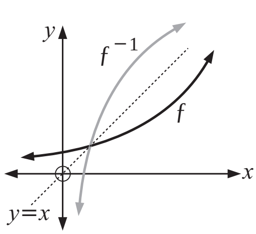
6 **a**

$$\begin{aligned} f(x) &= 4x+2 \\ \text{i.e.,}\quad y &= 4x+2 \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 4y+2 \\ \therefore\quad y &= \frac{x-2}{4} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x-2}{4} \end{aligned}$$

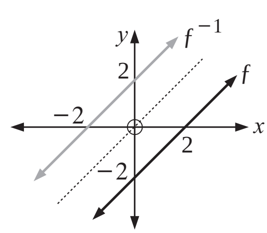
b

$$\begin{aligned} f(x) &= \frac{3-5x}{4} \\ \text{i.e.,}\quad y &= \frac{3-5x}{4} \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= \frac{3-5y}{4} \\ \therefore\quad 4x &= 3-5y \\ \therefore\quad y &= \frac{3-4x}{5} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{3-4x}{5} \end{aligned}$$

7 **a**



b

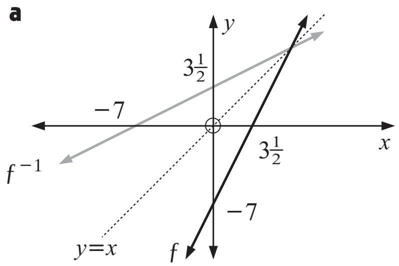


8

$$\begin{aligned} f(x) &= 2x+11 \\ \text{i.e.,}\quad y &= 2x+11 \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 2y+11 \\ \therefore\quad y &= \frac{x-11}{2} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x-11}{2} \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 \\ (g\circ f^{-1})(x) &= g(f^{-1}(x)) \\ &= g\left(\frac{x-11}{2}\right) \\ &= \left(\frac{x-11}{2}\right)^2 \\ \therefore\quad (g\circ f^{-1})(3) &= \left(\frac{3-11}{2}\right)^2 \\ &= (-4)^2 \\ &= 16 \end{aligned}$$

9 **a**



b $f(x)=2x-7$ passes through $(0,\,-7)$ and $(\frac{7}{2},\,0)$

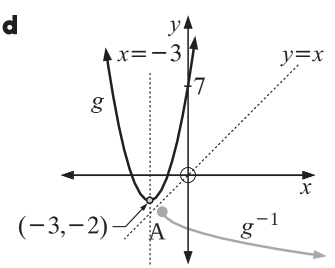
$\therefore$ $f^{-1}(x)$ passes through $(-7,\,0)$ and $(0,\,\frac{7}{2})$

$$\begin{aligned} f^{-1}(x)\ \text{has slope}\quad \frac{\frac{7}{2}-0}{0-(-7)} &= \frac{\frac{7}{2}}{7} \\ &= \tfrac{1}{2} \\ \text{so its equation is}\quad \frac{y-0}{x-(-7)} &= \frac{1}{2} \\ \therefore\quad y &= \frac{x+7}{2} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

c

$$\begin{aligned} f(x) &= 2x-7 \\ \text{i.e.,}\quad y &= 2x-7 \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 2y-7 \\ \therefore\quad y &= \frac{x+7}{2} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

10 **a**, **d**



b If $x \leqslant -3$, we have the graph to the left of $x=-3$, and any horizontal line through the graph cuts it no more than once.
Therefore it has an inverse function.

c

$$\begin{aligned} g(x) &= x^2+6x+7,\quad x\leqslant -3 \\ \text{i.e.,}\quad y &= x^2+6x+7,\quad x\leqslant -3 \\ \text{so}\quad g^{-1}(x)\ \text{is}\quad x &= y^2+6y+7,\quad y\leqslant -3 \\ &= (y+3)^2-9+7 \\ \therefore\quad x+2 &= (y+3)^2 \\ \therefore\quad y+3 &= \pm\sqrt{x+2} \\ \therefore\quad y &= -3\pm\sqrt{x+2} \\ \text{but}\ y\leqslant -3,\ \text{so}\quad y &= -3-\sqrt{x+2} \end{aligned}$$

11

$$\begin{aligned} h(x) &= (x-4)^2+3, \quad x \geqslant 4 \\ \text{i.e.,} \quad y &= (x-4)^2+3, \quad x \geqslant 4 \\ \text{so} \quad h^{-1}(x) \text{ is } \quad x &= (y-4)^2+3, \quad y \geqslant 4 \\ \therefore \quad x-3 &= (y-4)^2 \\ \therefore \quad y-4 &= \pm\sqrt{x-3} \\ \therefore \quad y &= 4 \pm \sqrt{x-3} \\ \text{but } y \geqslant 4, \quad \text{so} \quad y &= 4+\sqrt{x-3} \\ \text{i.e.,} \quad h^{-1}(x) &= 4+\sqrt{x-3}, \quad x \geqslant 3 \end{aligned}$$

12

$$\begin{aligned} f(x) &= 3x+6 \\ \text{i.e.,} \quad y &= 3x+6 \\ \text{so} \quad f^{-1}(x) \text{ is } \quad x &= 3y+6 \\ \therefore \quad y &= \frac{x-6}{3} \\ \text{i.e.,} \quad f^{-1}(x) &= \frac{x-6}{3} \end{aligned}$$

$$\begin{aligned} h(x) &= \frac{x}{3} \\ \text{i.e.,} \quad y &= \frac{x}{3} \\ \text{so} \quad h^{-1}(x) \text{ is } \quad x &= \frac{y}{3} \\ \therefore \quad y &= 3x \\ \text{i.e.,} \quad h^{-1}(x) &= 3x \end{aligned}$$

$$\begin{aligned} \text{Now} \quad (f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x)) \\ &= f^{-1}(3x) \\ &= \frac{3x-6}{3} \\ &= x-2 \end{aligned}$$

$$\begin{aligned} (h \circ f)(x) &= h(f(x)) \\ &= h(3x+6) \\ &= \frac{3x+6}{3} \\ \text{i.e.,} \quad y &= x+2 \\ \text{i.e.,} \quad (h \circ f)^{-1}(x) \text{ is } \quad x &= y+2 \\ \therefore \quad y &= x-2 \\ \text{i.e.,} \quad (h \circ f)^{-1}(x) &= x-2 \end{aligned}$$

i.e., $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required

Chapter 2

SEQUENCES AND SERIES

EXERCISE 2A

1 **a** 4, 13, 22, 31, ... **b** 45, 39, 33, 27, ... **c** 2, 6, 18, 54, ... **d** 96, 48, 24, 12, ...

2 **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.

b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.

c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.

d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.

e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.

f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.

g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.

h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.

i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.

3 **a** 79, 75 **b** 1280, 5120 **c** 81, 90

4 **a** Each term is the square of the number of the term. The next three terms are 25, 36 and 49.

b Each term is the cube of the number of the term. The next three terms are 125, 216 and 343.

c Each term is $n \times (n+1)$ where n is the number of the term. The next three terms are 30, 42 and 56.

5 **a** 625, 1296 **b** 13, 21 **c** 9, 11 **d** 13, 17 (primes) **e** 16, 22 **f** 14, 18

EXERCISE 2B

1 **a** $\{2n\}$ generates the sequence 2, 4, 6, 8, 10, ... (letting $n = 1, 2, 3, 4, 5, ...$)

b $\{2n+2\}$ generates the sequence 4, 6, 8, 10, 12, ... (letting $n = 1, 2, 3, 4, 5, ...$)

c $\{2n-1)$ generates the sequence 1, 3, 5, 7, 9, ... (letting $n = 1, 2, 3, 4, 5, ...$)

d $\{2n-3\}$ generates the sequence -1, 1, 3, 5, 7, ... (letting $n = 1, 2, 3, 4, 5, ...$)

e $\{2n+3\}$ generates the sequence 5, 7, 9, 11, 13, ... (letting $n = 1, 2, 3, 4, 5, ...$)

f $\{2n+11\}$ generates the sequence 13, 15, 17, 19, 21, ... (letting $n = 1, 2, 3, 4, 5, ...$)

g $\{3n+1\}$ generates the sequence 4, 7, 10, 13, 16, ... (letting $n = 1, 2, 3, 4, 5, ...$)

h $\{4n-3\}$ generates the sequence 1, 5, 9, 13, 17, ... (letting $n = 1, 2, 3, 4, 5, ...$)

i $\{5n+4\}$ generates the sequence 9, 14, 19, 24, 29, ... (letting $n = 1, 2, 3, 4, 5, ...$)

2 **a** $\{2^n\}$ generates the sequence 2, 4, 8, 16, 32, ... (letting $n = 1, 2, 3, 4, 5, ...$)

b $\{3 \times 2^n\}$ generates the sequence 6, 12, 24, 48, 96, ... (letting $n = 1, 2, 3, 4, 5, ...$)

c $\{6 \times (\frac{1}{2})^n\}$ generates the sequence 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$, ... (letting $n = 1, 2, 3, 4, 5, ...$)

d $\{(-2)^n\}$ generates the sequence -2, 4, -8, 16, -32, ... (letting $n = 1, 2, 3, 4, 5, ...$)

3 $\{15-(-2)^n\}$ generates the sequence with first five terms:

$t_1 = 15-(-2)^1 = 17$, $t_2 = 15-(-2)^2 = 11$, $t_3 = 15-(-2)^3 = 23$,
$t_4 = 15-(-2)^4 = -1$, $t_5 = 15-(-2)^5 = 47$

EXERCISE 2C

1 **a** $$\begin{aligned}17-6 &= 11\\ 28-17 &= 11\\ 39-28 &= 11\\ 50-39 &= 11\end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 11.
$\therefore$ the sequence is arithmetic with $u_1 = 6$, $d = 11$.

b $$\begin{aligned}u_n &= u_1 + (n-1)d\\ &= 6 + (n-1)11\end{aligned}$$
i.e., $u_n = 11n - 5$

c $$\begin{aligned}u_{50} &= 11(50) - 5\\ &= 545\end{aligned}$$

d Let $u_n = 325 = 11n - 5$
$$\begin{aligned}\therefore\ 330 &= 11n\\ \therefore\ n &= 30\end{aligned}$$
So, 325 is a member, i.e., u_{30}.

e Let $u_n = 761 = 11n - 5$
$$\begin{aligned}\therefore\ 766 &= 11n\\ \therefore\ n &= 69\tfrac{7}{11},\end{aligned}$$
but n is an integer, so 761 is not a member of the sequence.

2 **a** $$\begin{aligned}83-87 &= -4\\ 79-83 &= -4\\ 75-79 &= -4\end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by -4.
$\therefore$ the sequence is arithmetic with $u_1 = 87$, $d = -4$.

b $$\begin{aligned}u_n &= u_1 + (n-1)d\\ &= 87 + (n-1)(-4)\\ &= 87 - 4n + 4\end{aligned}$$
i.e., $u_n = 91 - 4n$

c $$\begin{aligned}u_{40} &= 91 - 4(40)\\ &= 91 - 160\\ &= -69\end{aligned}$$

d Let $u_n = -143 = 91 - 4n$
$$\begin{aligned}\therefore\ 4n &= 234\\ \therefore\ n &= 58\tfrac{1}{2}\end{aligned}$$
but n is an integer, so -143 is not a member of the sequence.

3 **a** $u_n = 3n - 2$ $\qquad u_1 = 3(1) - 2 = 1$
$$\begin{aligned}u_{n+1} &= 3(n+1) - 2\\ &= 3n + 1\end{aligned}$$
$$\begin{aligned}u_{n+1} - u_n &= (3n+1) - (3n-2)\\ &= 3 \quad \text{which is constant}\end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 3.
$\therefore$ the sequence is arithmetic with $u_1 = 1$ and $d = 3$.

b $u_1 = 1$, $d = 3$

c $$\begin{aligned}u_{57} &= 3(57) - 2\\ &= 169\end{aligned}$$

d Let $u_n = 450 = 3n - 2$
$$\begin{aligned}\text{i.e., } 3n &= 452\\ \therefore\ n &= 150\tfrac{2}{3}\end{aligned}$$

So, try the two values on either side of $n = 150\frac{2}{3}$, i.e., for $n = 150$ and $n = 151$:

$$\begin{aligned}u_{150} &= 3(150) - 2\\ &= 448\end{aligned} \quad \text{and} \quad \begin{aligned}u_{151} &= 3(151) - 2\\ &= 451\end{aligned}$$

i.e., $u_{151} = 451$ is the least term which is greater than 450.

4 **a** $u_n = \dfrac{71-7n}{2} = 35\frac{1}{2} - \frac{7}{2}n$ $\qquad u_1 = \dfrac{71-7(1)}{2} = 32$

$$\begin{aligned} u_{n+1} &= \frac{71-7(n+1)}{2} \\ &= \frac{71-7n-7}{2} \\ &= \frac{64-7n}{2} \\ &= 32 - \tfrac{7}{2}n \end{aligned}$$

$$\begin{aligned} u_{n+1} - u_n &= (32 - \tfrac{7}{2}n) - (35\tfrac{1}{2} - \tfrac{7}{2}n) \\ &= -\tfrac{7}{2} \quad \text{which is constant} \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by $-\frac{7}{2}$.

$\therefore$ the sequence is arithmetic with $u_1 = 32$, $d = -\frac{7}{2}$.

b $u_1 = 32, \quad d = -\frac{7}{2}$

c $u_{75} = \dfrac{71-7(75)}{2} = -227$

d Let $u_n = -200 = \dfrac{71-7n}{2}$ $\qquad$ i.e., $\begin{aligned} -400 &= 71 - 7n \\ \therefore\ 7n &= 471 \\ \therefore\ n &= 67\tfrac{2}{7} \end{aligned}$

So, try the two values on either side of $n = 67\frac{2}{7}$, i.e., for $n = 67$ and $n = 68$:

$u_{67} = \dfrac{71-7(67)}{2} = -199$ and $u_{68} = \dfrac{71-7(68)}{2} = -202\frac{1}{2}$

i.e., terms of the sequence are less than -200 for $n \geqslant 68$.

5 **a** Since the terms are consecutive,

$$k - 32 = 3 - k$$

{equating common differences}

$$\begin{aligned} \therefore\ 2k &= 35 \\ \therefore\ k &= 17\tfrac{1}{2} \end{aligned}$$

b Since the terms are consecutive,

$$7 - k = 10 - 7$$

{equating common differences}

$$\begin{aligned} \therefore\ 7 - k &= 3 \\ \therefore\ k &= 4 \end{aligned}$$

c Since the terms are consecutive,

$$(2k+1) - (k+1) = 13 - (2k+1)$$

{equating common differences}

$$\begin{aligned} \therefore\ k &= 12 - 2k \\ \therefore\ 3k &= 12 \\ \therefore\ k &= 4 \end{aligned}$$

d Since the terms are consecutive,

$$(2k+3) - (k-1) = (7-k) - (2k+3)$$

{equating common differences}

$$\begin{aligned} \therefore\ k + 4 &= 4 - 3k \\ \therefore\ 4k &= 0 \\ \therefore\ k &= 0 \end{aligned}$$

e Since the terms are consecutive,

$$k^2 - k = k^2 + 6 - k^2$$

{equating common differences}

$$\begin{aligned} \therefore\ k^2 - k &= 6 \\ \therefore\ k^2 - k - 6 &= 0 \\ \therefore\ (k-3)(k+2) &= 0 \\ \therefore\ k &= -2 \text{ or } 3 \end{aligned}$$

f Since the terms are consecutive,

$$k - 5 = k^2 - 8 - k$$

{equating common differences}

$$\begin{aligned} \therefore\ k^2 - 2k - 3 &= 0 \\ \therefore\ (k-3)(k+1) &= 0 \\ \therefore\ k &= -1 \text{ or } 3 \end{aligned}$$

6 **a** $u_7 = 41 \quad \therefore\ u_1 + 6d = 41$ .... (1)
$u_{13} = 77 \quad \therefore\ u_1 + 12d = 77$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -41 \\ u_1 + 12d &= 77 \\ \hline \therefore\ 6d &= 36 \\ \therefore\ d &= 6 \end{aligned}$$ {adding the equations}

So in (1), $u_1 + 6(6) = 41$
$\therefore\ u_1 + 36 = 41$
$\therefore\ u_1 = 5$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 5 + (n-1)6$
$\therefore\ u_n = 5 + 6n - 6$
$\therefore\ u_n = 6n - 1$

b $u_5 = -2 \quad \therefore\ u_1 + 4d = -2$ .... (1)
$u_{12} = -12\frac{1}{2} \quad \therefore\ u_1 + 11d = -12\frac{1}{2}$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 4d &= 2 \\ u_1 + 11d &= -12\tfrac{1}{2} \\ \hline \therefore\ 7d &= -10\tfrac{1}{2} \\ \therefore\ d &= -\tfrac{3}{2} \end{aligned}$$ {adding the equations}

So in (1), $u_1 + 4(-\frac{3}{2}) = -2$
$\therefore\ u_1 - 6 = -2$
$\therefore\ u_1 = 4$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 4 + (n-1)(-\frac{3}{2})$
$\therefore\ u_n = 4 - \frac{3}{2}n + \frac{3}{2}$
$\therefore\ u_n = -\frac{3}{2}n + \frac{11}{2}$

c $u_7 = 1 \quad \therefore\ u_1 + 6d = 1$ .... (1)
$u_{15} = -39 \quad \therefore\ u_1 + 14d = -39$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -1 \\ u_1 + 14d &= -39 \\ \hline \therefore\ 8d &= -40 \\ \therefore\ d &= -5 \end{aligned}$$ {adding the equations}

So in (1), $u_1 + 6(-5) = 1$
$\therefore\ u_1 - 30 = 1$
$\therefore\ u_1 = 31$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 31 + (n-1)(-5)$
$\therefore\ u_n = 31 - 5n + 5$
$\therefore\ u_n = -5n + 36$

d $u_{11} = -16 \quad \therefore\ u_1 + 10d = -16$ .... (1)
$u_8 = -11\frac{1}{2} \quad \therefore\ u_1 + 7d = -11\frac{1}{2}$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 10d &= 16 \\ u_1 + 7d &= -11\tfrac{1}{2} \\ \hline \therefore\ -3d &= 4\tfrac{1}{2} \\ \therefore\ d &= -\tfrac{3}{2} \end{aligned}$$ {adding the equations}

So in (1), $u_1 + 10(-\frac{3}{2}) = -16$
$\therefore\ u_1 - 15 = -16$
$\therefore\ u_1 = -1$

Now $u_n = u_1 + (n-1)d$
$\therefore\ u_n = -1 + (n-1)(-\frac{3}{2})$
$\therefore\ u_n = -1 - \frac{3}{2}n + \frac{3}{2}$
$\therefore\ u_n = -\frac{3}{2}n + \frac{1}{2}$

7 **a** Let the numbers be $5,\ 5+d,\ 5+2d,\ 5+3d,\ 10$.

Then $5 + 4d = 10$
$\therefore\ 4d = 5$
$\therefore\ d = \frac{5}{4} = 1\frac{1}{4}$

So the numbers are $5,\ 6\frac{1}{4},\ 7\frac{1}{2},\ 8\frac{3}{4},\ 10$.

b Let the numbers be $-1,\ -1+d,\ -1+2d,\ -1+3d,\ -1+4d,\ -1+5d,\ -1+6d,\ 32$.

Then $-1 + 7d = 32$
$\therefore\ 7d = 33$
$\therefore\ d = \frac{33}{7} = 4\frac{5}{7}$

So the numbers are $-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32$.

8 **a** $u_1 = 36,\quad 35\frac{1}{3} - 36 = -\frac{2}{3}$

$34\frac{2}{3} - 35\frac{1}{3} = -\frac{2}{3},\quad$ so $\quad d = -\frac{2}{3}$

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ -30 &= 36 + (n-1)(-\tfrac{2}{3}) \qquad \{\text{letting } u_n = -30, \text{ the last term of the sequence}\} \\ \therefore\ -66 &= -\tfrac{2}{3}n + \tfrac{2}{3} \\ \therefore\ \tfrac{2}{3}n &= 66\tfrac{2}{3} \\ \therefore\ n &= 100 \qquad \text{i.e., the sequence has 100 terms} \end{aligned}$$

9 $u_1 = 23,\quad 36 - 23 = 13$

$49 - 36 = 13$

$62 - 49 = 13,\quad$ so $\quad d = 13$

$$\begin{aligned} \therefore\ u_n &= u_1 + (n-1)d \\ \text{i.e., } u_n &= 23 + (n-1)13 \\ &= 23 + 13n - 13 \\ \therefore\ u_n &= 13n + 10 \end{aligned}$$

Let $\quad u_n = 100\,000 = 13n + 10$

$\therefore\ 99\,990 = 13n$

$\therefore\ n = 7691\frac{7}{13}$

So, try the two values on either side of $n = 7691\frac{7}{13}$, i.e., for $n = 7691$ and $n = 7692$:

i.e., $u_{7691} = 13(7691) + 10 = 99\,993$ and $u_{7692} = 13(7692) + 10 = 100\,006$

i.e., the first term to exceed $100\,000$ is $u_{7692} = 100\,006$.

EXERCISE 2D

1 **a** $\dfrac{6}{2} = 3 \quad \therefore\ r = 3,\quad u_1 = 2 \quad \therefore\ b = 6 \times 3 = 18$ and $c = 18 \times 3 = 54$

b $\dfrac{5}{10} = \dfrac{1}{2} \quad \therefore\ r = \frac{1}{2},\quad u_1 = 10 \quad \therefore\ b = 5 \times \frac{1}{2} = 2\frac{1}{2}$ and $c = 2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$

c $\dfrac{-6}{12} = -\dfrac{1}{2} \quad \therefore\ r = -\frac{1}{2},\quad u_1 = 12 \quad \therefore\ b = -6 \times -\frac{1}{2} = 3$ and $c = 3 \times -\frac{1}{2} = -1\frac{1}{2}$

2 **a** $\dfrac{10}{5} = 2$

$\dfrac{20}{10} = 2$

$\dfrac{40}{20} = 2$

So, assuming the pattern continues, consecutive terms have a common ratio of 2.

$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2$.

b
$$\begin{aligned} u_n &= u_1 r^{n-1} \\ \therefore\ u_n &= 5 \times 2^{n-1} \\ \text{so}\quad u_{15} &= 5 \times 2^{14} \\ &= 81\,920 \end{aligned}$$

3 **a** $\dfrac{-6}{12} = -\frac{1}{2}$

$\dfrac{3}{-6} = -\frac{1}{2}$

$\dfrac{-1.5}{3} = \dfrac{(-\frac{3}{2})}{3} = -\frac{1}{2}$

So, assuming the pattern continues, consecutive terms have a common ratio of $-\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = -\frac{1}{2}$.

b $u_n = u_1 r^{n-1}$
$\therefore\ u_n = 12(2^{-1})^{n-1}$
$= 12 \times 2^{-n+1}$
$= 3 \times 2^2 \times 2^{-n+1}$
$= 3 \times 2^{-n+3}$

so $u_{13} = 3 \times 2^{-13+3}$
$= 3 \times 2^{-10}$
$= 3 \times \frac{1}{1024}$
$= \frac{3}{1024}$

4 $\frac{-6}{8} = -\frac{3}{4}$

$\frac{4.5}{-6} = -\frac{(\frac{9}{2})}{6} = -\frac{3}{4}$

$\frac{-3.375}{4.5} = \frac{(-\frac{27}{8})}{(\frac{9}{2})} = -\frac{3}{4}$

So, assuming the pattern continues, consecutive terms have a common ratio of $-\frac{3}{4}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = -\frac{3}{4}$.

$u_n = u_1 r^{n-1} = 8 \times (-\frac{3}{4})^{n-1}$ i.e., $u_{10} = 8 \times (-\frac{3}{4})^9 \doteqdot -0.600\,677\,49$

5 $\frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

So, assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{\sqrt{2}}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

$$u_n = u_1 r^{n-1} = 8\left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n+\frac{1}{2}}$$

i.e., $u_n = 2^{\frac{7}{2}-\frac{n}{2}}$

6 **a** Since the terms are geometric,

$\frac{k}{7} = \frac{28}{k}$ $\therefore\ k^2 = 196$
$\therefore\ k = \pm 14$

b Since the terms are geometric,

$\frac{3k}{k} = \frac{20-k}{3k} = 3$
$\therefore\ 20 - k = 9k$
$\therefore\ 20 = 10k$
$\therefore\ k = 2$

c Since the terms are geometric,

$\frac{k+8}{k} = \frac{9k}{k+8}$
$\therefore\ (k+8)^2 = 9k^2$
$\therefore\ k^2 + 16k + 64 = 9k^2$
$\therefore\ 8k^2 - 16k - 64 = 0$
$\therefore\ k^2 - 2k - 8 = 0$
$\therefore\ (k+2)(k-4) = 0$ and so $k = -2$ or 4

7 **a** $u_4 = 24$ $\therefore\ u_1 \times r^3 = 24$ .... (1)
$u_7 = 192$ $\therefore\ u_1 \times r^6 = 192$ .... (2)

So, $\frac{u_1 r^6}{u_1 r^3} = \frac{192}{24}$ $\{(2) \div (1)\}$

$\therefore\ r^3 = 8$ $\therefore\ r = 2$

So in (1), $u_1 \times 2^3 = 24$
$\therefore\ u_1 \times 8 = 24$
$\therefore\ u_1 = 3$

$\therefore\ u_n = 3 \times 2^{n-1}$

b $u_3 = 8$ $\therefore\ u_1 \times r^2 = 8$ .... (1)
$u_6 = -1$ $\therefore\ u_1 \times r^5 = -1$ .... (2)

So, $\frac{u_1 r^5}{u_1 r^2} = -\frac{1}{8}$ $\{(2) \div (1)\}$

$\therefore\ r^3 = -\frac{1}{8}$ $\therefore\ r = -\frac{1}{2}$

So in (1), $u_1 \times (-\frac{1}{2})^2 = 8$
$\therefore\ u_1 \times \frac{1}{4} = 8$
$\therefore\ u_1 = 32$

$\therefore\ u_n = 32 \times (-\frac{1}{2})^{n-1}$

c $u_7 = 24 \quad \therefore \; u_1 \times r^6 = 24 \quad .... (1)$

$u_{15} = 384 \quad \therefore \; u_1 \times r^{14} = 384 \quad .... (2)$

So, $\dfrac{u_1 r^{14}}{u_1 r^6} = \dfrac{384}{24} \quad \{(2) \div (1)\}$

$\therefore \; r^8 = 16 \qquad \therefore \; r = \sqrt{2}$

So in (1), $u_1 \times (\sqrt{2})^6 = 24$

$\therefore \; u_1 \times 8 = 24$

$\therefore \; u_1 = 3$

Now $u_n = u_1 r^{n-1}$

$\therefore \; u_n = 3 \times (\sqrt{2})^{n-1}$

d $u_3 = 5 \quad \therefore \; u_1 \times r^2 = 5 \quad .... (1)$

$u_7 = \frac{5}{4} \quad \therefore \; u_1 \times r^6 = \frac{5}{4} \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^2} = \dfrac{(\frac{5}{4})}{5} \quad \{(2) \div (1)\}$

$\therefore \; r^4 = \frac{1}{4} \qquad \therefore \; r = \frac{1}{\sqrt{2}}$

So in (1), $u_1 \times (\frac{1}{\sqrt{2}})^2 = 5$

$\therefore \; u_1 \times \frac{1}{2} = 5$

$\therefore \; u_1 = 10$

Now $u_n = u_1 r^{n-1}$

$\therefore \; u_n = 10 \times (\frac{1}{\sqrt{2}})^{n-1}$

$\therefore \; u_n = 10 \times (\sqrt{2})^{1-n}$

8 **a** 2, 6, 18, 54, ... i.e., $u_1 = 2$ and $r = 3$

$u_n = u_1 r^{n-1} \qquad \therefore \; u_n = 2 \times 3^{n-1}$

Let $u_n = 10\,000 = 2 \times 3^{n-1}$ i.e., $5000 = 3^{n-1}$

$\therefore \; n \doteqdot 8.7527$ {using technology}

So, try the two values on either side of $n = 8.7527$, i.e., for $n = 8$ and $n = 9$:

$u_8 = 2 \times 3^7 = 4374$ and $u_9 = 2 \times 3^8 = 13\,122$

i.e., the first term to exceed 10 000 is $u_9 = 13\,122$.

b $4, 4\sqrt{3}, 12, 12\sqrt{3}$, ... i.e., $u_1 = 4$ and $r = \sqrt{3}$

$u_n = u_1 r^{n-1} \qquad \therefore \; u_n = 4 \times \left(\sqrt{3}\right)^{n-1}$

Let $u_n = 4800$ i.e., $1200 = \left(\sqrt{3}\right)^{n-1}$

$\therefore \; n \doteqdot 13.91$ {using technology}

So, try the two values on either side of $n \doteqdot 13.91$, i.e., for $n = 13$ and $n = 14$:

$u_{13} = 4 \times \left(\sqrt{3}\right)^{12} = 2916$ and $u_{14} = 4 \times \left(\sqrt{3}\right)^{13} \doteqdot 5050.66$

i.e., the first term to exceed 4800 is $u_{14} \doteqdot 5050.66$.

c 12, 6, 3, 1.5, ... i.e., $u_1 = 12$ and $r = \frac{1}{2}$

$u_n = u_1 r^{n-1} \qquad \therefore \; u_n = 12 \times (\frac{1}{2})^{n-1}$

Let $0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

i.e., $0.000\,008\overline{3} = \left(\frac{1}{2}\right)^{n-1}$

$\therefore \; n \doteqdot 17.87$ {using technology}

So, try the two values on either side of $n \doteqdot 17.87$, i.e., for $n = 17$ and $n = 18$:

$u_{17} = 12 \times \left(\frac{1}{2}\right)^{16} \doteqdot 0.000\,1831$ and $u_{18} = 12 \times \left(\frac{1}{2}\right)^{17} \doteqdot 0.000\,091\,55$

i.e., the first term of the sequence which is less than 0.0001 is $u_{18} \doteqdot 0.000\,091\,55$.

9 **a**
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 3000, \quad r = 1.1, \quad n = 3 \\ \text{i.e.,} \quad u_4 &= 3000 \times (1.1)^3 \\ &= 3993 \end{aligned}$$

So it amounts to \$3993.

b
$$\begin{aligned} \text{Interest} &= \text{amount after 3 years} - \text{initial amount} \\ &= \$3993 - \$3000 \\ &= \$993 \end{aligned}$$

10
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 20\,000, \quad r = 1.12, \quad n = 4 \\ \text{i.e.,} \quad u_5 &= 20\,000 \times (1.12)^4 \\ &= 31\,470.39 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= 31\,470.39 - 20\,000 \ \text{ Euro} \\ &= 11\,470.39 \text{ Euro} \end{aligned}$$

11 **a**
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 30\,000, \quad r = 1.1, \quad n = 4 \\ \text{i.e.,} \quad u_5 &= 30\,000 \times (1.1)^4 \\ &= 43\,923, \ \text{ i.e., the investment amounts to } 43\,923 \text{ Yen.} \end{aligned}$$

b
$$\begin{aligned} \text{Interest} &= \text{amount after 4 years} - \text{initial amount} \\ &= 43\,923 - 30\,000 \ \text{ Yen} \\ &= 13\,923 \text{ Yen} \end{aligned}$$

12
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 80\,000, \quad r = 1.09, \quad n = 3 \\ \text{i.e.,} \quad u_4 &= 80\,000 \times (1.09)^3 \\ &= 103\,602.32 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \text{amount after 3 years} - \text{initial amount} \\ &= \$103\,602.32 - \$80\,000 \\ &= \$23\,602.32 \end{aligned}$$

13
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 100\,000, \quad r = 1 + \frac{0.08}{2} = 1.04, \quad n = 10 \\ \text{i.e.,} \quad u_{11} &= 100\,000 \times (1.04)^{10} \\ &= 148\,024.43, \ \text{ i.e., it amounts to } 148\,024.43 \text{ Yen.} \end{aligned}$$

14
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_1 = 45\,000, \quad r = 1 + \frac{0.075}{4} = 1.018\,75, \\ \text{i.e.,} \quad u_{10} &= 45\,000 \times (1.018\,75)^7 \qquad\qquad n = 7 \\ &= 51\,249.06 \end{aligned}$$

i.e., it amounts to £51 249.06

15
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_{n+1} = 20\,000, \quad r = 1.075, \quad n = 4 \\ \therefore \quad 20\,000 &= u_1 \times (1.075)^4 \\ \therefore \quad u_1 &= 14\,976.01 \end{aligned}$$

So, \$14 976.01 should be invested now.

16
$$\begin{aligned} u_{n+1} &= u_1 \times r^n, \quad \text{where} \quad u_{n+1} = 15\,000, \quad r = 1.055, \quad n = \tfrac{60}{12} = 5 \\ \therefore \quad 15\,000 &= u_1 \times (1.055)^5 \\ \therefore \quad u_1 &= 11\,477.02 \end{aligned}$$

So, the initial investment required is £11 477.02.

17 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 25\,000$, $r = 1 + \dfrac{0.08}{4} = 1.02$, $n = 3 \times 4 = 12$

$\therefore\ 25\,000 = u_1 \times (1.02)^{12}$

$\therefore\ u_1 = 19\,712.33$

i.e., should invest 19 712.33 Euro now.

18 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 40\,000$, $r = 1 + \dfrac{0.09}{12} = 1.0075$, $n = 8 \times 12 = 96$

$\therefore\ 40\,000 = u_1 \times (1.0075)^{96}$

$\therefore\ u_1 = 19\,522.47$, i.e., initial investment should be 19 522.47 Yen.

19 $u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$

a **i** $u_{11} = 500 \times (1.12)^{10} \doteqdot 1552.92 \doteqdot 1550$ ants

ii $u_{21} = 500 \times (1.12)^{20} \doteqdot 4823.15 \doteqdot 4820$ ants

b For the population to reach 2000,

$u_{n+1} = 500 \times (1.12)^n = 2000$

$\therefore\ (1.12)^n = 4$

$\therefore\ n \doteqdot 12.23$ {using technology}

i.e., it will take approximately 12.2 weeks.

20 $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$

a $u_{16} = 555 \times (0.955)^{15}$

$\doteqdot 278.19$ i.e., the population is approximately 278 animals in the year 2000.

b For the population to have declined to 50,

$u_{n+1} = 555 \times (0.955)^n = 50$

$\therefore\ (0.955)^n = 0.0\overline{900}$

$\therefore\ n \doteqdot 52.3$ {using technology}

So, after approximately 52 years the population is 50, i.e., in the year 2037.

EXERCISE 2E.1

1 **a** **i** 3, 11, 19, 27, ... so $u_1 = 3$, $d = 8$, i.e., $u_n = 3 + (n-1)8 = 8n - 5$

i.e., $S_n = 3 + 11 + 19 + 27 + \ldots\ldots + (8n-5)$

ii $S_5 = 3 + 11 + 19 + 27 + 35 = 95$

b **i** 42, 37, 32, 27, ... so $u_1 = 42$, $d = -5$, i.e., $u_n = 42 + (n-1)(-5) = 47 - 5n$

i.e., $S_n = 42 + 37 + 32 + 27 + \ldots\ldots + (47 - 5n)$

ii $S_5 = 42 + 37 + 32 + 27 + 22 = 160$

c **i** 12, 6, 3, $1\frac{1}{2}$, ... so $u_1 = 12$, $r = \frac{1}{2}$, i.e., $u_n = 12 \times (\frac{1}{2})^{n-1}$

i.e., $S_n = 12 + 6 + 3 + 1\frac{1}{2} + \ldots\ldots + 12(\frac{1}{2})^{n-1}$

ii $S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$

d **i** 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ... so $u_1 = 2$, $r = \frac{3}{2}$, i.e., $u_n = 2 \times (\frac{3}{2})^{n-1}$

i.e., $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + \ldots\ldots + 2(\frac{3}{2})^{n-1}$

ii $S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$

e **i** 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... so $u_1 = 1$, $r = \frac{1}{2}$, i.e., $u_n = 1 \times (\frac{1}{2})^{n-1} = 2^{1-n}$

i.e., $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots + 2^{1-n}$

ii $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$

f **i** 1, 8, 27, 64, ...

i.e., $S_n = 1 + 8 + 27 + 64 + \ldots\ldots + n^3$ {since $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $64 = 4^3$}

ii $S_5 = 1 + 8 + 27 + 64 + 125 = 225$

EXERCISE 2E.2

1 **a** The series is arithmetic with

$u_1 = 3$, $d = 4$, $n = 20$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{20} = \frac{20}{2}(2 \times 3 + 19 \times 4) = 10(6 + 76) = 820$

b The series is arithmetic with

$u_1 = \frac{1}{2}$, $d = 2\frac{1}{2}$, $n = 50$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{50} = \frac{50}{2}\left(2 \times \frac{1}{2} + 49 \times \frac{5}{2}\right) = 25(1 + 122\frac{1}{2}) = 3087\frac{1}{2}$

c The series is arithmetic with

$u_1 = 100$, $d = -7$, $n = 40$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{40} = \frac{40}{2}(2 \times 100 + 39 \times (-7)) = 20(200 - 273) = -1460$

d The series is arithmetic with

$u_1 = 50$, $d = -1\frac{1}{2}$, $n = 80$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

So, $S_{80} = \frac{80}{2}\left(2 \times 50 + 79 \times (-\frac{3}{2})\right) = 40(100 - \frac{237}{2}) = -740$

2 **a** The series is arithmetic with

$u_1 = 5$, $d = 3$, $u_n = 101$

Since $u_n = 101$,

then $u_1 + (n-1)d = 101$

$\therefore$ $5 + 3(n-1) = 101$

$\therefore$ $5 + 3n - 3 = 101$

$\therefore$ $3n - 3 = 96$

$\therefore$ $3n = 99$

$\therefore$ $n = 33$

So, $S_n = \frac{n}{2}(u_1 + u_n) = \frac{33}{2}(5 + 101) = 1749$

b The series is arithmetic with

$u_1 = 50$, $d = -\frac{1}{2}$, $u_n = -20$

Since $u_n = -20$,

then $u_1 + (n-1)d = -20$

$\therefore$ $50 + (-\frac{1}{2})(n-1) = -20$

$\therefore$ $-\frac{1}{2}n + \frac{1}{2} = -70$

$\therefore$ $-\frac{1}{2}n = -\frac{141}{2}$

$\therefore$ $n = 141$

So, $S_n = \frac{n}{2}(u_1 + u_n) = \frac{141}{2}(50 + (-20)) = 2115$

c The series is arithmetic with

$u_1 = 8, \quad d = 2\frac{1}{2}, \quad u_n = 83$

$$\begin{aligned} \text{Since} \quad u_n &= 83, \\ \text{then} \quad u_1 + (n-1)d &= 83 \\ \therefore \quad 8 + \tfrac{5}{2}(n-1) &= 83 \\ \therefore \quad \tfrac{5}{2}n - \tfrac{5}{2} &= 75 \\ \therefore \quad \tfrac{5}{2}n &= \tfrac{155}{2} \\ \therefore \quad n &= 31 \end{aligned}$$

$$\begin{aligned} \text{So,} \quad S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{31}{2}(8 + 83) \\ &= 1410\tfrac{1}{2} \end{aligned}$$

3 $u_1 = 5, \quad n = 7,$
$u_n = 53$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{7}{2}(5 + 53) \\ &= 203 \end{aligned}$$

4 $u_1 = 6, \quad n = 11,$
$u_n = -27$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{11}{2}(6 + (-27)) \\ &= -115\tfrac{1}{2} \end{aligned}$$

5 The total number of bricks can be expressed as an arithmetic series:

$1 + 2 + 3 + 4 + \ldots\ldots + n$

We know that the total number of bricks is 171, i.e., $S_n = 171$.

Also, $u_1 = 1, \quad d = 1$ and we need to find n, the number of members (layers) of the series.

$$\begin{aligned} S_n = \frac{n}{2}(2u_1 + (n-1)d) &= 171 \\ \therefore \quad \frac{n}{2}(2 \times 1 + (n-1) \times 1) &= 171 \\ \therefore \quad n(2 + n - 1) &= 342 \\ \therefore \quad n(n+1) &= 342 \end{aligned}$$

$$\begin{aligned} \therefore \quad n^2 + n - 342 &= 0 \\ \therefore \quad (n-18)(n+19) &= 0 \\ \therefore \quad n &= -19 \text{ or } 18 \\ \text{but } n > 0, \quad \therefore \quad n &= 18 \end{aligned}$$

So, there are 18 layers placed.

6 The total number of seats can be expressed as an arithmetic series:

$22 + 23 + 24 + \ldots\ldots + u_n$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore \quad S_n &= \frac{n}{2}(2 \times 22 + 1(n-1)) \\ &= \frac{n}{2}(44 + n - 1) \quad \therefore \quad S_n = \frac{n}{2}(n + 43) \end{aligned}$$

which is the total number of seats in n rows.

a Number of seats in row 44 = total no. of seats in every row − no. of seats in 43 rows

$$\begin{aligned} &= S_{44} - S_{43} \\ &= \tfrac{44}{2}(44 + 43) - \tfrac{43}{2}(43 + 43) \\ &= 1914 - 1849 \\ &= 65 \end{aligned}$$

b Number of seats in a section $= S_{44} = 1914$ (from **a**)

c Number of seats in 25 sections $= S_{44} \times 25 = 1914 \times 25 = 47\,850$

7 **a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$11 + 22 + 33 + \ldots\ldots + u_{50}$ where $u_1 = 11, \quad d = 11, \quad n = 50$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\begin{aligned} \therefore \quad S_{50} &= \tfrac{50}{2}(2 \times 11 + 11(50 - 1)) \\ &= 25(22 + 539) \\ &= 14\,025 \end{aligned}$$

b The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$7+14+21+28+\ldots\ldots+u_n$ where $u_1=7$, $d=7$

To find u_n, we need to find the largest multiple of 7 less than 1000, i.e., $\frac{1000}{7} \doteqdot 142.9$, so $u_n = 142\times 7 = 994$

$$\begin{aligned} \text{Now} \quad u_n &= u_1+(n-1)d \\ \therefore \quad 994 &= 7+7(n-1) \\ \therefore \quad 987 &= 7n-7 \\ \therefore \quad 7n &= 994 \\ \therefore \quad n &= 142 \end{aligned}$$

So, $S_{142} = \frac{142}{2}(7+994) = 71\,071$

c The integers between 1 and 100 which are not divisible by 3 can be expressed as:

1, 2, 4, 5, 7, 8, ..., 100 where $u_1=1$, $u_n=100$.

Alternatively, these integers can be expressed as two separate arithmetic series:

i.e., $S_1 = 1+4+7+\ldots\ldots+97+100$ where $u_1=1$, $d=3$, $u_n=100$

and $S_2 = 2+5+8+\ldots\ldots+95+98$ where $u_1=2$, $d=3$, $u_n=98$

$$\begin{aligned} \text{Now for } S_1, \quad u_n &= u_1+(n-1)d \\ \therefore \quad 100 &= 1+3(n-1) \\ \therefore \quad 99 &= 3n-3 \\ \therefore \quad 3n &= 102 \\ \therefore \quad n &= 34 \end{aligned}$$

$$\begin{aligned} \text{and for } S_2, \quad u_n &= u_1+(n-1)d \\ \therefore \quad 98 &= 2+3(n-1) \\ \therefore \quad 96 &= 3n-3 \\ \therefore \quad 3n &= 99 \\ \therefore \quad n &= 33 \end{aligned}$$

i.e., $S_1 = \frac{34}{2}(1+100) = 1717$ and $S_2 = \frac{33}{2}(2+98) = 1650$

i.e., total sum $= S_1+S_2 = 1717+1650 = 3367$

8 We need to show that $1+2+3+4+\ldots\ldots+n = \dfrac{n(n+1)}{2}$

The sum of the first n positive integers can be expressed as an arithmetic series:

$1+2+3+4+\ldots\ldots+n$, where $u_1=1$, $d=1$, $u_n=n$.

So the sum of the series is
$$\begin{aligned} S_n &= \frac{n}{2}(u_1+u_n) \\ &= \frac{n}{2}(1+n) \end{aligned}$$
i.e., $S_n = \dfrac{n(n+1)}{2}$ as required.

9 The series of odd numbers can be expressed as an arithmetic series:

$1+3+5+7+\ldots\ldots$ where $u_1=1$, $d=2$

a Now $u_n = u_1+(n-1)d = 1+2(n-1)$

i.e., $u_n = 2n-1$

b We need to show that S_n is n^2.

The sum of the first n odd numbers can be expressed as an arithmetic series:

$1+3+5+7+\ldots\ldots(2n-1)$ {using $u_n = 2n-1$ from **a**}

$$\begin{aligned} \text{So,} \quad S_n &= \frac{n}{2}(u_1+u_n) \\ &= \frac{n}{2}\left(1+(2n-1)\right) \\ &= \frac{n}{2}(2n) \end{aligned}$$

i.e., $S_n = n^2$ as required

c
$S_1 = \frac{1}{2}(1+1) = 1 = 1^2 = n^2$ for $n=1$ ✓

$S_2 = \frac{2}{2}(1+3) = 4 = 2^2 = n^2$ for $n=2$ ✓

$S_3 = \frac{3}{2}(1+5) = 9 = 3^2 = n^2$ for $n=3$ ✓

$S_4 = \frac{4}{2}(1+7) = 16 = 4^2 = n^2$ for $n=4$ ✓

10 $u_6 = 21, \quad S_{17} = 0;$ so we need to find u_1 and u_2

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore \quad S_{17} = \tfrac{17}{2}(2u_1 + 16d) = 0$$
$$\therefore \quad u_1 + 8d = 0$$
$$\therefore \quad u_1 = -8d \quad \text{......} \ (1)$$

Also, $u_n = u_1 + (n-1)d$
$$\therefore \quad u_6 = u_1 + 5d$$
$$\therefore \quad 21 = -8d + 5d \quad \{\text{using } (1)\}$$
$$\therefore \quad 21 = -3d$$
$$\therefore \quad d = -7$$
and $u_1 = -8(-7) = 56$

so $u_2 = 56 - 7 = 49$

i.e., the first two terms are 56 and 49.

11 Let the three consecutive terms be $x - d$, x and $x + d$.

Now, sum of terms $= 12$
$$\text{i.e.,} \quad (x-d) + x + (x+d) = 12$$
$$\therefore \quad 3x = 12$$
$$\therefore \quad x = 4$$

So the terms are $4 - d$, 4, $4 + d$

Also, product of terms $= -80$
$$\text{i.e.,} \quad (4-d)4(4+d) = -80$$
$$\therefore \quad 4(4^2 - d^2) = -80$$
$$\therefore \quad 16 - d^2 = -20$$
$$\therefore \quad d^2 = 36$$
$$\therefore \quad d = \pm 6$$

So, the three terms could be $4-6,\ 4,\ 4+6$, i.e., $-2,\ 4,\ 10$

or $4-(-6),\ 4,\ 4+(-6)$, i.e., $10,\ 4,\ -2$

12 Let the five consecutive terms be $n-2d$, $n-d$, n, $n+d$, $n+2d$.

Now, sum of terms $= 40$
$$\text{i.e.,} \quad (n-2d) + (n-d) + n + (n+d) + (n+2d) = 40$$
$$\therefore \quad 5n = 40$$
$$\therefore \quad n = 8$$

So the terms are $8-2d$, $8-d$, 8, $8+d$, $8+2d$

Also, product of middle and end terms $= 8 \times (8-2d) \times (8+2d) = 224$
$$\therefore \quad 8(8^2 - 4d^2) = 224$$
$$\therefore \quad 64 - 4d^2 = 28$$
$$\therefore \quad 4d^2 = 36$$
$$\therefore \quad d^2 = 9$$
$$\therefore \quad d = \pm 3$$

So, the five terms could be $8-2(3),\ 8-3,\ 8,\ 8+3,\ 8+2(3)$, i.e., $2, 5, 8, 11, 14$

or $8-2(-3),\ 8-(-3),\ 8,\ 8+(-3),\ 8+2(-3)$, i.e., $14, 11, 8, 5, 2$

EXERCISE 2E.3

1 **a** The series is geometric with $u_1 = 12, \quad r = \frac{1}{2}, \quad n = 10$

Now $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$

$$\text{i.e.,} \quad S_{10} = \frac{12\left(1 - (\frac{1}{2})^{10}\right)}{1 - \frac{1}{2}}$$
$$\doteqdot 23.9766$$

b The series is geometric with $u_1 = \sqrt{7}, \quad r = \sqrt{7}, \quad n = 12$

Now $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$

$$\text{i.e.,} \quad S_{12} = \frac{\sqrt{7}\left((\sqrt{7})^{12} - 1\right)}{\sqrt{7} - 1}$$
$$\doteqdot 189\,134$$

c The series is geometric with
$u_1 = 6, \quad r = -\frac{1}{2}, \quad n = 15$

Now $\quad S_n = \dfrac{u_1(1-r^n)}{1-r}$

i.e., $\quad S_{15} = \dfrac{6\left(1-(-\frac{1}{2})^{15}\right)}{1-(-\frac{1}{2})}$

$\doteqdot 4.000$

d The series is geometric with
$u_1 = 1, \quad r = -\frac{1}{\sqrt{2}}, \quad n = 20$

Now $\quad S_n = \dfrac{u_1(1-r^n)}{1-r}$

i.e., $\quad S_{20} = \dfrac{1\left(1-(-\frac{1}{\sqrt{2}})^{20}\right)}{1-(-\frac{1}{\sqrt{2}})}$

$\doteqdot 0.5852$

2 **a** The series is geometric with
$u_1 = \sqrt{3}, \quad r = \sqrt{3}$

$$
\begin{aligned}
S_n &= \frac{u_1(r^n-1)}{r-1} \\
&= \frac{\sqrt{3}\left((\sqrt{3})^n - 1\right)}{\sqrt{3}-1} \times \left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) \\
&= \frac{\left(3+\sqrt{3}\right)\left((\sqrt{3})^n - 1\right)}{3-1} \\
&= \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)
\end{aligned}
$$

b The series is geometric with
$u_1 = 12, \quad r = \frac{1}{2}$

$$
\begin{aligned}
S_n &= \frac{u_1(1-r^n)}{1-r} \\
&= \frac{12\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \\
&= 24\left(1-(\tfrac{1}{2})^n\right)
\end{aligned}
$$

c The series is geometric with
$u_1 = 0.9, \quad r = 0.1$

$$
\begin{aligned}
S_n &= \frac{u_1(1-r^n)}{1-r} \\
&= \frac{0.9\left(1-(0.1)^n\right)}{1-0.1} \\
&= 1-(0.1)^n
\end{aligned}
$$

d The series is geometric with
$u_1 = 20, \quad r = -\frac{1}{2}$

$$
\begin{aligned}
S_n &= \frac{u_1(1-r^n)}{1-r} \\
&= \frac{20\left(1-(-\frac{1}{2})^n\right)}{1-(-\frac{1}{2})} \\
&= \frac{20\left(1-(-\frac{1}{2})^n\right)}{(\frac{3}{2})} \\
&= \tfrac{40}{3}\left(1-(-\tfrac{1}{2})^n\right)
\end{aligned}
$$

3 **a**

$$
\begin{aligned}
A_2 &= A_1 \times 1.06 + 2000 \\
&= (A_0 \times 1.06 + 2000) \times 1.06 + 2000 \\
&= (2000 \times 1.06 + 2000) \times 1.06 + 2000 \\
\text{i.e.,} \quad A_2 &= 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2 \quad \text{as required}
\end{aligned}
$$

b

$$
\begin{aligned}
A_3 &= A_2 \times 1.06 + 2000 \\
&= \left[2000 + 2000 \times 1.06 + 2000 \times (1.06)^2\right] \times 1.06 + 2000 \quad \{\text{from } \mathbf{a}\} \\
\text{i.e.,} \quad A_3 &= 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3\right] \quad \text{as required}
\end{aligned}
$$

c

$$
\begin{aligned}
A_9 &= 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 \right. \\
&\qquad \left. + (1.06)^8 + (1.06)^9\right] \\
\text{i.e.,} \quad A_9 &= 26\,361.59
\end{aligned}
$$

i.e., total bank balance after 10 years is $\$26\,361.59$

4 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$,
$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$

c
$$S_n = \frac{u_1(1-r^n)}{1-r}, \quad \text{where} \quad u_1 = \tfrac{1}{2}, \quad r = \tfrac{1}{2}$$
$$= \frac{\frac{1}{2}\left(1 - (\frac{1}{2})^n\right)}{1 - \frac{1}{2}}$$
$$\text{i.e.,} \quad S_n = 1 - (\tfrac{1}{2})^n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

d As n gets very large, i.e., as $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$, so $S_n \to 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
As $n \to \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches one whole unit.

5 **a**
$$\begin{aligned}\text{Total time of motion } &= 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1)\\ &\quad + (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + ...\\ &= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + ...\\ &= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ... \quad \text{as required}\end{aligned}$$

b
$$S_n = \frac{u_1(1-r^n)}{1-r}, \quad \text{where} \quad u_1 = 2(0.9), \quad r = 0.9, \quad \text{‘}n\text{’} = n-1$$

(since the term u_1, used in calculating S_n, is the second term of the series, not the first)

$$\text{i.e.,} \quad S_n = \frac{2(0.9)\left(1 - (0.9)^{n-1}\right)}{1 - 0.9} + 1$$
$$= \frac{1.8\left(1 - (0.9)^{n-1}\right)}{0.1} + 1$$
$$\text{i.e.,} \quad S_n = 1 + 18\left(1 - (0.9)^{n-1}\right)$$

c For the ball to come to rest, n must approach infinity,
thus $(0.9)^{n-1} \to 0$ so $\left(1 - (0.9)^{n-1}\right) \to 1$ (from below)
So, $S_n \to 1 + 18(1)$, i.e., $S_n \to 19$
i.e., it takes 19 seconds for the ball to come to rest.

6 **a** **i** $u_1 = \frac{3}{10}$ **ii** $r = \dfrac{(\frac{3}{100})}{(\frac{3}{10})} = \frac{1}{10}$

b We need to show that $0.\overline{3} = \frac{1}{3}$.
Now $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$
So, let $S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$
Since $n \to \infty$, then $S_\infty = \dfrac{u_1}{1-r} = \dfrac{\frac{3}{10}}{1 - (\frac{1}{10})} = \frac{1}{3}$
i.e., $0.\overline{3} = \frac{1}{3}$ as required

7 Checking **4 d**:

$$S_\infty = \frac{u_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

i.e., $S_\infty = 1$ ✓

Checking **5 c**:

$$S_\infty = \frac{u_1}{1-r} = \frac{2(0.9)}{1-0.9} + 1 \quad \{\text{since 1st term ‘1’ is not part of the series}\}$$

i.e., $S_\infty = 19$ ✓

EXERCISE 2F

1 **a** $\displaystyle\sum_{r=1}^{4} (3r-5) = -2+1+4+7 = 10$

b $\displaystyle\sum_{r=1}^{5} (11-2r) = 9+7+5+3+1 = 25$

c $\displaystyle\sum_{r=1}^{7} r(r+1) = 2+6+12+20+30+42+56 = 168$

d $\displaystyle\sum_{i=1}^{5} 10 \times 2^{i-1} = 10+20+40+80+160 = 310$

2 $u_n = 3n-1$

$u_1 + u_2 + u_3 + \ldots\ldots + u_{20} = 2+5+8+11+14+17+20+23+26+29+32+35 + 38+41+44+47+50+53+56+59$

This series is arithmetic with $u_1 = 2$, $d = 3$ and $n = 20$

so its sum is $\dfrac{n}{2}[2u_1 + (n-1)d]$

$= 10[4 + 19 \times 3]$

$= 610$

3 **a** $\displaystyle\sum_{r=1}^{10} (2r+5) = 7+9+11+\ldots\ldots+21+23+25$

This series is arithmetic with $u_1 = 7$, $d = 2$ and $n = 10$.

$\therefore$ sum $= \dfrac{n}{2}[2u_1 + (n-1)d]$

$= \frac{10}{2}[14 + 9 \times 2]$

$= 160$

b $\displaystyle\sum_{r=1}^{15} (r-50) = (-49)+(-48)+(-47)+\ldots\ldots+(-37)+(-36)+(-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$ and $n = 15$.

$\therefore$ sum $= \dfrac{n}{2}[2u_1 + (n-1)d]$

$= \frac{15}{2}[-98 + 14 \times 1]$

$= -630$

c $\displaystyle\sum_{r=1}^{20} \left(\frac{r+3}{2}\right) = 2 + \tfrac{5}{2} + 3 + \ldots\ldots + \tfrac{21}{2} + 11 + \tfrac{23}{2}$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 20$.

$\therefore$ sum $= \dfrac{n}{2}[2u_1 + (n-1)d]$

$= \frac{20}{2}[4 + 19 \times \frac{1}{2}]$

$= 135$

4 **a** $\sum_{r=1}^{10} \left(3 \times 2^{r-1}\right) = 3 + 6 + 12 + \text{......} + 384 + 768 + 1536$

This series is geometric with $u_1 = 3$, $r = 2$ and $n = 10$.

$\therefore$ sum $= \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{3(2^{10} - 1)}{1} = 3069$

b $\sum_{r=1}^{12} \left(\frac{1}{2}\right)^{r-2} = 2 + 1 + \frac{1}{2} + \text{......} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 12$.

$\therefore$ sum $= \dfrac{u_1(1 - r^n)}{1 - r} = \dfrac{2\left(1 - (\frac{1}{2})^{12}\right)}{\frac{1}{2}} = 4\left(1 - \left(\frac{1}{2}\right)^{12}\right)$

$\therefore$ sum $\doteqdot 3.999$

c $\sum_{r=1}^{25} (6 \times (-2)^r) = -12 + 24 + (-48) + \text{....} + 100\,663\,296 + (-201\,326\,592)$

This series is geometric with $u_1 = -12$, $r = -2$ and $n = 25$.

$\therefore$ sum $= \dfrac{u_1(1 - r^n)}{1 - r} = \dfrac{-12\left(1 - (-2)^{25}\right)}{1 + 2} = -4\left(1 - (-2)^{25}\right)$

$\therefore$ sum $= -134\,217\,732$

5 **a** $\sum_{k=1}^{5} k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$

Note: This series is neither arithmetic nor geometric so the sum is found by adding the 5 terms.

b $\sum_{k=6}^{12} \left(100(1.2)^{k-3}\right) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.318\,08 + 429.981\,696 + 515.978\,0352$

This series is geometric with $u_1 = 172.8$, $r = 1.2$ and $n = 7$

$\therefore$ sum $= \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{172.8\left((1.2)^7 - 1\right)}{0.2} \doteqdot 2231.87$

REVIEW SET 2A

1 **a** $u_n = 3^{n-2}$ $\therefore$ $u_1 = 3^{-1} = \frac{1}{3}$, $u_2 = 3^0 = 1$, $u_3 = 3^1 = 3$, $u_4 = 3^2 = 9$

b $u_n = \dfrac{3n + 2}{n + 3}$ $\therefore$ $u_1 = \frac{5}{4}$, $u_2 = \frac{8}{5}$, $u_3 = \frac{11}{6}$, $u_4 = \frac{14}{7} = 2$

c $u_n = 2^n - (-3)^n$

$u_1 = 2 - (-3) = 5$, $u_2 = 4 - 9 = -5$, $u_3 = 8 - (-27) = 35$, $u_4 = 16 - 81 = -65$

2 $u_n = 68 - 5n$

a
$$\begin{aligned} u_{n+1} - u_n &= 68 - 5(n+1) - [68 - 5n] \\ &= 68 - 5n - 5 - 68 + 5n \\ &= -5 \quad \text{for all } n \end{aligned}$$

b $u_1 = 68 - 5(1) = 63$, $d = -5$

c $u_{37} = 68 - 5(37) = -117$

d Let $u_n = -200$, and we need to find n.

$$\begin{aligned} u_n = 68 - 5n &= -200 \\ \therefore\ 5n &= 268 \\ \therefore\ n &= 53\tfrac{3}{5} \end{aligned}$$

So, try the two values on either side of $n = 53\frac{3}{5}$, i.e., for $n = 53$ and $n = 54$:

$$\begin{aligned} u_{53} &= 68 - 5(53) \\ &= -197 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u_{54} &= 68 - 5(54) \\ &= -202 \end{aligned}$$

i.e., the first term of the sequence less than -200 is $u_{54} = -202$.

3 **a** 3, 12, 48, 192, ...

$\frac{12}{3} = 4$, $\frac{48}{12} = 4$, $\frac{192}{48} = 4$

So, assuming the pattern continues, consecutive terms have a common ratio of 4.

$\therefore$ the sequence is geometric with $u_1 = 3$ and $r = 4$.

b

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ \therefore\ u_n &= 3 \times 4^{n-1} \\ \text{i.e.,}\ u_9 &= 3 \times 4^8 \\ &= 196\,608 \end{aligned}$$

4 Since the terms are consecutive,

$$\begin{aligned} \text{then}\quad (k-2) - 3k &= k + 7 - (k-2) \quad \{\text{equating common differences}\} \\ \therefore\ k - 2 - 3k &= k + 7 - k + 2 \\ \therefore\ -2 - 2k &= 9 \\ \therefore\ 2k &= -11 \\ \therefore\ k &= -\tfrac{11}{2} \end{aligned}$$

5 $u_7 = 31 \quad \therefore\ u_1 + 6d = 31 \quad$ ..... (1)

$u_{15} = -17 \quad \therefore\ u_1 + 14d = -17 \quad$ ..... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -31 \\ u_1 + 14d &= -17 \\ \hline \therefore\ 8d &= -48 \quad \{\text{adding the equations}\} \\ \therefore\ d &= -6 \end{aligned}$$

$$\begin{aligned} \text{So in (1),}\quad u_1 + 6(-6) &= 31 \\ \therefore\ u_1 - 36 &= 31 \\ \therefore\ u_1 &= 67 \end{aligned}$$

$$\begin{aligned} \text{Now}\quad u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 67 + (n-1)(-6) \\ \therefore\ u_n &= 67 - 6n + 6 \\ \therefore\ u_n &= -6n + 73 \\ \text{So}\quad u_{34} &= -6(34) + 73 = -131 \end{aligned}$$

6 $u_n = 6\left(\frac{1}{2}\right)^{n-1}$

a $\dfrac{u_{n+1}}{u_n} = \dfrac{6\left(\frac{1}{2}\right)^{n+1-1}}{6\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2}$ for all n

$\therefore$ $\{u_n\}$ is a geometric sequence.

b $u_1 = 6$, $r = \frac{1}{2}$

c

$$\begin{aligned} u_{16} &= 6\left(\tfrac{1}{2}\right)^{15} \\ &= 0.000\,183 \end{aligned}$$

7 28, 23, 18, 13, ...

$23 - 28 = -5$ So, assuming that the pattern continues, consecutive terms differ by -5.

$18 - 23 = -5$ $\therefore$ the sequence is arithmetic with $u_1 = 28,\ d = -5$.

$13 - 18 = -5$

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 28 + (n-1)(-5) \\ &= 28 - 5n + 5 \\ &= -5n + 33 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2 \times 28 + (n-1)(-5)) \\ &= \frac{n}{2}(56 - 5n + 5) \\ \text{i.e., } S_n &= \frac{n}{2}(61 - 5n) \end{aligned}$$

8 Since the terms are geometric, then $\dfrac{k}{4} = \dfrac{k^2-1}{k}$

$$\begin{aligned} \therefore\ k^2 &= 4(k^2-1) \\ \therefore\ k^2 &= 4k^2 - 4 \\ \therefore\ 3k^2 &= 4 \\ \therefore\ k^2 &= \tfrac{4}{3} \end{aligned}$$

$$\therefore\ k = \pm\frac{2}{\sqrt{3}} \qquad \therefore\ k = \pm\frac{2\sqrt{3}}{3}$$

9 $u_6 = \frac{16}{3}$ $\quad\therefore\ u_1 \times r^5 = \frac{16}{3}$ ..... (1)

$u_{10} = \frac{256}{3}$ $\quad\therefore\ u_1 \times r^9 = \frac{256}{3}$ ..... (2)

So $\dfrac{u_1 r^9}{u_1 r^5} = \dfrac{(\frac{256}{3})}{(\frac{16}{3})}$ $\{(2) \div (1)\}$

$$\therefore\ r^4 = 16 \qquad \therefore\ r = \pm 2$$

Substituting $r = 2$ into (1) gives

$$\begin{aligned} u_1 \times 2^5 &= \tfrac{16}{3} \\ \therefore\ u_1 \times 32 &= \tfrac{16}{3} \\ \therefore\ u_1 &= \tfrac{1}{6} \end{aligned}$$

Substituting $r = -2$ into (1) gives

$$\begin{aligned} u_1 \times (-2)^5 &= \tfrac{16}{3} \\ \therefore\ u_1 \times (-32) &= \tfrac{16}{3} \\ \therefore\ u_1 &= -\tfrac{1}{6} \end{aligned}$$

Now $u_n = u_1 r^{n-1}$ $\quad\therefore\ u_n = \frac{1}{6} \times 2^{n-1}$ *or* $-\frac{1}{6} \times 2^{n-1}$

REVIEW SET 2B

1 **a** 24, $23\frac{1}{4}$, $22\frac{1}{2}$, ..., -36 i.e., $u_1 = 24$, $u_n = -36$, and we need to find n.

The sequence is arithmetic with $d = -\frac{3}{4}$.

$$\begin{aligned} \text{Now}\quad u_n &= u_1 + (n-1)d \\ \therefore\ -36 &= 24 + (n-1)(-\tfrac{3}{4}) \\ \therefore\ -60 &= -\tfrac{3}{4}n + \tfrac{3}{4} \\ \therefore\ \tfrac{3}{4}n &= \tfrac{243}{4} \\ \therefore\ n &= 81 \end{aligned}$$

i.e., there are 81 terms in the sequence.

b
$$\begin{aligned} u_{35} &= 24 + (35-1)(-\tfrac{3}{4}) \\ &= 24 - \tfrac{102}{4} \\ &= -\tfrac{3}{2} \end{aligned}$$

c
$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ \text{i.e., } S_{81} &= \tfrac{81}{2}(24 + (-36)) \\ &= -486 \end{aligned}$$

2 Let the numbers be $23,\ 23+d,\ 23+2d,\ 23+3d,\ 23+4d,\ 23+5d,\ 23+6d,\ 9$

Then $23+7d = 9$

i.e., $7d = -14$

$\therefore\ d = -2$ i.e., the numbers are 23, 21, 19, 17, 15, 13, 11, 9.

3 **a** 86, 83, 80, 77, ... i.e., the sequence is arithmetic with $u_1 = 86,\ d = -3$

$u_n = u_1 + (n-1)d$

i.e., $u_n = 86 + (n-1)(-3) = 86 - 3n + 3$

i.e., $u_n = 89 - 3n$

b $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}$, ... which can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}$, ...

i.e., the numerator starts at 3 and increases by 2 each time,
whilst the denominator starts at 4 and increases by 1 each time.

i.e., the nth term is $\dfrac{2n+1}{n+3}$, i.e., $u_n = \dfrac{2n+1}{n+3}$

c 100, 90, 81, 72.9, ... i.e., the sequence is geometric with $u_1 = 100,\ r = \frac{90}{100} = 0.9$

$u_n = u_1 r^{n-1}$

i.e., $u_n = 100(0.9)^{n-1}$

4 **a** $\displaystyle\sum_{r=1}^{7} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

$= 1 + 4 + 9 + 16 + 25 + 36 + 49$

b $\displaystyle\sum_{r=1}^{8} \frac{r+3}{r+2} = \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$

5 **a** $4 + 11 + 18 + 25 + ...$

i.e., the series is arithmetic with $u_1 = 4,\ d = 7,$ $u_r = u_1 + (r-1)d$

$= 4 + 7(r-1)$

$= 7r - 3$

i.e., $\displaystyle\sum_{r=1}^{n} (7r-3)$

b $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$ i.e., the series is geometric with $u_1 = \frac{1}{4},\ R = \frac{1}{2}$,

$u_r = u_1 R^{r-1} = \frac{1}{4} \times \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^{r+1}$

i.e., $\displaystyle\sum_{r=1}^{n} \left(\tfrac{1}{2}\right)^{r+1}$

6 **a** $3 + 9 + 15 + 21 + ...$

i.e., the series is arithmetic with

$u_1 = 3,\quad d = 6,\quad n = 23$

Now $S_n = \dfrac{n}{2}\left(2u_1 + (n-1)d\right)$

i.e., $S_{23} = \frac{23}{2}\left(2 \times 3 + 6(23-1)\right)$

$= \frac{23}{2}(6 + 132)$

$= 1587$

b $24 + 12 + 6 + 3 + ...$

i.e., the series is geometric with

$u_1 = 24,\quad r = \frac{1}{2},\quad n = 12$

$S_n = \dfrac{u_1(1-r^n)}{1-r}$

i.e., $S_{12} = \dfrac{24\left(1 - (\frac{1}{2})^{12}\right)}{1 - \frac{1}{2}}$

$= 48\left(1 - (\tfrac{1}{2})^{12}\right)$

i.e., $S_{12} \doteqdot 47.99$

7 **a** $\sum_{r=1}^{8}\left(\frac{31-3r}{2}\right) = 14+12\frac{1}{2}+11+9\frac{1}{2}+8+6\frac{1}{2}+5+3\frac{1}{2}$

This series is arithmetic with $u_1 = 14$, $d = -1\frac{1}{2}$ and $n = 8$.

$\therefore$ the sum is $\frac{8}{2}\left[28+7(-1\frac{1}{2})\right] = 70$

b $\sum_{r=1}^{15} 50(0.8)^{r-1} \doteqdot 50+40+32+ \ldots\ldots + 3.436+2.749+2.199$

This series is geometric with $u_1 = 50$, $r = 0.8$ and $n = 15$.

$\therefore$ the sum is $\dfrac{50\left[1-(0.8)^{15}\right]}{1-0.8} \doteqdot 241.2$

8 5, 10, 20, 40, ... i.e., the sequence is geometric with $u_1 = 5$, $r = 2$

$u_n = u_1 r^{n-1} = 5 \times 2^{n-1}$

Let $u_n = 10\,000 = 5 \times 2^{n-1}$

$\therefore\ 2000 = 2^{n-1}$

$\therefore\ n \doteqdot 11.97$ {using technology}

So, try the two values on either side of $n \doteqdot 11.97$, i.e., for $n = 11$ and $n = 12$:

$u_{11} = 5 \times 2^{10} = 5120$ and $u_{12} = 5 \times 2^{11} = 10\,240$

i.e., the first term to exceed 10 000 is $u_{12} = 10\,240$.

9 **a** $u_6 = u_1 \times r^5$ is the amount after 5 years, where $r = 1.07$

$= 6000 \times (1.07)^5$

i.e., $u_6 = 8415.31$, i.e., the value of the investment will be 8415.31 Euro

b If interest is compounded quarterly, then $r = 1 + \frac{0.07}{4} = 1.0175$

and $n = 5 \times 4 = 20$

$u_{21} = u_1 \times r^{20}$

$= 6000 \times (1.0175)^{20}$

i.e., $u_{21} = 8488.67$, i.e., the value of the investment will be 8488.67 Euro

c If interest is compounded monthly, then $r = 1 + \frac{0.07}{12} = 1.005\,8\bar{3}$

and $n = 5 \times 12 = 60$

$u_{61} = u_1 \times r^{60}$

$= 6000 \times (1.005\,8\bar{3})^{60}$

i.e., $u_{61} = 8505.75$, i.e., the value of the investment will be 8505.75 Euro

REVIEW SET 2C

1 $u_6 = 24 \quad \therefore\ u_1 \times r^5 = 24$ ...... (1)

$u_{11} = 768 \quad \therefore\ u_1 \times r^{10} = 768$ ...... (2)

So $\dfrac{u_1 r^{10}}{u_1 r^5} = \dfrac{768}{24}$ {(2) ÷ (1)}

$\therefore\ r^5 = 32$

$\therefore\ r = 2$

Substituting $r = 2$ into (1) gives $u_1 \times 2^5 = 24 \quad \therefore\ u_1 = \dfrac{24}{32} = \dfrac{3}{4}$

$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$

a $u_{17} = \left(\frac{3}{4}\right) 2^{17-1}$
$= 49\,152$

b $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$
$= \dfrac{\frac{3}{4}(2^n - 1)}{2 - 1}$
$= \frac{3}{4}(2^n - 1)$
$\therefore\ S_{15} = \frac{3}{4}(2^{15} - 1)$
$= 24\,575.25$

2 $11 + 16 + 21 + 26 + ...$ i.e., the series is arithmetic with $u_1 = 11$, $d = 5$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$= \frac{n}{2}(2 \times 11 + 5(n-1))$$
$$= \frac{n}{2}(22 + 5n - 5)$$
$$= \frac{n}{2}(5n + 17)$$

Let $S_n = 450$ and we need to find n,

i.e., $S_n = \frac{n}{2}(5n + 17) = 450$

$\therefore\ \frac{5}{2}n^2 + \frac{17}{2}n - 450 = 0$

$\therefore\ 5n^2 + 17n - 900 = 0$

$\therefore\ n \doteqdot -15.2,\ 11.8$ {using technology}

but $n > 0$, $\therefore\ n \doteqdot 11.8$

So, try the two values on either side of $n \doteqdot 11.8$, i.e., for $n = 11$ and $n = 12$:

$S_{11} = \frac{11}{2}(5(11) + 17)$ and $S_{12} = \frac{12}{2}(5(12) + 17)$
$= 396$ $= 462$

i.e., 12 terms of the series are required to exceed a sum of 450.

3 24, 8, $\frac{8}{3}$, $\frac{8}{9}$, ... i.e., the sequence is geometric with $u_1 = 24$, $r = \frac{1}{3}$

$$u_n = u_1 r^{n-1}$$
$$= 24\left(\tfrac{1}{3}\right)^{n-1}$$

Let $u_n = 0.001$ and we need to find n.

i.e., $u_n = 24\left(\frac{1}{3}\right)^{n-1} = 0.001$

$\therefore\ \left(\frac{1}{3}\right)^{n-1} = \dfrac{0.001}{24}$

$\therefore\ n \doteqdot 10.18$ {using technology}

So, try the two values on either side of $n \doteqdot 10.18$, i.e., for $n = 10$ and $n = 11$:

$u_{10} = 24\left(\frac{1}{3}\right)^9$ and $u_{11} = 24\left(\frac{1}{3}\right)^{10}$
$= 0.001\,22$ $= 0.000\,406$

i.e., $u_{11} \doteqdot 0.000\,406$ is the first term of the sequence which is less than 0.001.

4 **a** 128, 64, 32, 16, ..., $\frac{1}{512}$

i.e., the sequence is geometric with

$u_1 = 128, \quad r = \frac{1}{2}, \quad u_n = \frac{1}{512}$

$$u_n = u_1 r^{n-1} = 128\left(\tfrac{1}{2}\right)^{n-1}$$

i.e., $\frac{1}{512} = 128\left(\frac{1}{2}\right)^{n-1}$

$\therefore \left(\frac{1}{2}\right)^{n-1} = \dfrac{1}{65\,536}$

$\therefore n = 17$ {using technology}

i.e., there are 17 terms in the sequence.

b $S_n = \dfrac{u_1(1-r^n)}{1-r}$

i.e., $S_{17} = \dfrac{128\left(1-(\frac{1}{2})^{17}\right)}{1-\frac{1}{2}}$

$= 255.998$

i.e., $S_{17} \doteqdot 256$

5 **a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500, \quad r = 1 + \dfrac{0.0825}{2} = 1.041\,25, \quad n = 5 \times 2 = 10$

i.e., $u_{n+1} = 12\,500 \times (1.041\,25)^{10}$

$= 18\,726.65$, i.e., the value of the investment is \$18 726.65

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500, \quad r = 1 + \dfrac{0.0825}{12} = 1.006\,875, \quad n = 5 \times 12 = 60$

i.e., $u_{n+1} = 12\,500 \times (1.006\,875)^{60}$

$= 18\,855.74$, i.e., the value of the investment is \$18 855.74

6 $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000, \quad r = 1 + \dfrac{0.09}{12} = 1.0075, \quad n = 4 \times 12 = 48$

i.e., $20\,000 = u_1 \times (1.0075)^{48}$

i.e., $u_1 = 13\,972.28$, i.e., \$13 972.28 should be invested

7 **a** $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000, \quad r = 1.05, \quad n = 3$

i.e., $u_{n+1} = 3000 \times (1.05)^3$

$= 3472.875$, i.e., approximately 3470 koalas

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000, \quad u_{n+1} = 5000, \quad r = 1.05$

i.e., $5000 = 3000 \times (1.05)^n$

i.e., $n \doteqdot 10.47$

i.e., after 10.47 years the population will exceed 5000, i.e., in the year 2008.

Chapter 3

EXPONENTS

EXERCISE 3A

1 **a** $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64$

b $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81$

c $5^1 = 5,\ 5^2 = 25,\ 5^3 = 125,\ 5^4 = 625$

d $7^1 = 7,\ 7^2 = 49,\ 7^3 = 343$

EXERCISE 3B

1 **a** $\begin{aligned}&(-1)^3\\ &= (-1)\times(-1)\times(-1)\\ &= 1\times(-1)\\ &= -1\end{aligned}$

b $\begin{aligned}&(-1)^4\\ &= (-1)^3\times(-1)\\ &= (-1)\times(-1)\\ &= 1\end{aligned}$

c $\begin{aligned}&(-1)^{12}\\ &= 1\end{aligned}$

d $\begin{aligned}&(-1)^{17}\\ &= -1\end{aligned}$

e $\begin{aligned}&(-1)^6\\ &= 1\end{aligned}$

f $\begin{aligned}&-1^6\\ &= -(1^6)\\ &= -1\end{aligned}$

g $\begin{aligned}&-(-1)^6\\ &= -(1)\\ &= -1\end{aligned}$

h $\begin{aligned}&(-2)^3\\ &= (-2)\times(-2)\times(-2)\\ &= 4\times(-2)\\ &= -8\end{aligned}$

i $\begin{aligned}&-2^3\\ &= -(2^3)\\ &= -8\end{aligned}$

j $\begin{aligned}&-(-2)^3\\ &= -(-8)\\ &= 8\end{aligned}$

k $\begin{aligned}&-(-5)^2\\ &= -(25)\\ &= -25\end{aligned}$

l $\begin{aligned}&-(-5)^3\\ &= -(-125)\\ &= 125\end{aligned}$

2 **a** 512 **b** -3125 **c** -243 **d** 16 807 **e** 512 **f** 6561 **g** -6561
h 5.117 264 691 **i** $-0.764\,479\,956$ **j** $-20.361\,584\,96$

3 **a** $0.\overline{142\,857}$ **b** $0.\overline{142\,857}$ **c** $0.\overline{1}$ **d** $0.\overline{1}$ **e** 0.015 625 **f** 0.015 625
g 1 **h** 1

We notice that $7^{-1} = \dfrac{1}{7^1}$, $3^{-2} = \dfrac{1}{3^2}$, $4^{-3} = \dfrac{1}{4^3}$ and $a^0 = 1$ for $a > 0$

4 $3^{33} = \underbrace{3^4\times 3^4\times 3^4\times \ldots\times 3^4}_{\text{8 of these}}\times 3^1$

But $3^4 = 81$ i.e., ends in a 1

$\therefore\ \underbrace{3^4\times 3^4\times 3^4\times \ldots\times 3^4}_{\text{8 of these}}$ ends in a 1

$\therefore\ 3^{33}$ ends in a 3

5 $7^1 = 7,\quad 7^2 = 49,\quad 7^3 = 343,\quad 7^4 = 2401,\quad 7^5 = 16\,807$

Now $7^{77} = \underbrace{7^4\times 7^4\times 7^4\times \ldots\times 7^4}_{\text{19 of these}}\times 7^1$

So, ends in a 1.

$\therefore\ 7^{77}$ ends in $1\times 7 = 7$.

EXERCISE 3C

1 **a** $7^3 \times 7^2 = 7^{3+2} = 7^5$

b $5^4 \times 5^3 = 5^{4+3} = 5^7$

c $a^7 \times a^2 = a^{7+2} = a^9$

d $a^4 \times a = a^4 \times a^1 = a^{4+1} = a^5$

e $b^8 \times b^5 = b^{8+5} = b^{13}$

f $a^3 \times a^n = a^{3+n}$

g $b^7 \times b^m = b^{7+m}$

h $m^4 \times m^2 \times m^3 = m^{4+2+3} = m^9$

2 **a** $\frac{5^9}{5^2} = 5^{9-2} = 5^7$

b $\frac{11^{13}}{11^9} = 11^{13-9} = 11^4$

c $7^7 \div 7^4 = 7^{7-4} = 7^3$

d $\frac{a^6}{a^2} = a^{6-2} = a^4$

e $\frac{b^{10}}{b^7} = b^{10-7} = b^3$

f $\frac{p^5}{p^m} = p^{5-m}$

g $\frac{y^a}{y^5} = y^{a-5}$

h $b^{2x} \div b = b^{2x} \div b^1 = b^{2x-1}$

3 **a** $(3^2)^4 = 3^{2\times 4} = 3^8$

b $(5^3)^5 = 5^{3\times 5} = 5^{15}$

c $(2^4)^7 = 2^{4\times 7} = 2^{28}$

d $(a^5)^2 = a^{5\times 2} = a^{10}$

e $(p^4)^5 = p^{4\times 5} = p^{20}$

f $(b^5)^n = b^{5\times n} = b^{5n}$

g $(x^y)^3 = x^{y\times 3} = x^{3y}$

h $(a^{2x})^5 = a^{2x\times 5} = a^{10x}$

4 **a** $8 = 2^3$

b $25 = 5^2$

c $27 = 3^3$

d $4^3 = (2^2)^3 = 2^6$

e $9^2 = (3^2)^2 = 3^4$

f $3^a \times 9 = 3^a \times 3^2 = 3^{a+2}$

g $5^t \div 5 = 5^t \div 5^1 = 5^{t-1}$

h $3^n \times 9^n = 3^n \times (3^2)^n = 3^{n+2n} = 3^{3n}$

i $\frac{16}{2^x} = \frac{2^4}{2^x} = 2^{4-x}$

j $\frac{3^{x+1}}{3^{x-1}} = 3^{(x+1)-(x-1)} = 3^{x+1-x+1} = 3^2$

k $(5^4)^{x-1} = 5^{4(x-1)} = 5^{4x-4}$

l $2^x \times 2^{2-x} = 2^{x+2-x} = 2^2$

m $\frac{2^y}{4^x} = \frac{2^y}{(2^2)^x} = \frac{2^y}{2^{2x}} = 2^{y-2x}$

n $\frac{4^y}{8^x} = \frac{(2^2)^y}{(2^3)^x} = \frac{2^{2y}}{2^{3x}} = 2^{2y-3x}$

o $\frac{3^{x+1}}{3^{1-x}} = 3^{(x+1)-(1-x)} = 3^{x+1-1+x} = 3^{2x}$

p $\frac{2^t \times 4^t}{8^{t-1}} = \frac{2^t \times (2^2)^t}{(2^3)^{t-1}} = \frac{2^t \times 2^{2t}}{2^{3t-3}} = 2^{3t-(3t-3)} = 2^3$

5 **a** $(ab)^3 = a^3b^3$

b $(ac)^4 = a^4c^4$

c $(bc)^5 = b^5c^5$

d $(abc)^3 = a^3b^3c^3$

e $(2a)^4 = 2^4a^4 = 16a^4$

f $(5b)^2 = 5^2b^2 = 25b^2$

g $(3n)^4 = 3^4n^4 = 81n^4$

h $(2bc)^3 = 2^3b^3c^3 = 8b^3c^3$

i $(4ab)^3 = 4^3a^3b^3 = 64a^3b^3$

j $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$

k $\left(\frac{m}{n}\right)^4 = \frac{m^4}{n^4}$

l $\left(\frac{2c}{d}\right)^5 = \frac{2^5c^5}{d^5}$ or $\frac{32c^5}{d^5}$

6 **a** $(2b^4)^3 = 2^3b^{12} = 8b^{12}$

b $\left(\frac{3}{x^2y}\right)^2 = \frac{3^2}{x^4y^2} = \frac{9}{x^4y^2}$

c $(5a^4b)^2 = 5^2a^8b^2 = 25a^8b^2$

d $\left(\frac{m^3}{2n^2}\right)^4 = \frac{m^{12}}{2^4n^8} = \frac{m^{12}}{16n^8}$

e $\left(\frac{3a^3}{b^5}\right)^3 = \frac{3^3a^9}{b^{15}} = \frac{27a^9}{b^{15}}$

f $(2m^3n^2)^5 = 2^5m^{15}n^{10} = 32m^{15}n^{10}$

g $\left(\frac{4a^4}{b^2}\right)^2 = \frac{4^2a^8}{b^4} = \frac{16a^8}{b^4}$

h $(5x^2y^3)^3 = 5^3x^6y^9 = 125x^6y^9$

7 **a** $(-2a)^2 = (-2)^2a^2 = 4a^2$

b $(-6b^2)^2 = (-6)^2b^4 = 36b^4$

c $(-2a)^3 = (-2)^3a^3 = -8a^3$

d $(-3m^2n^2)^3 = (-3)^3m^6n^6 = -27m^6n^6$

e $(-2ab^4)^4 = (-2)^4a^4b^{16} = 16a^4b^{16}$

f $\left(\frac{-2a^2}{b^2}\right)^3 = \frac{(-2)^3a^6}{b^6} = -\frac{8a^6}{b^6}$

g $\left(\frac{-4a^3}{b}\right)^2 = \frac{(-4)^2a^6}{b^2} = \frac{16a^6}{b^2}$

h $\left(\frac{-3p^2}{q^3}\right)^2 = \frac{(-3)^2p^4}{q^6} = \frac{9p^4}{q^6}$

8 **a** $\frac{a^3}{a} = \frac{a^3}{a^1} = a^{3-1}$ or a^2

b $4b^2 \times 2b^3 = 8b^{2+3} = 8b^5$

c $\frac{m^5n^4}{m^2n^3} = m^{5-2}n^{4-3} = m^3n^1 = m^3n$

d $\frac{14a^7}{2a^2} = \frac{14}{2}a^{7-2} = 7a^5$

e $\frac{12a^2b^3}{3ab} = \frac{12}{3}a^{2-1}b^{3-1} = 4ab^2$

f $\frac{18m^7a^3}{4m^4a^3} = \frac{18}{4}m^{7-4}a^{3-3} = \frac{9}{2}m^3a^0 = \frac{9}{2}m^3$ or $\frac{9m^3}{2}$

g $10hk^3 \times 4h^4 = 40 \times h^{1+4}k^3 = 40h^5k^3$

h $\frac{m^{11}}{(m^2)^8} = \frac{m^{11}}{m^{16}} = m^{11-16} = m^{-5}$ or $\frac{1}{m^5}$

i $\frac{p^2 \times p^7}{(p^3)^2} = \frac{p^{2+7}}{p^6} = \frac{p^9}{p^6} = p^{9-6} = p^3$

9 **a** $5^0 = 1$ **b** $3^{-1} = \frac{1}{3}$ **c** $6^{-1} = \frac{1}{6}$ **d** $8^0 = 1$

e $2^2 = 4$ **f** $2^{-2} = \dfrac{1}{2^2} = \frac{1}{4}$ **g** $2^3 = 8$ **h** $2^{-3} = \dfrac{1}{2^3} = \frac{1}{8}$

i $5^2 = 25$ **j** $5^{-2} = \dfrac{1}{5^2} = \frac{1}{25}$ **k** $10^2 = 100$ **l** $10^{-2} = \dfrac{1}{10^2} = \frac{1}{100}$

10 **a** $(\frac{2}{3})^0 = 1$ **b** $\dfrac{4^3}{4^3} = 1$

c $3y^0 = 3 \times 1 = 3$

d $(3y)^0 = 1$

e $2 \times 3^0 = 2 \times 1 = 2$

f $6^0 = 1$

g $\dfrac{5^2}{5^4} = 5^{2-4} = 5^{-2} = \dfrac{1}{5^2}$ or $\frac{1}{25}$

h $\dfrac{2^{10}}{2^{15}} = 2^{10-15} = 2^{-5} = \dfrac{1}{2^5}$ or $\frac{1}{32}$

i $(\frac{1}{3})^{-1} = (\frac{3}{1})^1 = 3$

j $(\frac{2}{5})^{-1} = (\frac{5}{2})^1 = 2\frac{1}{2}$

k $(\frac{4}{3})^{-1} = (\frac{3}{4})^1 = \frac{3}{4}$

l $(\frac{1}{12})^{-1} = (\frac{12}{1})^1 = 12$

m $(\frac{2}{3})^{-2} = (\frac{3}{2})^2 = \frac{9}{4}$ or $2\frac{1}{4}$

n $5^0 - 5^{-1} = 1 - \frac{1}{5} = \frac{4}{5}$

o $7^{-1} + 7^0 = \frac{1}{7} + 1 = 1\frac{1}{7}$ or $\frac{8}{7}$

p $2^0 + 2^1 + 2^{-1} = 1 + 2 + \frac{1}{2} = 3\frac{1}{2}$ or $\frac{7}{2}$

11 **a** $(2a)^{-1} = \dfrac{1}{2a}$

b $2a^{-1} = \dfrac{2}{1} \times \dfrac{1}{a} = \dfrac{2}{a}$

c $3b^{-1} = \dfrac{3}{1} \times \dfrac{1}{b} = \dfrac{3}{b}$

d $(3b)^{-1} = \dfrac{1}{3b}$

e $\left(\dfrac{2}{b}\right)^{-2} = \left(\dfrac{b}{2}\right)^2 = \dfrac{b^2}{4}$

f $(2b)^{-2} = \dfrac{1}{(2b)^2} = \dfrac{1}{4b^2}$

g $(3n)^{-2} = \dfrac{1}{(3n)^2} = \dfrac{1}{9n^2}$

h $(3n^{-2})^{-1} = 3^{-1}n^2 = \frac{1}{3}n^2 = \dfrac{n^2}{3}$

i $ab^{-1} = \dfrac{a}{1} \times \dfrac{1}{b} = \dfrac{a}{b}$

j $(ab)^{-1} = \dfrac{1}{ab}$

k $ab^{-2} = a \times \dfrac{1}{b^2} = \dfrac{a}{b^2}$

l $(ab)^{-2} = \dfrac{1}{(ab)^2} = \dfrac{1}{a^2b^2}$

m $(2ab)^{-1} = \dfrac{1}{2ab}$

n $2(ab)^{-1} = \dfrac{2}{1} \times \dfrac{1}{ab} = \dfrac{2}{ab}$

o $2ab^{-1} = \dfrac{2a}{1} \times \dfrac{1}{b} = \dfrac{2a}{b}$

p $\dfrac{(ab)^2}{b^{-1}} = \dfrac{a^2b^2}{b^{-1}} = a^2b^{2-(-1)} = a^2b^3$

12 **a** $\frac{1}{3} = 3^{-1}$ **b** $\frac{1}{2} = 2^{-1}$ **c** $\frac{1}{5} = 5^{-1}$ **d** $\dfrac{1}{4} = \dfrac{1}{2^2} = 2^{-2}$

e $\dfrac{1}{27} = \dfrac{1}{3^3} = 3^{-3}$

f $\dfrac{1}{25} = \dfrac{1}{5^2} = 5^{-2}$

g $\dfrac{1}{8^x} = \dfrac{1}{(2^3)^x} = \dfrac{1}{2^{3x}} = 2^{-3x}$

h $\dfrac{1}{16^y} = \dfrac{1}{(2^4)^y} = \dfrac{1}{2^{4y}} = 2^{-4y}$

i $\dfrac{1}{81^a} = \dfrac{1}{(3^4)^a} = \dfrac{1}{3^{4a}} = 3^{-4a}$

j $\dfrac{9}{3^4} = \dfrac{3^2}{3^4} = 3^{2-4} = 3^{-2}$

k $25 \times 5^{-4} = 5^2 \times 5^{-4} = 5^{2+(-4)} = 5^{-2}$

l $\dfrac{5^{-1}}{5^2} = 5^{-1-2} = 5^{-3}$

m $2 \div 2^{-3} = 2^1 \div 2^{-3} = 2^{1-(-3)} = 2^4$

n $1 = 2^0$ or 3^0 or 5^0

o $6^{-3} = (2 \times 3)^{-3} = 2^{-3} \times 3^{-3}$

p $4 \times 10^2 = 2^2 \times (2 \times 5)^2 = 2^2 \times 2^2 \times 5^2 = 2^4 \times 5^2$

13 1 less day, i.e., 25 days

14 For 1 coin, 1 sum
For 2 coins, 3 sums
For 3 coins, 7 sums
For 4 coins, 15 sums

For coins A, B, C

ABC	AB	A
	AC	B
	BC	C

For coins A, B, C, D

ABCD	ABC	AB	A
	ABD	AC	B
	ACD	AD	C
	BCD	BC	D
		BD	
		CD	

and $1 = 2^1 - 1$
$3 = 2^2 - 1$
$7 = 2^3 - 1$
$15 = 2^4 - 1$ So, for 6 coins we expect $2^6 - 1 = 63$ different sums.

15 **a** $5^3 = 21 + 23 + 25 + 27 + 29$ **b** $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$
c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

16 We notice that 175 and 75 have a common factor of 25.
So 2^{175} and 5^{75}

$= (2^7)^{25}$ $= (5^3)^{25}$
$= (128)^{25}$ $= (125)^{25}$ $\therefore$ as $125 < 128$, 5^{75} is smaller.

EXERCISE 3D

1 **a** $\sqrt[5]{2} = 2^{\frac{1}{5}}$

b $\dfrac{1}{\sqrt[5]{2}} = \dfrac{1}{2^{\frac{1}{5}}} = 2^{-\frac{1}{5}}$

c $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$

d $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}$

e $\dfrac{1}{\sqrt[3]{2}} = \dfrac{1}{2^{\frac{1}{3}}} = 2^{-\frac{1}{3}}$

f $2\times\sqrt[3]{2}$
$=2^1\times 2^{\frac{1}{3}}$
$=2^{\frac{4}{3}}$

g $\dfrac{4}{\sqrt{2}}$
$=\dfrac{2^2}{2^{\frac{1}{2}}}$
$=2^{2-\frac{1}{2}}$
$=2^{\frac{3}{2}}$

h $(\sqrt{2})^3$
$=\left(2^{\frac{1}{2}}\right)^3$
$=2^{\frac{1}{2}\times 3}$
$=2^{\frac{3}{2}}$

i $\dfrac{1}{\sqrt[3]{16}}$
$=\dfrac{1}{(2^4)^{\frac{1}{3}}}$
$=\dfrac{1}{2^{\frac{4}{3}}}$
$=2^{-\frac{4}{3}}$

j $\dfrac{1}{\sqrt{8}}$
$=\dfrac{1}{(2^3)^{\frac{1}{2}}}$
$=\dfrac{1}{2^{\frac{3}{2}}}$
$=2^{-\frac{3}{2}}$

2 **a** $\sqrt[3]{3}$
$=3^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt[3]{3}}$
$=\dfrac{1}{3^{\frac{1}{3}}}$
$=3^{-\frac{1}{3}}$

c $\sqrt[4]{3}$
$=3^{\frac{1}{4}}$

d $3\sqrt{3}$
$=3^1\times 3^{\frac{1}{2}}$
$=3^{1\frac{1}{2}}$
$=3^{\frac{3}{2}}$

e $\dfrac{1}{9\sqrt{3}}$
$=\dfrac{1}{3^2 3^{\frac{1}{2}}}$
$=\dfrac{1}{3^{2\frac{1}{2}}}$
$=3^{-\frac{5}{2}}$

3 **a** $\sqrt[3]{7}$
$=7^{\frac{1}{3}}$

b $\sqrt[4]{27}$
$=\left(3^3\right)^{\frac{1}{4}}$
$=3^{\frac{3}{4}}$

c $\sqrt[5]{16}$
$=\left(2^4\right)^{\frac{1}{5}}$
$=2^{\frac{4}{5}}$

d $\sqrt[3]{32}$
$=\left(2^5\right)^{\frac{1}{3}}$
$=2^{\frac{5}{3}}$

e $\sqrt[7]{49}$
$=\left(7^2\right)^{\frac{1}{7}}$
$=7^{\frac{2}{7}}$

f $\dfrac{1}{\sqrt[3]{7}}$
$=\dfrac{1}{7^{\frac{1}{3}}}$
$=7^{-\frac{1}{3}}$

g $\dfrac{1}{\sqrt[4]{27}}$
$=\dfrac{1}{(3^3)^{\frac{1}{4}}}$
$=\dfrac{1}{3^{\frac{3}{4}}}$
$=3^{-\frac{3}{4}}$

h $\dfrac{1}{\sqrt[5]{16}}$
$=\dfrac{1}{(2^4)^{\frac{1}{5}}}$
$=\dfrac{1}{2^{\frac{4}{5}}}$
$=2^{-\frac{4}{5}}$

i $\dfrac{1}{\sqrt[3]{32}}$
$=\dfrac{1}{(2^5)^{\frac{1}{3}}}$
$=\dfrac{1}{2^{\frac{5}{3}}}$
$=2^{-\frac{5}{3}}$

j $\dfrac{1}{\sqrt[7]{49}}$
$=\dfrac{1}{(7^2)^{\frac{1}{7}}}$
$=\dfrac{1}{7^{\frac{2}{7}}}$
$=7^{-\frac{2}{7}}$

4 **a** 2.280 **b** 1.834 **c** 0.794 **d** 0.435

5 **a** 3 **b** 1.682 **c** 1.933 **d** 0.523

6 **a** $4^{\frac{3}{2}}$
$=\left(2^2\right)^{\frac{3}{2}}$
$=2^3$
$=8$

b $8^{\frac{5}{3}}$
$=\left(2^3\right)^{\frac{5}{3}}$
$=2^5$
$=32$

c $16^{\frac{3}{4}}$
$=\left(2^4\right)^{\frac{3}{4}}$
$=2^3$
$=8$

d $25^{\frac{3}{2}}$
$=\left(5^2\right)^{\frac{3}{2}}$
$=5^3$
$=125$

e $32^{\frac{2}{5}}$
$=\left(2^5\right)^{\frac{2}{5}}$
$=2^2$
$=4$

f $4^{-\frac{1}{2}}$
$=\left(2^2\right)^{-\frac{1}{2}}$
$=2^{-1}$
$=\frac{1}{2}$

g $9^{-\frac{3}{2}}$
$=\left(3^2\right)^{-\frac{3}{2}}$
$=3^{-3}$
$=\dfrac{1}{3^3}$
$=\frac{1}{27}$

h $8^{-\frac{4}{3}}$
$=\left(2^3\right)^{-\frac{4}{3}}$
$=2^{-4}$
$=\dfrac{1}{2^4}$
$=\frac{1}{16}$

i $27^{-\frac{4}{3}}$
$=\left(3^3\right)^{-\frac{4}{3}}$
$=3^{-4}$
$=\dfrac{1}{3^4}$
$=\frac{1}{81}$

j $125^{-\frac{2}{3}}$
$=\left(5^3\right)^{-\frac{2}{3}}$
$=5^{-2}$
$=\dfrac{1}{5^2}$
$=\frac{1}{25}$

EXERCISE 3E

1 **a** $x^2(x^3+2x^2+1)$
$=x^5+2x^4+x^2$

b $2^x(2^x+1)$
$=2^{2x}+2^x$

c $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$
$=x^1+x^0$
$=x+1$

d $e^x(e^x+2)$
$=e^{2x}+2e^x$

e $3^x(2-3^{-x})$
$=2\times 3^x-3^0$
$=2\times 3^x-1$

f $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$
$=x^2+2x^1+3x^0$
$=x^2+2x+3$

g $2^{-x}(2^x+5)$
$=2^0+5\times 2^{-x}$
$=1+5\times 2^{-x}$

h $5^{-x}(5^{2x}+5^x)$
$=5^{-x+2x}+5^{-x+x}$
$=5^x+5^0$
$=5^x+1$

i $x^{-\frac{1}{2}}(x^2+x^1+x^{\frac{1}{2}})$
$=x^{1\frac{1}{2}}+x^{\frac{1}{2}}+x^0$
$=x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$

2 **a** $(2^x+1)(2^x+3)$
$=2^{2x}+3\times 2^x+2^x+3$
$=2^{2x}+4\times 2^x+3$
$=4^x+2^2\times 2^x+3$
$=4^x+2^{2+x}+3$

b $(3^x+2)(3^x+5)$
$=3^{2x}+5\times 3^x+2\times 3^x+10$
$=9^x+7\times 3^x+10$

c $(5^x-2)(5^x-4)$
$=5^{2x}-4\times 5^x-2\times 5^x+8$
$=25^x-6\times 5^x+8$

d $(2^x+3)^2$
$=(2^x)^2+2\times 2^x\times 3+3^2$
$=2^{2x}+3\times 2^{1+x}+9$
$=4^x+3\times 2^{x+1}+9$

e $(3^x-1)^2$
$=(3^x)^2-2\times 3^x+1$
$=3^{2x}-2\times 3^x+1$
$=9^x-2\times 3^x+1$

f $(4^x+7)^2$
$=(4^x)^2+2\times 4^x\times 7+7^2$
$=4^{2x}+7\times 2\times 2^{2x}+49$
$=2^{4x}+7\times 2^{2x+1}+49$

g $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$
$=(x^{\frac{1}{2}})^2-2^2$
$=x-4$

h $(2^x+3)(2^x-3)$
$=(2^x)^2-3^2$
$=2^{2x}-9$

i $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$
$=x^1-x^{-1}$
$=x-\dfrac{1}{x}$

j $\left(x+\dfrac{2}{x}\right)^2$
$=x^2+2\times x\times\dfrac{2}{x}+\dfrac{4}{x^2}$
$=x^2+4+\dfrac{4}{x^2}$

k $(e^x-e^{-x})^2$
$=(e^x)^2-2e^xe^{-x}+e^{-2x}$
$=e^{2x}-2+e^{-2x}$

l $(5-2^{-x})^2$
$=5^2-2\times 5\times 2^{-x}+(2^{-x})^2$
$=25-5\times 2^{1-x}+2^{-2x}$

EXERCISE 3F

1 **a** $2^x=2$
$\therefore\ 2^x=2^1$
$\therefore\ x=1$

b $2^x=4$
$\therefore\ 2^x=2^2$
$\therefore\ x=2$

c $3^x=27$
$\therefore\ 3^x=3^3$
$\therefore\ x=3$

d $2^x=1$
$\therefore\ 2^x=2^0$
$\therefore\ x=0$

e
$$\begin{aligned} 2^x &= \tfrac{1}{2} \\ \therefore\ 2^x &= 2^{-1} \\ \therefore\ x &= -1 \end{aligned}$$

f
$$\begin{aligned} 3^x &= \tfrac{1}{3} \\ \therefore\ 3^x &= 3^{-1} \\ \therefore\ x &= -1 \end{aligned}$$

g
$$\begin{aligned} 2^x &= \tfrac{1}{8} \\ \therefore\ 2^x &= 2^{-3} \\ \therefore\ x &= -3 \end{aligned}$$

h
$$\begin{aligned} 2^{x+1} &= 8 \\ \therefore\ 2^{x+1} &= 2^3 \\ \therefore\ x+1 &= 3 \\ \therefore\ x &= 2 \end{aligned}$$

i
$$\begin{aligned} 2^{x-2} &= \tfrac{1}{4} \\ \therefore\ 2^{x-2} &= 2^{-2} \\ \therefore\ x-2 &= -2 \\ \therefore\ x &= 0 \end{aligned}$$

j
$$\begin{aligned} 3^{x+1} &= \tfrac{1}{27} \\ \therefore\ 3^{x+1} &= 3^{-3} \\ \therefore\ x+1 &= -3 \\ \therefore\ x &= -4 \end{aligned}$$

k
$$\begin{aligned} 2^{x+1} &= 64 \\ \therefore\ 2^{x+1} &= 2^6 \\ \therefore\ x+1 &= 6 \\ \therefore\ x &= 5 \end{aligned}$$

l
$$\begin{aligned} 2^{1-2x} &= \tfrac{1}{2} \\ \therefore\ 2^{1-2x} &= 2^{-1} \\ \therefore\ 1-2x &= -1 \\ \therefore\ -2x &= -2 \\ \therefore\ x &= 1 \end{aligned}$$

2 **a**
$$\begin{aligned} 4^x &= 32 \\ \therefore\ 2^{2x} &= 2^5 \\ \therefore\ 2x &= 5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

b
$$\begin{aligned} 8^x &= \tfrac{1}{4} \\ \therefore\ 2^{3x} &= 2^{-2} \\ \therefore\ 3x &= -2 \\ \therefore\ x &= -\tfrac{2}{3} \end{aligned}$$

c
$$\begin{aligned} 9^x &= \tfrac{1}{3} \\ \therefore\ 3^{2x} &= 3^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

d
$$\begin{aligned} 49^x &= \tfrac{1}{7} \\ \therefore\ 7^{2x} &= 7^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

e
$$\begin{aligned} 4^x &= \tfrac{1}{8} \\ \therefore\ 2^{2x} &= 2^{-3} \\ \therefore\ 2x &= -3 \\ \therefore\ x &= -\tfrac{3}{2} \end{aligned}$$

f
$$\begin{aligned} 25^x &= \tfrac{1}{5} \\ \therefore\ 5^{2x} &= 5^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

g
$$\begin{aligned} 8^{x+2} &= 32 \\ \therefore\ 2^{3(x+2)} &= 2^5 \\ \therefore\ 3x+6 &= 5 \\ \therefore\ 3x &= -1 \\ \therefore\ x &= -\tfrac{1}{3} \end{aligned}$$

h
$$\begin{aligned} 8^{1-x} &= \tfrac{1}{4} \\ \therefore\ 2^{3(1-x)} &= 2^{-2} \\ \therefore\ 3-3x &= -2 \\ \therefore\ -3x &= -5 \\ \therefore\ x &= \tfrac{5}{3} \end{aligned}$$

i
$$\begin{aligned} 4^{2x-1} &= \tfrac{1}{2} \\ \therefore\ 2^{2(2x-1)} &= 2^{-1} \\ \therefore\ 4x-2 &= -1 \\ \therefore\ 4x &= 1 \\ \therefore\ x &= \tfrac{1}{4} \end{aligned}$$

j
$$\begin{aligned} 9^{x-3} &= 3 \\ \therefore\ 3^{2(x-3)} &= 3^1 \\ \therefore\ 2x-6 &= 1 \\ \therefore\ 2x &= 7 \\ \therefore\ x &= \tfrac{7}{2} \end{aligned}$$

k
$$\begin{aligned} \left(\tfrac{1}{2}\right)^{x+1} &= 2 \\ \therefore\ \left(2^{-1}\right)^{x+1} &= 2^1 \\ \therefore\ -x-1 &= 1 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

l
$$\begin{aligned} \left(\tfrac{1}{3}\right)^{x+2} &= 9 \\ \therefore\ \left(3^{-1}\right)^{x+2} &= 3^2 \\ \therefore\ -x-2 &= 2 \\ \therefore\ -x &= 4 \\ \therefore\ x &= -4 \end{aligned}$$

m
$$\begin{aligned} 4^x &= 8^{-x} \\ \therefore\ 2^{2x} &= \left(2^3\right)^{-x} \\ \therefore\ 2x &= -3x \\ \therefore\ 5x &= 0 \\ \therefore\ x &= 0 \end{aligned}$$

n
$$\begin{aligned} \left(\tfrac{1}{4}\right)^{1-x} &= 8 \\ \therefore\ \left(2^{-2}\right)^{1-x} &= 2^3 \\ \therefore\ -2+2x &= 3 \\ \therefore\ 2x &= 5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

o
$$\begin{aligned} \left(\tfrac{1}{7}\right)^{x} &= 49 \\ \therefore\ \left(7^{-1}\right)^{x} &= 7^2 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

p
$$\begin{aligned} \left(\tfrac{1}{2}\right)^{x+1} &= 32 \\ \therefore\ \left(2^{-1}\right)^{x+1} &= 2^5 \\ \therefore\ -x-1 &= 5 \\ \therefore\ -x &= 6 \\ \therefore\ x &= -6 \end{aligned}$$

3 **a**
$$\begin{aligned} 4^{2x+1} &= 8^{1-x} \\ \therefore\ \left(2^2\right)^{2x+1} &= \left(2^3\right)^{1-x} \\ \therefore\ 4x+2 &= 3-3x \\ \therefore\ 7x &= 1 \\ \therefore\ x &= \tfrac{1}{7} \end{aligned}$$

b
$$\begin{aligned} 9^{2-x} &= \left(\tfrac{1}{3}\right)^{2x+1} \\ \therefore\ \left(3^2\right)^{2-x} &= \left(3^{-1}\right)^{2x+1} \\ \therefore\ 4-2x &= -2x-1 \\ \therefore\ 4 &= -1 \end{aligned}$$
So, no solutions exist.

c
$$\begin{aligned} 2^x \times 8^{1-x} &= \tfrac{1}{4} \\ \therefore\ 2^x \times \left(2^3\right)^{1-x} &= 2^{-2} \\ \therefore\ x+3-3x &= -2 \\ \therefore\ -2x &= -5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

4 **a**
$$\begin{aligned} 3 \times 2^x &= 24 \\ \therefore\ 2^x &= 8 \\ \therefore\ 2^x &= 2^3 \\ \therefore\ x &= 3 \end{aligned}$$

b
$$\begin{aligned} 7 \times 2^x &= 56 \\ \therefore\ 2^x &= 8 \\ \therefore\ 2^x &= 2^3 \\ \therefore\ x &= 3 \end{aligned}$$

c
$$\begin{aligned} 3 \times 2^{x+1} &= 24 \\ \therefore\ 2^{x+1} &= 8 \\ \therefore\ 2^{x+1} &= 2^3 \\ \therefore\ x+1 &= 3 \\ \therefore\ x &= 2 \end{aligned}$$

d
$$\begin{aligned} 12 \times 3^{-x} &= \tfrac{4}{3} \\ \therefore\ 3^{-x} &= \tfrac{4}{3} \div 12 \\ \therefore\ 3^{-x} &= \tfrac{4}{3} \times \tfrac{1}{12} \\ \therefore\ 3^{-x} &= \tfrac{1}{9} \\ \therefore\ 3^{-x} &= 3^{-2} \\ \therefore\ x &= 2 \end{aligned}$$

e
$$\begin{aligned} 4 \times \left(\tfrac{1}{3}\right)^x &= 36 \\ \therefore\ \left(\tfrac{1}{3}\right)^x &= 9 \\ \therefore\ \left(3^{-1}\right)^x &= 3^2 \\ \therefore\ 3^{-x} &= 3^2 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

f
$$\begin{aligned} 5 \times \left(\tfrac{1}{2}\right)^x &= 20 \\ \therefore\ \left(\tfrac{1}{2}\right)^x &= 4 \\ \therefore\ \left(2^{-1}\right)^x &= 2^2 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

EXERCISE 3G

1 **a** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$

from point A, $y \doteqdot 1.4$

$\therefore\ 2^{\frac{1}{2}} \doteqdot 1.4$

b When $x = 0.8$, $y = 2^{0.8}$

from point B, $y \doteqdot 1.7$

$\therefore\ 2^{0.8} \doteqdot 1.7$

c When $x = 1.5$, $y = 2^{1.5}$

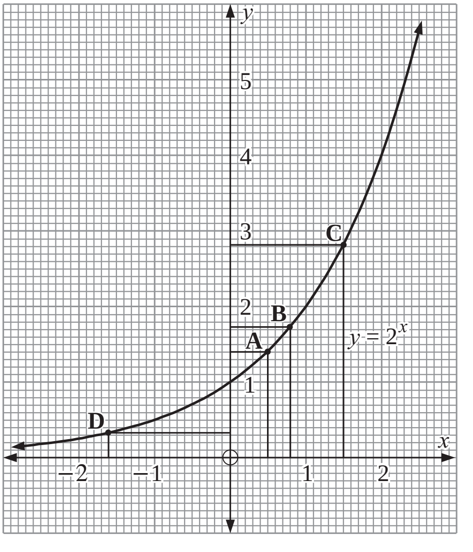
from point C, $y \doteqdot 2.8$

$\therefore\ 2^{1.5} \doteqdot 2.8$

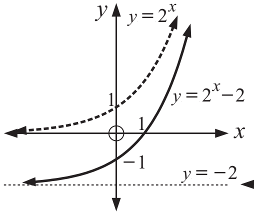
d When $x = -1.6$, $y = 2^{-1.6}$

from point D, $y \doteqdot 0.3$

$\therefore\ 2^{-1.6} \doteqdot 0.3$

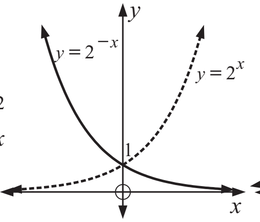


2 **a**



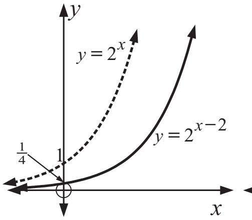
a vertical translation of 2 units downwards
$y = -2$ is a HA

b



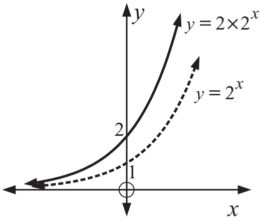
a reflection in the y-axis

c



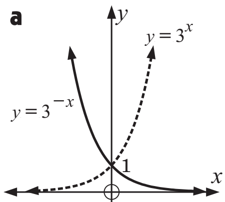
a vertical dilation of factor $\frac{1}{4}$

d



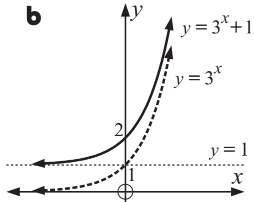
a vertical dilation of factor 2

4 **a**



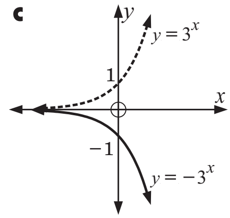
a reflection in the y-axis

b



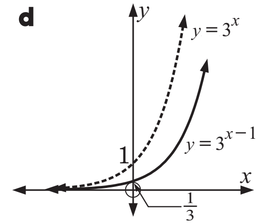
a vertical translation of 1 unit
$y = 1$ is the HA

c



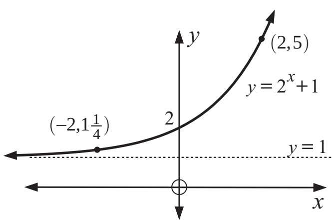
a reflection in the x-axis

d



a vertical dilation of factor $\frac{1}{3}$

5 **a**

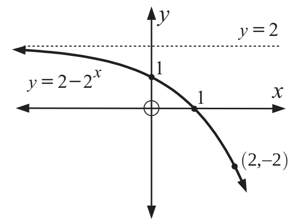


a vertical translation of 1 unit
Horizontal asymptote is $y = 1$.

When $x = 2$, $y = 4 + 1$
$= 5$

When $x = -2$, $y = \frac{1}{4} + 1$
$= 1\frac{1}{4}$

b



The horizontal asymptote is $y = 2$.

When $x = 0$, $y = 2 - 2^0 = 2 - 1 = 1$

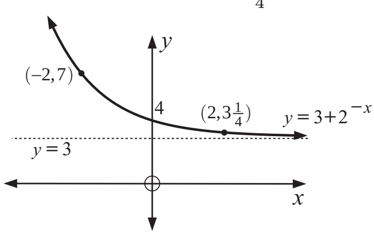
$\therefore$ y-intercept is 1

When $x = 1$, $y = 2 - 2 = 0$

When $x = 2$, $y = 2 - 4 = -2$

When $x = -2$, $y = 2 - \frac{1}{4} = 1\frac{3}{4}$

c



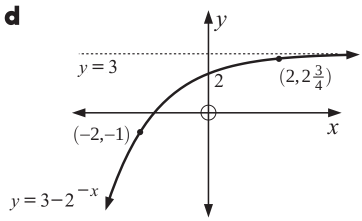
The HA is $y = 3$.

When $x = 0$, $y = 3 + 1 = 4$

When $x = 2$, $y = 3 + \frac{1}{4} = 3\frac{1}{4}$

When $x = -2$, $y = 3 + 2^2 = 7$

d



The HA is $y = 3$.

When $x = 0$, $y = 3 - 1 = 2$

When $x = 2$, $y = 3 - \frac{1}{4} = 2\frac{3}{4}$

When $x = -2$, $y = 3 - 4 = -1$

EXERCISE 3H

1 **a** When $t = 0$, $W_0 = 100$ grams

b **i** When $t = 4$,

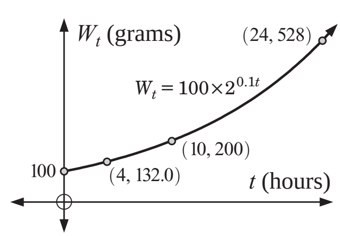
$W_4 = 100 \times 2^{0.1 \times 4}$
$= 100 \times 2^{0.4}$
$\doteqdot 132$ grams

ii When $t = 10$,

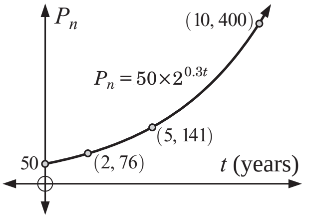
$W_{10} = 100 \times 2^1$
$= 200$ grams

iii When $t = 24$,

$W_{24} = 100 \times 2^{0.1 \times 24}$
$= 100 \times 2^{2.4}$
$\doteqdot 528$ grams



2 **a** $P_0 = 50$ (the initial population)



b **i** When $t = 2$,

$$\begin{aligned} P_2 &= 50 \times 2^{0.3\times2} \\ &= 50 \times 2^{0.6} \\ &\doteqdot 75.785.... \end{aligned}$$

So, the expected population is 76 possums.

ii When $t = 5$,

$$\begin{aligned} P_5 &= 50 \times 2^{0.3\times5} \\ &= 50 \times 2^{1.5} \\ &\doteqdot 141.421.... \end{aligned}$$

So, the expected population is 141 possums.

iii When $t = 10$,

$$\begin{aligned} P_{10} &= 50 \times 2^{0.3\times10} \\ &= 50 \times 2^{3} \\ &= 400 \end{aligned}$$

So, the expected population is 400 possums.

3 **a** When $t = 0$

$$\begin{aligned} V_0 &= V_0 \times 2^0 \\ &= V_0 \end{aligned}$$

So, the speed is V_0.

b When $t = 20$

$$\begin{aligned} V_{20} &= V_0 \times 2^{0.05\times20} \\ &= V_0 \times 2^1 \\ &= 2V_0 \end{aligned}$$

So, the speed is $2V_0$.

c V_0 becomes $2V_0$
a 100% increase.

d

$$\begin{aligned} &\left(\frac{V_{50} - V_{20}}{V_{20}}\right) \times 100\% \\ &= \left(\frac{V_0 \times 2^{2.5} - V_0 \times 2^1}{V_0 \times 2^1}\right) \times 100\% \\ &= \left(\frac{2^{2.5} - 2^1}{2^1}\right) \times 100\% \\ &\doteqdot 183\% \end{aligned}$$

This expression is the percentage increase in the speed from a speed at 20^{o}C increased to the speed at 50^{o}C.

$V_{50} - V_{20}$ is the increase in speed.

4 **a** $B_0 = 6$ pairs $= 12$ bears

b At year 2018, $t = 20$

$$\begin{aligned} \therefore\ B_{20} &= 12 \times 2^{0.18\times20} \\ &= 12 \times 2^{3.6} \\ &\doteqdot 145.508.... \\ &\doteqdot 146 \text{ bears} \end{aligned}$$

c At year 2008, $t = 10$

$$\begin{aligned} \therefore\ \%\text{ increase} &= \left(\frac{B_{20} - B_{10}}{B_{10}}\right) \times 100\% \\ &= \left(\frac{12 \times 2^{3.6} - 12 \times 2^{1.8}}{12 \times 2^{1.8}}\right) \times 100\% \\ &= \left(\frac{2^{3.6} - 2^{1.8}}{2^{1.8}}\right) \times 100\% \\ &\doteqdot 248\% \end{aligned}$$

EXERCISE 3I

1 $W_t = 250 \times (0.998)^t$ grams

a

$$\begin{aligned} W_0 &= 250 \times (0.998)^0 \\ &= 250 \times 1 \\ &= 250 \text{ grams} \end{aligned}$$

b **i** When $t = 400$

$$\begin{aligned} &W_{400} \\ &= 250 \times (0.998)^{400} \\ &\doteqdot 112 \text{ grams} \end{aligned}$$

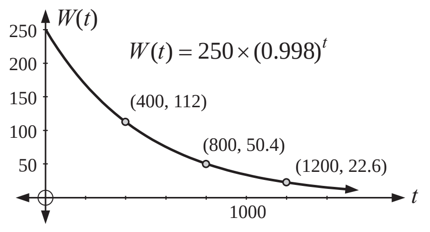
ii When $t = 800$

$$\begin{aligned} &W_{800} \\ &= 250 \times (0.998)^{800} \\ &\doteqdot 50.4 \text{ grams} \end{aligned}$$

iii When $t = 1200$

$$\begin{aligned} &W_{1200} \\ &= 250 \times (0.998)^{1200} \\ &\doteqdot 22.6 \text{ grams} \end{aligned}$$

c



d When $W(t) = 125$

$250 \times (0.998)^t = 125$

$\therefore \quad (0.998)^t = 0.5$

$\therefore \quad t \doteqdot 346.2$ {technology}

i.e., it takes approximately 346 years

2 $R = 8000 \times (0.837)^t$

1 When $t = 0$, $R = 8000 \times (0.837)^0$

$= 8000 \times 1$

$= 8000$ rabbits

2 'to the power 3.5' means that

$(0.837)^{3.5} = (0.837)^3 \times \sqrt{0.837}$

3 When $R = 80$,

$8000 \times (0.837)^t = 80$

$\therefore \quad (0.837)^t = 0.01$

and by trial-and-error methods, $t \doteqdot 26$

i.e., it would take 26 weeks (approx)

4 Yes, after a very long time.

Notice that when $R = 1$

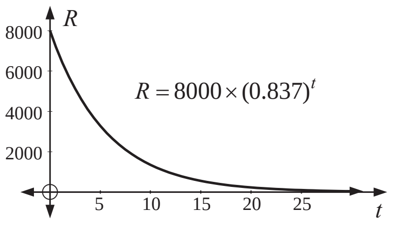
$8000(0.837)^t = 1$

$\therefore \quad (0.837)^t = 0.000\,125$

and by trial and error $t \doteqdot 51$

i.e., after 51 weeks

5



3 $T_t = 100 \times 2^{-0.02t}$

a $T_0 = 100 \times 2^0$

$= 100 \times 1$

$= 100^o$C

b **i** $T_{15} = 100 \times 2^{-0.02 \times 15}$

$= 100 \times 2^{-0.3}$

$\doteqdot 81.2^o$C

ii $T_{20} = 100 \times 2^{-0.02 \times 20}$

$= 100 \times 2^{-0.4}$

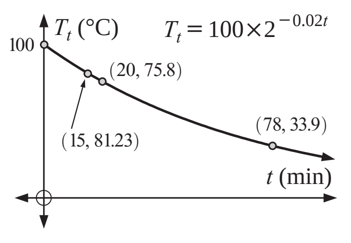
$\doteqdot 75.8^o$C

iii $T_{78} = 100 \times 2^{-0.02 \times 78}$

$= 100 \times 2^{-1.56}$

$\doteqdot 33.9^o$C

c



4 $W_t = 1000 \times 2^{-0.03t}$

a $W_0 = 1000 \times 2^0$

$= 1000 \times 1$

$= 1000$ grams

b **i** W_{10}

$= 1000 \times 2^{-0.3}$

$\doteqdot 812$ g

ii W_{100}

$= 1000 \times 2^{-3}$

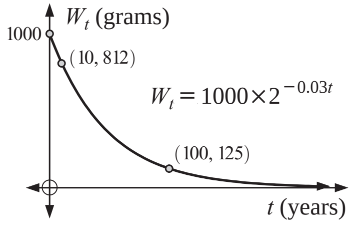
$= 125$ g

iii W_{1000}

$= 1000 \times 2^{-30}$

$\doteqdot 9.31 \times 10^{-7}$ g

c



5 **a** When $t = 0$, $W_0 = W_0 2^0 = W_0$

$\therefore$ the original weight was W_0

b $\%\text{ loss} = \left(\dfrac{W_{1000} - W_0}{W_0}\right) \times 100\%$

$= \left(\dfrac{W_0 \times 2^{-0.2} - W_0}{W_0}\right) \times 100\%$

$= (2^{-0.2} - 1) \times 100\%$

$\doteqdot -12.9\%$

i.e., a 12.9% weight loss

6 **a** When $t = 0$, $I_0 = I_0 \times 2^0 = I_0 \times 1 = I_0$

$\therefore$ the original current is I_0 amps

b When $t = 1$, $I_1 = I_0 \times 2^{-0.02} \doteqdot 0.9862 \times I_0$

c $\%\text{ change} = \left(\dfrac{I_1 - I_0}{I_0}\right) \times 100\%$

$\doteqdot \left(\dfrac{0.986I_0 - I_0}{I_0}\right) \times 100\%$

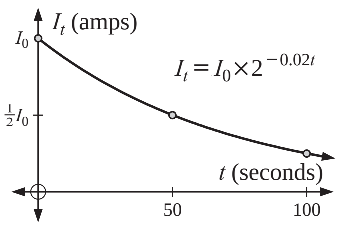
$\doteqdot (0.9862 - 1) \times 100\%$

$\doteqdot -0.0138 \times 100\%$

$\doteqdot 1.38\%$ loss

d $I_{50} = I_0 \times 2^{-1} = \frac{1}{2}I_0$

$I_{100} = I_0 \times 2^{-2} = \frac{1}{4}I_0$



REVIEW SET 3A

1 **a** $-(-1)^{10}$
$= -1 \quad \{(-1)^{10} = 1\}$

b $-(-3)^3$
$= -[-27]$
$= 27$

c $3^0 - 3^{-1}$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

2 **a** $a^4b^5 \times a^2b^2$
$= a^{4+2} \times b^{5+2}$
$= a^6b^7$

b $6xy^5 \div 9x^2y^5$
$= \frac{6}{9}x^{1-2}y^{5-5}$
$= \frac{2}{3}x^{-1}y^0$
$= \dfrac{2}{3x}$

c $\dfrac{5(x^2y)^2}{(5x^2)^2}$
$= \dfrac{5 \times x^4y^2}{25x^4}$
$= \frac{1}{5}x^0y^2$ or $\dfrac{y^2}{5}$

3 **a** 2×2^{-4}
$= 2^1 \times 2^{-4}$
$= 2^{1+(-4)}$
$= 2^{-3}$

b $16 \div 2^{-3}$
$= 2^4 \div 2^{-3}$
$= 2^{4-(-3)}$
$= 2^7$

c 8^4
$= (2^3)^4$
$= 2^{12}$

4 **a** b^{-3}
$= \dfrac{1}{b^3}$

b $(ab)^{-1}$
$= \left(\dfrac{1}{ab}\right)$
$= \dfrac{1}{ab}$

c ab^{-1}
$= \dfrac{a}{1} \times \dfrac{1}{b}$
$= \dfrac{a}{b}$

5 **a** $2^{x-3} = \frac{1}{32}$
$\therefore 2^{x-3} = 2^{-5}$
$\therefore x - 3 = -5$
$\therefore x = -2$

b $9^x = 27^{2-2x}$
$\therefore (3^2)^x = (3^3)^{2-2x}$
$\therefore 2x = 6 - 6x$
$\therefore 8x = 6$
$\therefore x = \frac{6}{8} = \frac{3}{4}$

6 **a** $8^{\frac{2}{3}} = \left(2^3\right)^{\frac{2}{3}} = 2^2 = 4$ **b** $27^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}} = 3^{-2} = \dfrac{1}{3^2} = \frac{1}{9}$

7 **a** 2.28 **b** 0.517 **c** 3.16

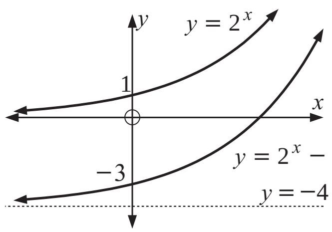
8 $f(x) = 3 \times 2^x$

a $f(0) = 3 \times 2^0 = 3 \times 1 = 3$

b $f(3) = 3 \times 2^3 = 3 \times 8 = 24$

c $f(-2) = 3 \times 2^{-2} = 3 \times \dfrac{1}{2^2} = \frac{3}{4}$

9



a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$

b $y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$

10 $T = 80 \times (0.913)^t$ ^{o}C

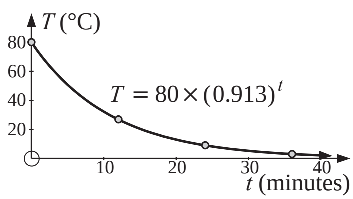
a When $t = 0$, $T = 80 \times (0.913)^0 = 80 \times 1 = 80$ $\therefore$ initial temperature is 80^{o}C

b **i** When $t = 12$, $T = 80 \times (0.913)^{12} \doteqdot 26.8^o$C

ii When $t = 24$, $T = 80 \times (0.913)^{24} \doteqdot 9.00^o$C

iii When $t = 36$, $T = 80 \times (0.913)^{36} \doteqdot 3.02^o$C

c



d When $T = 25$

$80 \times (0.913)^t = 25$

$\therefore\ 0.913^t = 0.3125$

$\therefore\ t \doteqdot 12.8$ sec {technology}

REVIEW SET 3B

1 **a** $-(-2)^3 = -[-8] = 8$

b $5^{-1} - 5^0 = \frac{1}{5} - 1 = -\frac{4}{5}$

2 **a** $(a^7)^3 = a^{7\times3} = a^{21}$

b $pq^2 \times p^3q^4 = p^{1+3}q^{2+4} = p^4q^6$

c $\dfrac{8ab^5}{2a^4b^4} = \frac{8}{2}a^{1-4}b^{5-4} = 4a^{-3}b^1 = \dfrac{4}{1} \times \dfrac{1}{a^3} \times b = \dfrac{4b}{a^3}$

3 **a** $\frac{1}{16} = \dfrac{1}{2^4} = 2^{-4}$

b $2^x \times 4 = 2^x \times 2^2 = 2^{x+2}$

c $4^x \div 8 = (2^2)^x \div 2^3 = 2^{2x} \div 2^3 = 2^{2x-3}$

4 **a** $x^{-2} \times x^{-3}$
$= x^{-2+(-3)}$
$= x^{-5}$
$= \dfrac{1}{x^5}$

b $2(ab)^{-2}$
$= 2 \times \dfrac{1}{(ab)^2}$
$= \dfrac{2}{a^2b^2}$

c $2ab^{-2}$
$= 2a \times \left(\dfrac{1}{b^2}\right)$
$= \dfrac{2a}{b^2}$

5 **a**
$$\begin{aligned} 2^{x+1} &= 32 \\ \therefore\ 2^{x+1} &= 2^5 \\ \therefore\ x+1 &= 5 \\ \therefore\ x &= 4 \end{aligned}$$

b
$$\begin{aligned} 4^{x+1} &= \left(\tfrac{1}{8}\right)^x \\ \therefore\ (2^2)^{x+1} &= (2^{-3})^x \\ \therefore\ 2x+2 &= -3x \\ \therefore\ 5x &= -2 \\ \therefore\ x &= -\tfrac{2}{5} \end{aligned}$$

6 **a** $81 = 3^4$ **b** $1 = 3^0$ **c** $\dfrac{1}{27} = \dfrac{1}{3^3} = 3^{-3}$ **d** $\dfrac{1}{243} = \dfrac{1}{3^5} = 3^{-5}$

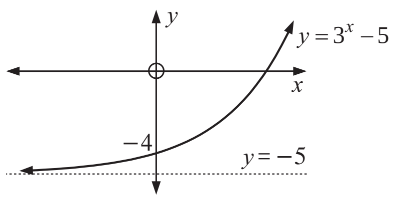
7 **a**
$$\begin{aligned} \frac{27}{9^a} &= \frac{3^3}{(3^2)^a} \\ &= 3^{3-2a} \end{aligned}$$

b
$$\begin{aligned} \left(\sqrt{3}\right)^{1-x} \times 9^{1-2x} &= \left(3^{\frac{1}{2}}\right)^{1-x} \times (3^2)^{1-2x} \\ &= 3^{\frac{1}{2}-\frac{1}{2}x+2-4x} \\ &= 3^{\frac{5}{2}-\frac{9}{2}x} \end{aligned}$$

8 **a**
When $x = 0$, $y = 3^0 - 5 = 1 - 5 = -4$
When $x = 1$, $y = 3^1 - 5 = 3 - 5 = -2$
When $x = 2$, $y = 3^2 - 5 = 9 - 5 = 4$
When $x = -1$, $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$
When $x = -2$, $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

b as $x \to \infty$,
$3^x \to \infty$, $\therefore$ $y \to \infty$
as $x \to -\infty$,
$3^x \to 0$, $\therefore$ $y \to -5$
(from above)

c



d $y = -5$ is the horizontal asymptote

9 **a**
$$\begin{aligned} 27^x &= 3 \\ \therefore\ (3^3)^x &= 3^1 \\ \therefore\ 3x &= 1 \\ \therefore\ x &= \tfrac{1}{3} \end{aligned}$$

b
$$\begin{aligned} 9^{1-x} &= 27^{x+2} \\ \therefore\ (3^2)^{1-x} &= (3^3)^{x+2} \\ \therefore\ 2-2x &= 3x+6 \\ \therefore\ -5x &= 4 \quad \text{and so } x = -\tfrac{4}{5} \end{aligned}$$

10
$$\begin{aligned} 4^x \times 2^y &= 16 \\ \therefore\ (2^2)^x \times 2^y &= 2^4 \\ \therefore\ 2x+y &= 4 \quad \text{..... (1)} \end{aligned}$$

$$\begin{aligned} 8^x &= 2^{\frac{y}{2}} \\ \therefore\ (2^3)^x &= 2^{\frac{y}{2}} \\ \therefore\ 3x &= \frac{y}{2} \quad \text{..... (2)} \end{aligned}$$

From (2), $y = 6x$
$\therefore$ in (1), $2x + 6x = 4$
$\therefore$ $8x = 4$
$\therefore$ $x = \frac{1}{2}$
and so $y = 6 \times \frac{1}{2} = 3$

REVIEW SET 3C

1 **a** $\frac{1}{4}$
$= \dfrac{1}{2^2}$
$= 2^{-2}$

b $32 = 2^5$

c $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

d $\sqrt{8}$
$= (2^3)^{\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

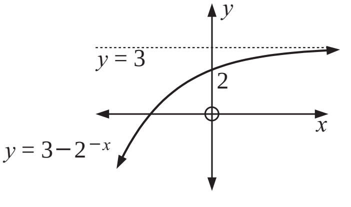
2 **a**
$$\begin{aligned}&4^a \times 8^b\\ &= (2^2)^a \times (2^3)^b\\ &= 2^{2a} \times 2^{3b}\\ &= 2^{2a+3b}\end{aligned}$$

b
$$\begin{aligned}&\frac{2^{1-x}}{4^{x+2}}\\ &= \frac{2^{1-x}}{(2^2)^{x+2}}\\ &= \frac{2^{1-x}}{2^{2x+4}}\\ &= 2^{1-x-(2x+4)}\\ &= 2^{1-x-2x-4}\\ &= 2^{-3x-3}\end{aligned}$$

3 **a**
When $x = 0$, $y = 3 - 2^0 = 3 - 1 = 2$
When $x = 1$, $y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2}$
When $x = 2$, $y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4}$
When $x = -1$, $y = 3 - 2^1 = 3 - 2 = 1$
When $x = -2$, $y = 3 - 2^2 = 3 - 4 = -1$

b as $x \to \infty$,
$2^{-x} \to 0$, $\therefore$ $y \to 3$ (below)
as $x \to -\infty$,
$2^{-x} \to \infty$, $\therefore$ $y \to -\infty$

c



d HA is $y = 3$

4 **a**
$$\begin{aligned}8^x &= \frac{1}{\sqrt{2}}\\ \therefore\ (2^3)^x &= \frac{1}{2^{\frac{1}{2}}}\\ \therefore\ 2^{3x} &= 2^{-\frac{1}{2}}\\ \therefore\ 3x &= -\tfrac{1}{2}\\ \therefore\ x &= -\tfrac{1}{6}\end{aligned}$$

b
$$\begin{aligned}4^{2x+1} &= \left(\tfrac{1}{2}\right)^{2-x}\\ \therefore\ (2^2)^{2x+1} &= (2^{-1})^{2-x}\\ \therefore\ 4x+2 &= -2+x\\ \therefore\ 3x &= -4\\ \therefore\ x &= -\tfrac{4}{3}\end{aligned}$$

5 **a**
$$\begin{aligned}&mn^{-2}\\ &= m \times \frac{1}{n^2}\\ &= \frac{m}{n^2}\end{aligned}$$

b
$$\begin{aligned}&(mn)^{-3}\\ &= \frac{1}{(mn)^3}\\ &= \frac{1}{m^3n^3}\end{aligned}$$

c
$$\begin{aligned}&\frac{m^2n^{-1}}{p^{-2}}\\ &= m^2\left(\frac{1}{n}\right)p^2\\ &= \frac{m^2p^2}{n}\end{aligned}$$

d
$$\begin{aligned}&(4m^{-1}n)^2\\ &= 4^2m^{-2}n^2\\ &= \frac{16n^2}{m^2}\end{aligned}$$

6 **a**
$$\begin{aligned}&\sqrt[4]{5}\\ &= 5^{\frac{1}{4}}\end{aligned}$$

b
$$\begin{aligned}&\sqrt{27}\\ &= \left(3^3\right)^{\frac{1}{2}}\\ &= 3^{\frac{3}{2}}\end{aligned}$$

c
$$\begin{aligned}&\frac{1}{\sqrt[3]{16}}\\ &= \frac{1}{(2^4)^{\frac{1}{3}}}\\ &= \frac{1}{2^{\frac{4}{3}}}\\ &= 2^{-\frac{4}{3}}\end{aligned}$$

d
$$\begin{aligned}&\frac{1}{\sqrt{125}}\\ &= \frac{1}{(5^3)^{\frac{1}{2}}}\\ &= \frac{1}{5^{\frac{3}{2}}}\\ &= 5^{-\frac{3}{2}}\end{aligned}$$

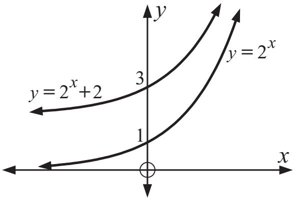
7 **a** $8^{\frac{4}{3}} = \left(2^3\right)^{\frac{4}{3}} = 2^4 = 16$

b $27^{\frac{4}{3}} = \left(3^3\right)^{\frac{4}{3}} = 3^4 = 81$

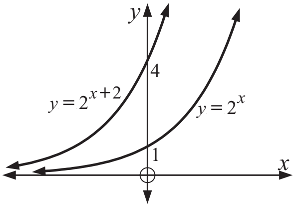
c $32^{-\frac{2}{5}} = \left(2^5\right)^{-\frac{2}{5}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

d $(25)^{-\frac{3}{2}} = \left(5^2\right)^{-\frac{3}{2}} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

8 **a**



b



9 $y = 2^{-x} - 5$

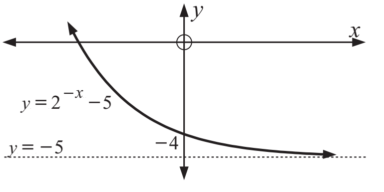
When $x = 0$, $y = 2^0 - 5 = 1 - 5 = -4$

$\therefore$ y-intercept is -4

as $x \to \infty$, $2^{-x} \to 0$ $\therefore$ $y \to -5$

So, $y = -5$ is a HA

as $x \to -\infty$, $2^{-x} \to \infty$ $\therefore$ $y \to \infty$



10 $3^x \times 3^y = 9$

$\therefore$ $3^{x+y} = 3^2$

$\therefore$ $x + y = 2$ ..... (1)

$9^x = 3^{y-2}$

$\therefore$ $(3^2)^x = 3^{y-2}$

$\therefore$ $2x = y - 2$ ..... (2)

From (1), $y = 2 - x$

$\therefore$ $2x = 2 - x - 2$

$\therefore$ $3x = 0$

$\therefore$ $x = 0$

So, $x = 0$, $y = 2$

REVIEW SET 3D

1 **a** $4 \times 2^n = 2^2 \times 2^n = 2^{2+n}$

b $7^{-1} - 7^0 = \frac{1}{7} - 1 = -\frac{6}{7}$

c $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8} = 3\frac{3}{8}$

d $\left(\dfrac{2a^{-1}}{b^2}\right)^2 = \dfrac{2^2a^{-2}}{b^4} = \dfrac{4}{a^2b^4}$

2 **a**

$$\begin{array}{r|r} 2 & 288 \\ \hline 2 & 144 \\ \hline 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

$\therefore$ $288 = 2^5 \times 3^2$

b $\dfrac{2^{x+1}}{2^{1-x}} = 2^{x+1-(1-x)} = 2^{x+1-1+x} = 2^{2x}$

3 **a** $1 = 5^0$

b $5\sqrt{5} = 5^1 \times 5^{\frac{1}{2}} = 5^{1\frac{1}{2}} = 5^{\frac{3}{2}}$

c $\dfrac{1}{\sqrt[4]{5}} = \dfrac{1}{5^{\frac{1}{4}}} = 5^{-\frac{1}{4}}$

d $25^{a+3} = (5^2)^{a+3} = 5^{2a+6}$

4 **a** $-(-2)^2 = -[4] = -4$

b $\left(-\frac{1}{2}a^{-3}\right)^2 = \left(-\frac{1}{2}\right)^2 a^{-6} = \frac{1}{4}a^{-6} = \frac{1}{4} \times \frac{1}{a^6} = \frac{1}{4a^6}$

c $(-3b^{-1})^{-3} = (-3)^{-3}b^3 = \dfrac{1}{(-3)^3}\, b^3 = \dfrac{1}{-27}\, b^3 = -\dfrac{b^3}{27}$

5 **a** $e^x(e^{-x} + e^x) = e^0 + e^{2x} = 1 + e^{2x}$

b $(2^x + 5)^2 = (2^x)^2 + 2 \times 2^x \times 5 + 5^2 = 2^{2x} + 5 \times 2^{x+1} + 25 = 4^x + 5 \times 2^{x+1} + 25$ (or $2^{2x} + 10 \times 2^x + 25$)

c $\left(x^{\frac{1}{2}} - 7\right)\left(x^{\frac{1}{2}} + 7\right) = \left(x^{\frac{1}{2}}\right)^2 - 7^2 = x^1 - 49 = x - 49$

6 **a** $(3 - 2^a)^2 = 3^2 - 2 \times 3 \times 2^a + (2^a)^2 = 9 - 3 \times 2^{a+1} + 2^{2a}$ (or $9 - 6 \times 2^a + 2^{2a}$)

b $(\sqrt{x} + 2)(\sqrt{x} - 2) = (\sqrt{x})^2 - 2^2 = x - 4$

c $2^{-x}(2^{2x} + 2^x) = 2^{-x+2x} + 2^{-x+x} = 2^x + 2^0 = 2^x + 1$

7 **a** $6 \times 2^x = 192$
$\therefore\ 2^x = 32$
$\therefore\ 2^x = 2^5$
$\therefore\ x = 5$

b $4 \times \left(\frac{1}{3}\right)^x = 324$
$\therefore\ \left(\frac{1}{3}\right)^x = 81$
$\therefore\ (3^{-1})^x = 3^4$
$\therefore\ 3^{-x} = 3^4$
$\therefore\ x = -4$

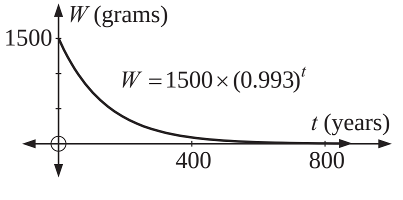
8 $W = 1500 \times (0.993)^t$ grams

a When $t = 0$,
$W = 1500 \times (0.993)^0 = 1500 \times 1 = 1500$ grams

b **i** When $t = 400$,
$W = 1500 \times (0.993)^{400} \doteqdot 90.3$ grams

ii When $t = 800$,
$W = 1500 \times (0.993)^{800} \doteqdot 5.4$ grams

c



d When $W = 100$,
$1500 \times (0.993)^t = 100$
$\therefore\ (0.993)^t \doteqdot 0.0667$

By trial and error
$(0.993)^{300} \doteqdot 0.1216$
$(0.993)^{350} \doteqdot 0.085\,56$
$(0.993)^{380} \doteqdot 0.069\,30$
$(0.993)^{385} \doteqdot 0.066\,91$
$(0.993)^{386} \doteqdot 0.066\,44$

i.e., about 386 years

Chapter 4
LOGARITHMS

EXERCISE 4A

1 **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$ **d** $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$
f $3^{1.5} = \sqrt{27}$

2 **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$ **c** $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$
f $\log_3(\frac{1}{27}) = -3$

3 **a** As $10^5 = 100\,000$ then $\log_{10}(100\,000) = 5$

b As $10^{-2} = 0.01$ then $\log_{10}(0.01) = -2$

c As $3^{\frac{1}{2}} = \sqrt{3}$ then $\log_3(\sqrt{3}) = \frac{1}{2}$

d As $2^3 = 8$ then $\log_2 8 = 3$

e As $2^6 = 64$ then $\log_2 64 = 6$

f As $2^7 = 128$ then $\log_2 128 = 7$

g As $5^2 = 25$ then $\log_5 25 = 2$

h As $5^3 = 125$ then $\log_5 125 = 3$

i As $2^{-3} = \frac{1}{8} = 0.125$ then $\log_2(0.125) = -3$

j As $9^{\frac{1}{2}} = 3$ then $\log_9 3 = \frac{1}{2}$

k As $4^2 = 16$ then $\log_4 16 = 2$

l As $36^{\frac{1}{2}} = \sqrt{36} = 6$ then $\log_{36} 6 = \frac{1}{2}$

m As $243 = 3^5$ then $\log_3 243 = 5$

n As $\sqrt[3]{2} = 2^{\frac{1}{3}}$ then $\log_2 \sqrt[3]{2} = \frac{1}{3}$

o As $a^n = a^n$ then $\log_n a^n = n$

p As $2 = 8^{\frac{1}{3}}$ then $\log_8 2 = \frac{1}{3}$

q As $\dfrac{1}{t} = t^{-1}$ then $\log_t\left(\dfrac{1}{t}\right) = -1$

r As $6\sqrt{6} = 6^1 \times 6^{\frac{1}{2}} = 6^{1\frac{1}{2}}$ then $\log_6(6\sqrt{6}) = 1\frac{1}{2}$

s As $1 = 4^0$ then $\log_4 1 = 0$

t As $9 = 9^1$ then $\log_9 9 = 1$

4 **a** $\doteqdot 2.18$ **b** $\doteqdot 1.40$ **c** $\doteqdot 1.87$ **d** $\doteqdot -0.0969$

5 **a**
$$\begin{aligned} \log_2 x &= 3 \\ \therefore\ x &= 2^3 \\ \therefore\ x &= 8 \end{aligned}$$

b
$$\begin{aligned} \log_4 x &= \tfrac{1}{2} \\ \therefore\ x &= 4^{\frac{1}{2}} \\ \therefore\ x &= 2 \end{aligned}$$

c
$$\begin{aligned} \log_x 81 &= 4 \\ \therefore\ 81 &= x^4 \\ \therefore\ x &= \pm\sqrt[4]{81} \\ \therefore\ x &= \pm 3 \\ \text{but}\ x &> 0 \\ \therefore\ x &= 3 \end{aligned}$$

d
$$\begin{aligned} \log_2(x-6) &= 3 \\ \therefore\ x-6 &= 2^3 \\ \therefore\ x-6 &= 8 \\ \therefore\ x &= 14 \end{aligned}$$

6 **a**
$$\begin{aligned} &\log_2 4 \\ &= \log_2 2^2 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} &\log_3(\tfrac{1}{3}) \\ &= \log_3 3^{-1} \\ &= -1 \end{aligned}$$

c
$$\begin{aligned} &\log_{10}(0.001) \\ &= \log_{10} 10^{-3} \\ &= -3 \end{aligned}$$

d
$$\begin{aligned} &\log_3\left(\tfrac{1}{\sqrt{3}}\right) \\ &= \log_3\left(\frac{1}{3^{\frac{1}{2}}}\right) \\ &= \log_3 3^{-\frac{1}{2}} \\ &= -\tfrac{1}{2} \end{aligned}$$

e $\log_{10}\sqrt[3]{100}$
$= \log_{10}\left(10^2\right)^{\frac{1}{3}}$
$= \log_{10} 10^{\frac{2}{3}}$
$= \frac{2}{3}$

f $\log_2\left(2\sqrt{2}\right)$
$= \log_2\left(2^1 2^{\frac{1}{2}}\right)$
$= \log_2 2^{\frac{3}{2}}$
$= \frac{3}{2}$

g $\log_5\left(25\sqrt{5}\right)$
$= \log_5\left(5^2 5^{\frac{1}{2}}\right)$
$= \log_5 5^{\frac{5}{2}}$
$= \frac{5}{2}$

h $\log_2\left(\frac{1}{\sqrt[3]{2}}\right)$
$= \log_2 2^{-\frac{1}{3}}$
$= -\frac{1}{3}$

EXERCISE 4B

1 **a** $\log 10\,000$
$= \log_{10} 10^4$
$= 4$

b $\log 0.001$
$= \log_{10} 10^{-3}$
$= -3$

c $\log 10$
$= \log_{10} 10^1$
$= 1$

d $\log 1$
$= \log_{10} 10^0$
$= 0$

e $\log\sqrt{10}$
$= \log_{10} 10^{\frac{1}{2}}$
$= \frac{1}{2}$

f $\log\sqrt[3]{10}$
$= \log_{10} 10^{\frac{1}{3}}$
$= \frac{1}{3}$

g $\log\left(\frac{1}{\sqrt[4]{10}}\right)$
$= \log_{10} 10^{-\frac{1}{4}}$
$= -\frac{1}{4}$

h $\log 10\sqrt{10}$
$= \log_{10} 10^1 10^{\frac{1}{2}}$
$= \log_{10} 10^{\frac{3}{2}}$
$= \frac{3}{2}$

i $\log\sqrt[3]{100}$
$= \log_{10}\left(10^2\right)^{\frac{1}{3}}$
$= \log_{10} 10^{\frac{2}{3}}$
$= \frac{2}{3}$

j $\log\left(\frac{100}{\sqrt{10}}\right)$
$= \log_{10}\left(\frac{10^2}{10^{\frac{1}{2}}}\right)$
$= \log_{10} 10^{\frac{3}{2}}$
$= \frac{3}{2}$

k $\log\left(10\times\sqrt[3]{10}\right)$
$= \log_{10}\left(10^1\times 10^{\frac{1}{3}}\right)$
$= \log_{10} 10^{\frac{4}{3}}$
$= \frac{4}{3}$

l $\log 1000\sqrt{10}$
$= \log_{10}\left(10^3\times 10^{\frac{1}{2}}\right)$
$= \log_{10} 10^{\frac{7}{2}}$
$= \frac{7}{2}$

m $\log 10^n$
$= \log_{10} 10^n$
$= n$

n $\log\left(10^a\times 100\right)$
$= \log_{10}\left(10^a\times 10^2\right)$
$= \log_{10}\left(10^{a+2}\right)$
$= a+2$

o $\log\left(\frac{10}{10^m}\right)$
$= \log_{10}\left(10^{1-m}\right)$
$= 1-m$

p $\log\left(\frac{10^a}{10^b}\right)$
$= \log_{10}\left(10^{a-b}\right)$
$= a-b$

2 Use:
a [log] 10 000 [ENTER]
b [log] 0.001 [ENTER]
c [log] [2nd] [√] 10 [)] [)] [ENTER]
d [log] 10 [∧] [(] 1 [÷] 3 [)] [)] [ENTER]
e [log] 100 [∧] [(] 1 [÷] 3 [)] [)] [ENTER]
f [log] 10 [×] [2nd] [√] 10 [)] [)] [ENTER]
g [log] 1 [÷] [2nd] [√] 10 [)] [)] [ENTER]
h [log] 1 [÷] 10 [∧] 0.25 [)] [ENTER]

3 **a** 6
$= 10^{\log 6}$
$\doteq 10^{0.7782}$

b 60
$= 10^{\log 60}$
$\doteq 10^{1.7782}$

c 6000
$= 10^{\log 6000}$
$\doteq 10^{3.7782}$

d 0.6
$= 10^{\log(0.6)}$
$= 10^{-0.2218}$

e 0.006
$= 10^{\log(0.006)}$
$= 10^{-2.2218}$

f 15
$= 10^{\log 15}$
$\doteq 10^{1.1761}$

g 1500
$= 10^{\log 1500}$
$= 10^{3.1761}$

h 1.5
$= 10^{\log 1.5}$
$= 10^{0.1761}$

i 0.15
$= 10^{\log(0.15)}$
$= 10^{-0.8239}$

j $0.000\,15$
$= 10^{\log(0.000\,15)}$
$= 10^{-3.8239}$

4 **a** **i** $\log 3 \doteq 0.477$ **ii** $\log 300 \doteq 2.477$

b
$$\begin{aligned} 300 &= 3 \times 10^2 \\ &= 10^{\log 3} \times 10^2 \\ &= 10^{\log 3 + 2} \\ \therefore \log 300 &= \log 3 + 2 \end{aligned}$$

5 **a** **i** $\log 5 \doteq 0.699$ **ii** $\log 0.05 = -1.301$

b
$$\begin{aligned} 0.05 &= 5 \times 10^{-2} \\ &= 10^{\log 5} \times 10^{-2} \\ &= 10^{\log 5 - 2} \\ \therefore \log(0.05) &= \log 5 - 2 \end{aligned}$$

6 **a** $\log x = 2$, $\therefore x = 10^2$, $\therefore x = 100$

b $\log x = 1$, $\therefore x = 10^1$, $\therefore x = 10$

c $\log x = 0$, $\therefore x = 10^o$, $\therefore x = 1$

d $\log x = -1$, $\therefore x = 10^{-1}$, $\therefore x = \frac{1}{10}$

e $\log x = \frac{1}{2}$, $\therefore x = 10^{\frac{1}{2}}$, $\therefore x = \sqrt{10}$

f $\log x = -\frac{1}{2}$, $\therefore x = 10^{-\frac{1}{2}}$, $\therefore x = \dfrac{1}{10^{\frac{1}{2}}}$, $\therefore x = \frac{1}{\sqrt{10}}$

g $\log x = 4$, $\therefore x = 10^4$, $\therefore x = 10\,000$

h $\log x = -5$, $\therefore x = 10^{-5}$, $\therefore x = 0.000\,01$

i $\log x \doteq 0.8351$, $\therefore x \doteq 10^{0.8351}$, $\therefore x \doteq 6.84$

j $\log x \doteq 2.1457$, $\therefore x \doteq 10^{2.1457}$, $\therefore x \doteq 140$

k $\log x \doteq -1.378$, $\therefore x \doteq 10^{-1.378}$, $\therefore x \doteq 0.0419$

l $\log x \doteq -3.1997$, $\therefore x \doteq 10^{-3.1997}$, $\therefore x \doteq 0.000\,631$

EXERCISE 4C

1 **a** $\log 8 + \log 2 = \log(8 \times 2) = \log 16$

b $\log 8 - \log 2 = \log(\frac{8}{2}) = \log 4$

c $\log 40 - \log 5 = \log(\frac{40}{5}) = \log 8$

d $\log 4 + \log 5 = \log(4 \times 5) = \log 20$

e $\log 5 + \log(0.4) = \log(5 \times 0.4) = \log 2$

f $\log 2 + \log 3 + \log 4 = \log(2 \times 3 \times 4) = \log 24$

g $1 + \log 3 = \log 10^1 + \log 3 = \log(10 \times 3) = \log 30$

h $\log 4 - 1 = \log 4 - \log 10^1 = \log(\frac{4}{10}) = \log(0.4)$

i $\log 5 + \log 4 - \log 2 = \log\left(\frac{5 \times 4}{2}\right) = \log 10$

j $2 + \log 2 = \log 10^2 + \log 2 = \log(100 \times 2) = \log 200$

k $\log 40 - 2 = \log 40 - \log 10^2 = \log\left(\frac{40}{100}\right) = \log(0.4)$

l $\log 6 - \log 2 - \log 3 = \log(6 \div 2 \div 3) = \log 1$

m $\log 50 - 4 = \log 50 - \log 10^4 = \log\left(\frac{50}{10^4}\right) = \log(0.005)$

n $3 - \log 50 = \log 10^3 - \log 50 = \log\left(\frac{1000}{50}\right) = \log 20$

o $\log(\frac{4}{3}) + \log 3 + \log 7 = \log\left(\frac{4}{3} \times 3 \times 7\right) = \log 28$

2 **a** $5\log 2 + \log 3 = \log 2^5 + \log 3 = \log(2^5 \times 3) = \log 96$

b $2\log 3 + 3\log 2 = \log 3^2 + \log 2^3 = \log(9 \times 8) = \log 72$

c $3\log 4 - \log 8 = \log 4^3 - \log 8 = \log(\frac{64}{8}) = \log 8$

d $2\log 5 - 3\log 2$
$= \log 5^2 - \log 2^3$
$= \log\left(\frac{25}{8}\right)$

e $\frac{1}{2}\log 4 + \log 3$
$= \log 4^{\frac{1}{2}} + \log 3$
$= \log(2 \times 3)$
$= \log 6$

f $\frac{1}{3}\log\left(\frac{1}{8}\right)$
$= \log\left(\frac{1}{8}\right)^{\frac{1}{3}}$
$= \log\left(2^{-3}\right)^{\frac{1}{3}}$
$= \log 2^{-1}$
$= \log(\frac{1}{2})$ or $-\log 2$

g $3 - \log 2 - 2\log 5$
$= \log 10^3 - \log 2 - \log 5^2$
$= \log(1000 \div 2 \div 25)$
$= \log 20$

h $1 - 3\log 2 + \log 20$
$= \log 10^1 - \log 2^3 + \log 20$
$= \log(10 \div 8 \times 20)$
$= \log 25$

i $2 - \frac{1}{2}\log 4 - \log 5$
$= \log 10^2 - \log 4^{\frac{1}{2}} - \log 5$
$= \log(100 \div 2 \div 5)$
$= \log 10$
$= 1$

3 **a** $\dfrac{\log 4}{\log 2} = \dfrac{\log 2^2}{\log 2} = \dfrac{2\log 2}{\log 2} = 2$

b $\dfrac{\log 27}{\log 9} = \dfrac{\log 3^3}{\log 3^2} = \dfrac{3\log 3}{2\log 3} = \frac{3}{2}$

c $\dfrac{\log 8}{\log 2} = \dfrac{\log 2^3}{\log 2} = \dfrac{3\log 2}{\log 2} = 3$

d $\dfrac{\log 3}{\log 9} = \dfrac{\log 3}{\log 3^2} = \dfrac{\log 3}{2\log 3} = \frac{1}{2}$

e $\dfrac{\log 25}{\log(0.2)} = \dfrac{\log 5^2}{\log 5^{-1}} = \dfrac{2\log 5}{-1\log 5} = -2$

f $\dfrac{\log 8}{\log(0.25)} = \dfrac{\log 2^3}{\log 2^{-2}}$ $\{0.25 = \frac{1}{4} = \frac{1}{2^2}\}$ $= \dfrac{3\log 2}{-2\log 2} = -\frac{3}{2}$

4 **a** $\log 9 = \log 3^2$
$= 2\log 3$

b $\log\sqrt{2} = \log 2^{\frac{1}{2}}$
$= \frac{1}{2}\log 2$

c $\log\left(\frac{1}{8}\right) = \log\left(\dfrac{1}{2^3}\right)$
$= \log 2^{-3}$
$= -3\log 2$

d $\log\left(\frac{1}{5}\right) = \log 5^{-1}$
$= -1\log 5$
$= -\log 5$

e $\log 5 = \log\left(\frac{10}{2}\right)$
$= \log 10^1 - \log 2$
$= 1 - \log 2$

f $\log 5000 = \log\left(\frac{10\,000}{2}\right)$
$= \log 10^4 - \log 2$
$= 4 - \log 2$

5 **a** $y = 2^x$
$\therefore \log y = \log 2^x$
$\therefore \log y = x\log 2$

b $y = 20b^3$
$\therefore \log y = \log(20b^3)$
$\therefore \log y = \log 20 + \log b^3$
$\therefore \log y = \log 20 + 3\log b$

c $M = ad^4$
$\therefore \log M = \log(ad^4)$
$\therefore \log M = \log a + \log d^4$
$\therefore \log M = \log a + 4\log d$

d $T = 5\sqrt{d} = 5d^{\frac{1}{2}}$
$\therefore \log T = \log(5d^{\frac{1}{2}})$
$\therefore \log T = \log 5 + \log d^{\frac{1}{2}}$
$\therefore \log T = \log 5 + \frac{1}{2}\log d$

e
$$\begin{aligned} R &= b\sqrt{l} = bl^{\frac{1}{2}} \\ \therefore\ \log R &= \log(bl^{\frac{1}{2}}) \\ \therefore\ \log R &= \log b + \log l^{\frac{1}{2}} \\ \therefore\ \log R &= \log b + \tfrac{1}{2}\log l \end{aligned}$$

f
$$\begin{aligned} Q &= \frac{a}{b^n} \\ \therefore\ \log Q &= \log\left(\frac{a}{b^n}\right) \\ \therefore\ \log Q &= \log a - \log b^n \\ \therefore\ \log Q &= \log a - n\log b \end{aligned}$$

g
$$\begin{aligned} y &= ab^x \\ \therefore\ \log y &= \log(ab^x) \\ \therefore\ \log y &= \log a + \log b^x \\ \therefore\ \log y &= \log a + x\log b \end{aligned}$$

h
$$\begin{aligned} F &= \frac{20}{\sqrt{n}} = \frac{20}{n^{\frac{1}{2}}} \\ \therefore\ \log F &= \log\left(\frac{20}{n^{\frac{1}{2}}}\right) \\ \therefore\ \log F &= \log 20 - \log n^{\frac{1}{2}} \\ \therefore\ \log F &= \log 20 - \tfrac{1}{2}\log n \end{aligned}$$

i
$$\begin{aligned} L &= \frac{ab}{c} \\ \therefore\ \log L &= \log\left(\frac{ab}{c}\right) \\ \therefore\ \log L &= \log ab - \log c \\ \therefore\ \log L &= \log a + \log b - \log c \end{aligned}$$

j
$$\begin{aligned} N &= \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore\ \log N &= \log\left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore\ \log N &= \tfrac{1}{2}\log\left(\frac{a}{b}\right) \\ \therefore\ \log N &= \tfrac{1}{2}\{\log a - \log b\} \end{aligned}$$

k
$$\begin{aligned} S &= 200 \times 2^t \\ \therefore\ \log S &= \log(200 \times 2^t) \\ \therefore\ \log S &= \log 200 + \log 2^t \\ \therefore\ \log S &= \log 200 + t\log 2 \end{aligned}$$

l
$$\begin{aligned} y &= \frac{a^m}{b^n} \\ \therefore\ \log y &= \log\left(\frac{a^m}{b^n}\right) \\ \therefore\ \log y &= \log a^m - \log b^n \\ \therefore\ \log y &= m\log a - n\log b \end{aligned}$$

6 **a**
$$\begin{aligned} \log D &= \log e + \log 2 \\ &= \log(e \times 2) \\ \therefore\ D &= 2e \end{aligned}$$

b
$$\begin{aligned} \log F &= \log 5 - \log t \\ &= \log\left(\frac{5}{t}\right) \\ \therefore\ F &= \frac{5}{t} \end{aligned}$$

c
$$\begin{aligned} \log P &= \tfrac{1}{2}\log x \\ &= \log x^{\frac{1}{2}} \\ \therefore\ P &= \sqrt{x} \end{aligned}$$

d
$$\begin{aligned} \log M &= 2\log b + \log c \\ &= \log b^2 + \log c \\ &= \log(b^2c) \\ \therefore\ M &= b^2c \end{aligned}$$

e
$$\begin{aligned} \log B &= 3\log m - 2\log n \\ &= \log m^3 - \log n^2 \\ &= \log\left(\frac{m^3}{n^2}\right) \\ \therefore\ B &= \frac{m^3}{n^2} \end{aligned}$$

f
$$\begin{aligned} \log N &= -\tfrac{1}{3}\log p \\ &= \log p^{-\frac{1}{3}} \\ &= \log\left(\frac{1}{\sqrt[3]{p}}\right) \\ \therefore\ N &= \frac{1}{\sqrt[3]{p}} \end{aligned}$$

g
$$\begin{aligned} \log P &= 3\log x + 1 \\ &= \log x^3 + \log 10^1 \\ &= \log(10x^3) \\ \therefore\ P &= 10x^3 \end{aligned}$$

h
$$\begin{aligned} \log Q &= 2 - \log x \\ &= \log 10^2 - \log x \\ &= \log\left(\frac{100}{x}\right) \quad \therefore\ Q = \frac{100}{x} \end{aligned}$$

7 **a**
$$\begin{aligned} &\log_b 6 \\ &= \log_b(2 \times 3) \\ &= \log_b 2 + \log_b 3 \\ &= p + q \end{aligned}$$

b
$$\begin{aligned} &\log_b 108 \\ &= \log_b(2^2 3^3) \\ &= 2\log_b 2 + 3\log_b 3 \\ &= 2p + 3q \end{aligned}$$

c
$$\begin{aligned} &\log_b 45 \\ &= \log_b(3^2 5) \\ &= 2\log_b 3 + \log_b 5 \\ &= 2q + r \end{aligned}$$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$
$= \log_b(5 \times 3^{\frac{1}{2}}) - \log_b 2$
$= \log_b 5 + \frac{1}{2}\log_b 3 - \log_b 2$
$= r + \frac{1}{2}q - p$

e $\log_b(\frac{5}{32})$
$= \log_b 5 - \log_b 2^5$
$= \log_b 5 - 5\log_b 2$
$= r - 5p$

f $\log_b(0.\bar{2})$
$= \log_b(\frac{2}{9})$
$= \log_b 2 - \log_b 3^2$
$= p - 2q$

8 **a** $\log_2(PR)$
$= \log_2 P + \log_2 R$
$= x + z$

b $\log_2(RQ^2)$
$= \log_2 R + \log_2 Q^2$
$= \log_2 R + 2\log_2 Q$
$= z + 2y$

c $\log_2\left(\frac{PR}{Q}\right)$
$= \log_2(PR) - \log_2 Q$
$= \log_2 P + \log_2 R - \log_2 Q$
$= x + z - y$

d $\log_2\left(P^2\sqrt{Q}\right)$
$= \log_2 P^2 + \log_2 Q^{\frac{1}{2}}$
$= 2\log_2 P + \frac{1}{2}\log_2 Q$
$= 2x + \frac{1}{2}y$

e $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$
$= \log_2 Q^3 - \log_2 R^{\frac{1}{2}}$
$= 3\log_2 Q - \frac{1}{2}\log_2 R$
$= 3y - \frac{1}{2}z$

f $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$
$= \log_2 R^2 + \log_2 Q^{\frac{1}{2}} - \log_2 P^3$
$= 2\log_2 R + \frac{1}{2}\log_2 Q - 3\log_2 P$
$= 2z + \frac{1}{2}y - 3x$

9 **a**
$$\begin{aligned} \log_t N^2 &= 1.72 \\ \therefore\ 2\log_t N &= 1.72 \\ \therefore\ \log_t N &= 1.72 \div 2 \\ &= 0.86 \end{aligned}$$

b $\log_t(MN)$
$= \log_t M + \log_t N$
$= 1.29 + 0.86$
$= 2.15$

c $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$
$= \log_t N^2 - \log_t M^{\frac{1}{2}}$
$= 2\log_t N - \frac{1}{2}\log_t M$
$= 2(0.86) - \frac{1}{2}(1.29)$
$= 1.075$

10 **a**
$$\begin{aligned} \log_3 27 + \log_3(\tfrac{1}{3}) &= \log_3 x \\ \therefore\ \log_3(27 \times \tfrac{1}{3}) &= \log_3 x \\ \therefore\ \log_3 9 &= \log_3 x \\ \therefore\ x &= 9 \end{aligned}$$

b
$$\begin{aligned} \log_5 x &= \log_5 8 - \log_5(6-x) \\ \therefore\ \log_5 x &= \log_5\left(\frac{8}{6-x}\right) \\ \therefore\ x &= \frac{8}{6-x} \\ \therefore\ 6x - x^2 &= 8 \\ \therefore\ x^2 - 6x + 8 &= 0 \\ \therefore\ (x-2)(x-4) &= 0 \\ \therefore\ x &= 2 \text{ or } 4 \end{aligned}$$

Note: $x > 0$
and $6 - x > 0$
i.e., $0 < x < 6$

c
$$\begin{aligned} \log_5 125 - \log_5\sqrt{5} &= \log_5 x \\ \therefore\ \log_5\left(\frac{125}{\sqrt{5}}\right) &= \log_5 x \\ \therefore\ x &= \tfrac{125}{\sqrt{5}} \text{ or } 25\sqrt{5} \end{aligned}$$

d
$$\begin{aligned} \log_{20} x &= 1 + \log_{20} 10 \\ \therefore\ \log_{20} x &= \log_{20} 20^1 + \log_{20} 10 \\ &= \log_{20} 200 \\ \therefore\ x &= 200 \end{aligned}$$

e
$$\begin{aligned} \log x + \log(x+1) &= \log 30 \\ \therefore\ \log[x(x+1)] &= \log 30 \\ \therefore\ x^2 + x &= 30 \\ \therefore\ x^2 + x - 30 &= 0 \\ \therefore\ (x+6)(x-5) &= 0 \\ \therefore\ x &= -6 \text{ or } 5 \end{aligned}$$

but $x > 0$ for $\log x$ to exist
$\therefore\ x = 5$

f
$$\begin{aligned} \log(x+2) - \log(x-2) &= \log 5 \\ \therefore\ \log\left(\frac{x+2}{x-2}\right) &= \log 5 \\ \therefore\ \frac{x+2}{x-2} &= 5 \\ \therefore\ x + 2 &= 5x - 10 \\ \therefore\ -4x &= -12 \\ \therefore\ x &= 3 \end{aligned}$$

Note: $x + 2 > 0$ and $x - 2 > 0$ $\therefore\ x > 2$ ✓

EXERCISE 4D

1 **a**
$$\begin{aligned} 2^x &= 10 \\ \therefore\ \log 2^x &= \log 10 \\ \therefore\ x\log 2 &= \log 10 \\ \therefore\ x &= \frac{\log 10}{\log 2} \doteqdot 3.32 \end{aligned}$$

b
$$\begin{aligned} 3^x &= 20 \\ \therefore\ \log 3^x &= \log 20 \\ \therefore\ x\log 3 &= \log 20 \\ \therefore\ x &= \frac{\log 20}{\log 3} \doteqdot 2.73 \end{aligned}$$

c
$$\begin{aligned} 4^x &= 100 \\ \therefore\ \log 4^x &= \log 100 \\ \therefore\ x\log 4 &= \log 100 \\ \therefore\ x &= \frac{\log 100}{\log 4} \doteqdot 3.32 \end{aligned}$$

d
$$\begin{aligned} (1.2)^x &= 1000 \\ \therefore\ x\log(1.2) &= \log 1000 \\ \therefore\ x &= \frac{\log 1000}{\log(1.2)} \doteqdot 37.9 \end{aligned}$$

e
$$\begin{aligned} 2^x &= 0.08 \\ \therefore\ \log 2^x &= \log(0.08) \\ \therefore\ x\log 2 &= \log(0.08) \\ \therefore\ x &= \frac{\log(0.08)}{\log 2} \doteqdot -3.64 \end{aligned}$$

f
$$\begin{aligned} 3^x &= 0.000\,25 \\ \therefore\ \log 3^x &= \log(0.000\,25) \\ \therefore\ x\log 3 &= \log(0.000\,25) \\ \therefore\ x &= \frac{\log(0.000\,25)}{\log 3} \doteqdot -7.55 \end{aligned}$$

g
$$\begin{aligned} \left(\tfrac{1}{2}\right)^x &= 0.005 \\ \therefore\ \log(0.5)^x &= \log(0.005) \\ \therefore\ x\log(0.5) &= \log(0.005) \\ \therefore\ x &= \frac{\log(0.005)}{\log(0.5)} \doteqdot 7.64 \end{aligned}$$

h
$$\begin{aligned} \left(\tfrac{3}{4}\right)^x &= 10^{-4} \\ \therefore\ \log(0.75)^x &= -4 \\ \therefore\ x\log(0.75) &= -4 \\ \therefore\ x &= \frac{-4}{\log(0.75)} \doteqdot 32.0 \end{aligned}$$

i
$$\begin{aligned} (0.99)^x &= 0.000\,01 \\ \therefore\ \log(0.99)^x &= \log(0.000\,01) \\ \therefore\ x\log(0.99) &= \log(0.000\,01) \\ \therefore\ x &= \frac{\log(0.000\,01)}{\log(0.99)} \\ \therefore\ x &\doteqdot 1146 \text{ or } 1150 \text{ (3 s.f.)} \end{aligned}$$

2 **a**
$$\begin{aligned} 200\times 2^{0.25t} &= 600 \\ \therefore\ 2^{0.25t} &= 3 \\ \therefore\ \log(2^{0.25t}) &= \log 3 \\ \therefore\ 0.25t\log 2 &= \log 3 \\ \therefore\ t &= \frac{\log 3}{0.25\times\log 2} \\ \therefore\ t &\doteqdot 6.34 \end{aligned}$$

b
$$\begin{aligned} 20\times 2^{0.06t} &= 450 \\ \therefore\ 2^{0.06t} &= 22.5 \\ \therefore\ \log(2^{0.06t}) &= \log(22.5) \\ \therefore\ 0.06t\log 2 &= \log(22.5) \\ \therefore\ t &= \frac{\log(22.5)}{0.06\times\log 2} \\ \therefore\ t &\doteqdot 74.9 \end{aligned}$$

c
$$\begin{aligned} 30\times 3^{-0.25t} &= 3 \\ \therefore\ 3^{-0.25t} &= \tfrac{1}{10} \\ \therefore\ \log 3^{-0.25t} &= \log(0.1) \\ \therefore\ -0.25t\log 3 &= \log(0.1) \\ \therefore\ t &= \frac{\log(0.1)}{-0.25\times\log 3} \doteqdot 8.38 \end{aligned}$$

d
$$\begin{aligned} 12\times 2^{-0.05t} &= 0.12 \\ \therefore\ 2^{-0.05t} &= \tfrac{1}{100} \\ \therefore\ \log 2^{-0.05t} &= \log(0.01) \\ \therefore\ -0.05t\log 2 &= \log(0.01) \\ \therefore\ t &= \frac{\log(0.01)}{-0.05\times\log 2} \doteqdot 133 \end{aligned}$$

e
$$\begin{aligned} 50\times 5^{-0.02t} &= 1 \\ \therefore\ 5^{-0.02t} &= \tfrac{1}{50} = 0.02 \\ \therefore\ \log 5^{-0.02t} &= \log(0.02) \\ \therefore\ -0.02t\log 5 &= \log(0.02) \\ \therefore\ t &= \frac{\log(0.02)}{-0.02\times\log 5} \\ \therefore\ t &\doteqdot 122 \end{aligned}$$

f
$$\begin{aligned} 300\times 2^{0.005t} &= 1000 \\ \therefore\ 2^{0.005t} &= \tfrac{10}{3} \\ \therefore\ \log 2^{0.005t} &= \log(\tfrac{10}{3}) \\ \therefore\ 0.005t\log 2 &= \log(\tfrac{10}{3}) \\ \therefore\ t &= \frac{\log(\frac{10}{3})}{0.005\times\log 2} \\ \therefore\ t &\doteqdot 347 \end{aligned}$$

EXERCISE 4E

1 $W_t = 20 \times 2^{0.15t}$ grams

a
$$\begin{aligned} \text{When } W_t &= 30, \\ 20 \times 2^{0.15t} &= 30 \\ \therefore\ 2^{0.15t} &= 1.5 \\ \therefore\ \log 2^{0.15t} &= \log(1.5) \\ \therefore\ 0.15t \log 2 &= \log(1.5) \\ \therefore\ t &= \frac{\log(1.5)}{0.15 \times \log 2} \\ \therefore\ t &\doteqdot 3.90 \text{ hours} \end{aligned}$$

b
$$\begin{aligned} \text{When } W_t &= 100, \\ 20 \times 2^{0.15t} &= 100 \\ \therefore\ 2^{0.15t} &= 5 \\ \therefore\ \log 2^{0.15t} &= \log 5 \\ \therefore\ 0.15t \log 2 &= \log 5 \\ \therefore\ t &= \frac{\log 5}{0.15 \times \log 2} \\ \therefore\ t &\doteqdot 15.5 \text{ hours} \end{aligned}$$

2 $T = 100 \times 2^{-0.03t}$ ^{o}C

a
$$\begin{aligned} \text{When } T &= 25, \\ 100 \times 2^{-0.03t} &= 25 \\ \therefore\ 2^{-0.03t} &= \tfrac{1}{4} = 2^{-2} \\ \therefore\ -0.03t &= -2 \\ \therefore\ t &= \frac{2}{0.03} \\ \therefore\ t &\doteqdot 66.7 \text{ min} \end{aligned}$$

b
$$\begin{aligned} \text{When } T &= 1, \\ 100 \times 2^{-0.03t} &= 1 \\ \therefore\ 2^{-0.03t} &= 0.01 \\ \therefore\ \log 2^{-0.03t} &= \log(0.01) \\ \therefore\ -0.03t \log 2 &= \log(0.01) \\ \therefore\ t &= \frac{\log(0.01)}{-0.03 \times \log 2} \doteqdot 221 \text{ min} \end{aligned}$$

3 $W_t = 1000 \times 2^{-0.04t}$ has $W_0 = 1000 \times 2^0 = 1000$ grams

a For the weight to halve,
$$\begin{aligned} W_t &= 500 \\ \therefore\ 1000 \times 2^{-0.04t} &= 500 \\ \therefore\ 2^{-0.04t} &= \tfrac{1}{2} = 2^{-1} \\ \therefore\ -0.04t &= -1 \\ \therefore\ t &= \frac{1}{0.04} \\ \therefore\ t &= 25 \text{ years} \end{aligned}$$

b
$$\begin{aligned} \text{For } W_t &= 20, \\ 1000 \times 2^{-0.04t} &= 20 \\ \therefore\ 2^{-0.04t} &= 0.02 \\ \therefore\ \log 2^{-0.04t} &= \log(0.02) \\ \therefore\ -0.04t \log 2 &= \log(0.02) \\ \therefore\ t &= \frac{\log(0.02)}{-0.04 \times \log 2} \\ \therefore\ t &\doteqdot 141 \text{ years} \end{aligned}$$

c
$$\begin{aligned} \text{When } W_t &= 1\% \text{ of } 1000 \text{ grams} = 10 \text{ g}, \\ 1000 \times 2^{-0.04t} &= 10 \\ \therefore\ 2^{-0.04t} &= 0.01 \\ \therefore\ \log 2^{-0.04t} &= \log(0.01) \end{aligned}$$

$$\begin{aligned} \text{and so } -0.04 \log 2 &= \log(0.01) \\ \therefore\ t &= \frac{\log(0.01)}{-0.04 \times \log 2} \\ \therefore\ t &\doteqdot 166 \text{ years} \end{aligned}$$

4 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,
$$\begin{aligned} W &= \tfrac{1}{4} \text{ of } W_0 \\ \therefore\ \cancel{W_0} \times 2^{-0.0002t} &= \tfrac{1}{4} \times \cancel{W_0} \\ \therefore\ 2^{-0.0002t} &= 2^{-2} \\ \therefore\ 0.0002t &= 2 \\ \therefore\ t &= \frac{2}{0.0002} \\ \therefore\ t &= 10\,000 \end{aligned}$$

$\therefore$ it would take 10 000 years

b When W is 0.1% of original,
$$\begin{aligned} W &= \tfrac{0.1}{100} \text{ of } W_0 \\ \therefore\ \cancel{W_0} \times 2^{-0.0002t} &= \tfrac{1}{1000} \times \cancel{W_0} \\ \therefore\ \log 2^{-0.0002t} &= \log(0.001) \\ \therefore\ -0.0002t \log 2 &= \log(0.001) \\ \therefore\ t &= \frac{\log(0.001)}{-0.0002 \times \log 2} \\ \therefore\ t &\doteqdot 49\,829 \end{aligned}$$

i.e., it would take about 49 800 years

5 $V = V_0 \times 2^{0.1t}$

When $t = 0$, $V = V_0 \times 2^0 = V_0$

So, we need to find t when

$$\begin{aligned} V &= 3V_0 \\ \text{i.e., } \cancel{V_0} \times 2^{0.1t} &= 3\cancel{V_0} \\ \therefore \log 2^{0.1t} &= \log 3 \\ \therefore 0.1t \log 2 &= \log 3 \\ \therefore t &= \frac{\log 3}{0.1 \times \log 2} \doteqdot 15.8 \end{aligned}$$

$\therefore$ the temperature would be 15.8^oC.

6 $I = I_0 \times 2^{-0.02t}$ amps

When $t = 0$, $I = I_0 \times 2^0 = I_0$ amps

So, we need to find t when

$$\begin{aligned} I &= 10\% \text{ of } I_0 \\ \therefore \cancel{I_0} \times 2^{-0.02t} &= 0.1 \times \cancel{I_0} \\ \therefore \log 2^{-0.02t} &= \log(0.1) \\ \therefore -0.02t \log 2 &= \log(0.1) \\ \therefore t &= \frac{\log(0.1)}{-0.02 \times \log 2} \doteqdot 166 \end{aligned}$$

i.e., it would take 166 seconds.

7 $V = 50(1 - 2^{-0.2t})$

$$\begin{aligned} \text{So when } V = 40, \quad 50(1 - 2^{-0.2t}) &= 40 \\ \therefore 1 - 2^{-0.2t} &= 0.8 \\ \therefore 2^{-0.2t} &= 0.2 \\ \therefore \log 2^{-0.2t} &= \log(0.2) \\ \therefore -0.2t \log 2 &= \log(0.2) \\ \therefore t &= \frac{\log(0.2)}{-0.2 \times \log 2} \doteqdot 11.6 \end{aligned}$$

i.e., it would take 11.6 sec.

EXERCISE 4F

1

$$\begin{aligned} r &= 107.5\%, \\ &= 1.075 \\ u_1 &= 160\,000 \\ u_{n+1} &= 250\,000 \\ u_{n+1} &= u_1 \times r^n \\ \therefore 250\,000 &= 160\,000 \times (1.075)^n \\ \therefore (1.075)^n &= \tfrac{25}{16} \\ \therefore \log(1.075)^n &= \log(\tfrac{25}{16}) \\ \therefore n\log(1.075) &= \log(\tfrac{25}{16}) \\ \therefore n &= \frac{\log(\frac{25}{16})}{\log(1.075)} \doteqdot 6.1709\ldots. \end{aligned}$$

i.e., it would take 6.17 years
($\doteqdot$ 6 years, 62 days)

2

$$\begin{aligned} u_1 &= 10\,000 \\ u_{n+1} &= 15\,000 \\ r &= 104.8\% \\ &= 1.048 \\ u_{n+1} &= u_1 \times r^n \\ \therefore 15\,000 &= 10\,000 \times (1.048)^n \\ \therefore (1.048)^n &= 1.5 \\ \therefore \log(1.048)^n &= \log(1.5) \\ \therefore n\log(1.048) &= \log(1.5) \\ \therefore n &= \frac{\log(1.5)}{\log(1.048)} \\ \therefore n &\doteqdot 8.648\ldots. \end{aligned}$$

i.e., it would take 8.65 years
($\doteqdot$ 8 years, 237 days)

3 **a** 8.4% p.a. compounded monthly

is $\dfrac{8.4\%}{12} = 0.7\%$ a month

$$\begin{aligned} \text{So } T &= 100\% + 0.7\% \\ &= 100.7\% \\ &= 1.007 \end{aligned}$$

b $u_1 = 15\,000$ and $u_{n+1} = 25\,000$

$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore 25\,000 &= 15\,000 \times (1.007)^n \\ \therefore (1.007)^n &= \tfrac{25}{15} = \tfrac{5}{3} \\ \therefore \log(1.007)^n &= \log(\tfrac{5}{3}) \\ \therefore n\log(1.007) &= \log(\tfrac{5}{3}) \\ \therefore n &= \frac{\log(\frac{5}{3})}{\log(1.007)} \doteqdot 73.23\ldots. \end{aligned}$$

i.e., it would take 74 months.

EXERCISE 4G

1 **a**
$$\log_3 12 = \frac{\log 12}{\log 3} \doteqdot 2.26$$

b
$$\log_{\frac{1}{2}} 1250 = \frac{\log 1250}{\log(0.5)} \doteqdot -10.3$$

c
$$\log_3(0.067) = \frac{\log(0.067)}{\log 3} \doteqdot -2.46$$

d
$$\log_{0.4}(0.006\,984) = \frac{\log(0.006\,984)}{\log(0.4)} \doteqdot 5.42$$

2 **a**
$$\begin{aligned} 2^x &= 0.051 \\ \therefore\ x &= \log_2(0.051) \\ \therefore\ x &= \frac{\log(0.051)}{\log 2} \\ \therefore\ x &\doteqdot -4.29 \end{aligned}$$

b
$$\begin{aligned} 4^x &= 213.8 \\ \therefore\ x &= \log_4 213.8 \\ \therefore\ x &= \frac{\log(213.8)}{\log 4} \\ \therefore\ x &\doteqdot 3.87 \end{aligned}$$

c
$$\begin{aligned} 3^{2x+1} &= 4.069 \\ \therefore\ 2x+1 &= \log_3(4.069) \\ \therefore\ 2x+1 &= \frac{\log(4.069)}{\log 3} \\ \therefore\ 2x+1 &\doteqdot 1.2774\ldots. \\ \therefore\ 2x &\doteqdot 0.2774\ldots. \\ \therefore\ x &\doteqdot 0.139 \end{aligned}$$

3 **a**
$$\begin{aligned} 25^x - 3\times 5^x &= 0 \\ \therefore\ 5^{2x} - 3\times 5^x &= 0 \\ \therefore\ 5^x(5^x-3) &= 0 \\ \therefore\ 5^x &= 3 \\ \{\text{as } 5^x > 0 &\text{ for all } x\} \\ \therefore\ x &= \log_5 3 \\ \therefore\ x &= \frac{\log 3}{\log 5} \\ \therefore\ x &\doteqdot 0.683 \end{aligned}$$

b
$$\begin{aligned} 8\times 9^x - 3^x &= 0 \\ \therefore\ 8\times 3^{2x} - 3^x &= 0 \\ \therefore\ 3^x(8\times 3^x - 1) &= 0 \\ \therefore\ 8\times 3^x - 1 &= 0 \\ \{\text{as } 3^x > 0 &\text{ for all } x\} \\ \therefore\ 3^x &= \tfrac{1}{8} \\ \therefore\ x &= \log_3(\tfrac{1}{8}) \\ \therefore\ x &= \frac{\log(\frac{1}{8})}{\log 3} \doteqdot -1.89 \end{aligned}$$

4 **a**
$$\begin{aligned} \log_4 x^3 + \log_2 \sqrt{x} &= 8 \\ \therefore\ \frac{\log x^3}{\log 4} + \frac{\log x^{\frac{1}{2}}}{\log 2} &= 8 \\ \therefore\ \frac{3\log x}{2\log 2} + \frac{\frac{1}{2}\log x}{\log 2} &= 8 \\ \therefore\ \frac{3\log x}{2\log 2} + \frac{\log x}{2\log 2} &= 8 \\ \therefore\ \frac{4\log x}{2\log 2} &= 8 \\ \therefore\ \log x &= 4\log 2 \\ \therefore\ \log x &= \log 2^4 \\ \therefore\ x &= 16 \end{aligned}$$

b
$$\begin{aligned} \log_{16} x^5 &= \log_{64} 125 - \log_4 \sqrt{x} \\ \therefore\ \frac{\log x^5}{\log 16} &= \frac{\log 125}{\log 64} - \frac{\log x^{\frac{1}{2}}}{\log 4} \\ \therefore\ \frac{5\log x}{4\log 2} &= \frac{\log 125}{6\log 2} - \frac{\frac{1}{2}\log x}{2\log 2} \\ \therefore\ \frac{15\log x}{12\log 2} &= \frac{2\log 125}{12\log 2} - \frac{3\log x}{12\log 2} \\ \therefore\ 15\log x &= 2\log 125 - 3\log x \\ \therefore\ 18\log x &= 2\log 125 \\ \therefore\ \log x &= \tfrac{1}{9}\log 125 \\ \therefore\ \log x &= \log 125^{\frac{1}{9}} \\ \therefore\ x &= 125^{\frac{1}{9}} \quad \textit{or} \quad x \doteqdot 1.71 \end{aligned}$$

5
$$\begin{aligned} 4^x \times 5^{4x+3} &= 10^{2x+3} \\ \therefore\ \log(4^x \times 5^{4x+3}) &= \log 10^{2x+3} \\ \therefore\ x\log 4 + (4x+3)\log 5 &= 2x+3 \\ \therefore\ x\log 4 + 4x\log 5 + 3\log 5 &= 2x+3 \\ \therefore\ x[\log 4 + 4\log 5 - 2] &= 3 - 3\log 5 \end{aligned}$$

$$\begin{aligned} \therefore\ x &= \frac{3-3\log 5}{\log 4 + 4\log 5 - 2} \\ \therefore\ x &= \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2} \\ \therefore\ x &= \frac{\log(\frac{1000}{125})}{\log\left(\frac{4\times 5^4}{10^2}\right)} \\ \therefore\ x &= \frac{\log 8}{\log 25} \quad \textit{or} \quad \log_{25} 8 \end{aligned}$$

REVIEW SET 4A

1

a $\log_4 64 = \log_4 4^3 = 3$

b $\log_2 256 = \log_2 2^8 = 8$

c $\log_2(0.25) = \log_2(\frac{1}{4}) = \log_2 2^{-2} = -2$

d $\log_{25} 5 = \log_{25} 25^{\frac{1}{2}} = \frac{1}{2}$

e $\log_8 1 = \log_8 8^0 = 0$

f $\log_6 6 = \log_6 6^1 = 1$

g $\log_{81} 3 = \log_{81} 81^{\frac{1}{4}} = \frac{1}{4}$

h $\log_9(0.\bar{1}) = \log_9(\frac{1}{9}) = \log_9 9^{-1} = -1$

i $\log_{27} 3 = \log_{27} 27^{\frac{1}{3}} = \frac{1}{3}$

j $\log_k \sqrt{k} = \log_k k^{\frac{1}{2}} = \frac{1}{2}$ $(k > 0,\ k \neq 1)$

k $\log_m \sqrt{m^5} = \log_m m^{\frac{5}{2}} = \frac{5}{2}$ $(m > 0,\ m \neq 1)$

l $\log_n \left(\frac{1}{n^2}\right) = \log_n(n^{-2}) = -2$ $(n > 0,\ n \neq 1)$

2

a $\log \sqrt{10} = \log 10^{\frac{1}{2}} = \frac{1}{2}$

b $\log\left(\frac{1}{\sqrt[3]{10}}\right) = \log 10^{-\frac{1}{3}} = -\frac{1}{3}$

c $\log(10^a \times 10^{b+1}) = \log 10^{a+b+1} = a+b+1$

3

a $32 = 10^{\log 32} \doteqdot 10^{1.505}$

b $0.0013 = 10^{\log(0.0013)} \doteqdot 10^{-2.886}$

c $8.963 \times 10^{-5} = 10^{\log(8.963)} \times 10^{-5} \doteqdot 10^{0.952} \times 10^{-5} \doteqdot 10^{-4.048}$

4

a $\log x = -3$
$\therefore\ x = 10^{-3}$
$\therefore\ x = 0.001$

b $\log x \doteqdot 2.743$
$\therefore\ x \doteqdot 10^{2.743}$
$\therefore\ x \doteqdot 553$

c $\log x \doteqdot -3.145$
$\therefore\ x \doteqdot 10^{-3.145}$
$\therefore\ x \doteqdot 0.000\,716$

5

a $\log 4 + 2\log 5 = \log 4 + \log 5^2 = \log(4 \times 25) = \log 100$ (or 2)

b $\log 24 + \log 4 - \log 16 = \log\left(\frac{24 \times 4}{16}\right) = \log 6$

c $2 - \log 25 = \log 10^2 - \log 25 = \log(\frac{100}{25}) = \log 4$

6

a $P = 3 \times b^x$
$\therefore\ \log P = \log(3 \times b^x)$
$\therefore\ \log P = \log 3 + \log b^x$
$\therefore\ \log P = \log 3 + x\log b$

b $m = \frac{n^3}{p^2}$
$\therefore\ \log m = \log\left(\frac{n^3}{p^2}\right)$
$\therefore\ \log m = \log n^3 - \log p^2$
$\therefore\ \log m = 3\log n - 2\log p$

7

a $\log k \doteqdot 1.699 + x$
$\therefore\ k \doteqdot 10^{1.699+x}$
$\therefore\ k \doteqdot 10^{1.699} \times 10^x$
$\therefore\ k \doteqdot 50 \times 10^x$

b $\log Q = 3\log P + \log R = \log(P^3 \times R)$
$\therefore\ Q = P^3 R$

c $\log A \doteqdot 5\log B - 2.602$
$\therefore\ \log A - \log B^5 \doteqdot -2.602$
$\therefore\ \log\left(\frac{A}{B^5}\right) \doteqdot -2.602$
$\therefore\ \frac{A}{B^5} \doteqdot 10^{-2.602} \doteqdot 0.0025$
$\therefore\ A \doteqdot \frac{B^5}{400}$

8 **a**
$$\begin{aligned} 5^x &= 7 \\ \therefore\ \log 5^x &= \log 7 \\ \therefore\ x\log 5 &= \log 7 \\ \therefore\ x &= \frac{\log 7}{\log 5} \\ \therefore\ x &\doteqdot 1.21 \end{aligned}$$

b
$$\begin{aligned} 20\times 2^{2x+1} &= 500 \\ \therefore\ 2^{2x+1} &= 25 \\ \therefore\ \log 2^{2x+1} &= \log 25 \\ \therefore\ (2x+1)\log 2 &= \log 25 \\ \therefore\ 2x+1 &= \frac{\log 25}{\log 2} \doteqdot 4.6438\ldots. \\ \therefore\ 2x &\doteqdot 3.6438\ldots. \\ \therefore\ x &\doteqdot 1.82 \end{aligned}$$

9 $W_t = 2500\times 3^{-\frac{t}{3000}}$ grams

a
$$\begin{aligned} W_0 &= 2500\times 3^0 \\ &= 2500\times 1 \\ &= 2500 \text{ grams} \end{aligned}$$

b We need t when $W_t = 30\%$ of 2500 g

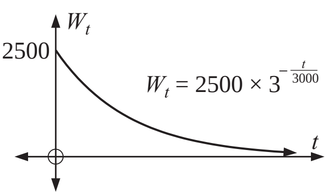
$$\begin{aligned} \text{i.e.,}\quad 2500\times 3^{-\frac{t}{3000}} &= 0.3\times 2500 \\ \therefore\ \log 3^{-\frac{t}{3000}} &= \log(0.3) \\ \therefore\ -\frac{t}{3000}\times\log 3 &= \log(0.3) \\ \therefore\ t &= \frac{-\log(0.3)\times 3000}{\log 3} \\ \therefore\ t &\doteqdot 3287.7\ldots. \end{aligned}$$

i.e., about 3290 years

c % loss
$$\begin{aligned} &= \left(\frac{W_{1500}-W_0}{W_0}\right)\times 100\% \\ &= \left(\frac{2500\times 3^{-\frac{1}{2}}-2500}{2500}\right)\times 100\% \\ &= (3^{-\frac{1}{2}}-1)\times 100\% \\ &\doteqdot -42.3\% \end{aligned}$$

i.e., a loss of 42.3%

d



REVIEW SET 4B

1 **a**
$$\begin{aligned} &\log\sqrt{1000} \\ &= \log\left(10^3\right)^{\frac{1}{2}} \\ &= \log 10^{\frac{3}{2}} \\ &= \tfrac{3}{2} \end{aligned}$$

b
$$\begin{aligned} &\log\left(\frac{10}{\sqrt[3]{10}}\right) \\ &= \log\left(\frac{10^1}{10^{\frac{1}{3}}}\right) \\ &= \log 10^{\frac{2}{3}} \\ &= \tfrac{2}{3} \end{aligned}$$

c
$$\begin{aligned} &\log\left(\frac{10^a}{10^{-b}}\right) \\ &= \log\left(10^{a-(-b)}\right) \\ &= \log 10^{a+b} \\ &= a+b \end{aligned}$$

2 **a**
$$\begin{aligned} &200 \\ &= 10^{\log 200} \\ &\doteqdot 10^{2.30} \end{aligned}$$

b
$$\begin{aligned} &568\,000 \\ &= 10^{\log(568\,000)} \\ &\doteqdot 10^{5.75} \end{aligned}$$

c
$$\begin{aligned} &3.69\times 10^{-4} \\ &= 10^{\log(3.69)}\times 10^{-4} \\ &\doteqdot 10^{0.567}\times 10^{-4} \\ &\doteqdot 10^{0.567-4} \\ &\doteqdot 10^{-3.43} \end{aligned}$$

3 **a**
$$\begin{aligned} \log x &= 3 \\ \therefore\ x &= 10^3 \\ \therefore\ x &= 1000 \end{aligned}$$

b
$$\begin{aligned} \log(x+2) &= 1.732 \\ \therefore\ x+2 &= 10^{1.732} \\ \therefore\ x+2 &\doteqdot 53.951 \\ \therefore\ x &\doteqdot 51.951 \\ \therefore\ x &\doteqdot 52.0 \end{aligned}$$

c
$$\begin{aligned} \log\left(\frac{x}{10}\right) &= -0.671 \\ \therefore\ \frac{x}{10} &= 10^{-0.671} \\ \therefore\ \frac{x}{10} &\doteqdot 0.213\,30 \\ \therefore\ x &\doteqdot 2.13 \end{aligned}$$

4 **a**
$$\begin{aligned}&\log 16 + 2\log 3\\ &= \log 16 + \log 3^2\\ &= \log(16\times 9)\\ &= \log 144\end{aligned}$$

b
$$\begin{aligned}&\log 16 - 2\log 3\\ &= \log 16 - \log 3^2\\ &= \log(\tfrac{16}{9})\end{aligned}$$

c
$$\begin{aligned}&2 + \log 5\\ &= \log 10^2 + \log 5\\ &= \log(100\times 5)\\ &= \log 500\end{aligned}$$

5 **a**
$$\begin{aligned}M &= ab^n\\ \therefore\ \log M &= \log(ab^n)\\ \therefore\ \log M &= \log a + \log b^n\\ \therefore\ \log M &= \log a + n\log b\end{aligned}$$

b
$$\begin{aligned}T &= \frac{5}{\sqrt{l}}\\ \therefore\ \log T &= \log\left(\frac{5}{l^{\frac{1}{2}}}\right)\\ \therefore\ \log T &= \log 5 - \log l^{\frac{1}{2}}\\ \therefore\ \log T &= \log 5 - \tfrac{1}{2}\log l\end{aligned}$$

c
$$\begin{aligned}G &= \frac{a^2b}{c}\\ \therefore\ \log G &= \log\left(\frac{a^2b}{c}\right)\\ \therefore\ \log G &= \log(a^2b) - \log c\\ \therefore\ \log G &= \log a^2 + \log b - \log c\\ \therefore\ \log G &= 2\log a + \log b - \log c\end{aligned}$$

6 **a**
$$\begin{aligned}\log T &= 2\log x - \log y\\ \therefore\ \log T &= \log x^2 - \log y\\ \therefore\ \log T &= \log\left(\frac{x^2}{y}\right)\\ \therefore\ T &= \frac{x^2}{y}\end{aligned}$$

b
$$\begin{aligned}\log K &= \log n + \tfrac{1}{2}\log t\\ \therefore\ \log K &= \log n + \log t^{\frac{1}{2}}\\ \therefore\ \log K &= \log(n\times\sqrt{t})\\ \therefore\ K &= n\sqrt{t}\end{aligned}$$

7 **a**
$$\begin{aligned}3^x &= 300\\ \therefore\ \log 3^x &= \log 300\\ \therefore\ x\log 3 &= \log 300\\ \therefore\ x &= \frac{\log 300}{\log 3}\\ \therefore\ x &\doteqdot 5.19\end{aligned}$$

b
$$\begin{aligned}30\times 5^{1-x} &= 0.15\\ \therefore\ 5^{1-x} &= 0.005\\ \therefore\ \log 5^{1-x} &= \log(0.005)\\ \therefore\ (1-x)\log 5 &= \log(0.005)\\ \therefore\ 1-x &= \frac{\log(0.005)}{\log 5}\\ \therefore\ 1-x &\doteqdot -3.292\\ \therefore\ x &\doteqdot 4.29\end{aligned}$$

c
$$\begin{aligned}3^{x+2} &= 2^{1-x}\\ \therefore\ \log 3^{x+2} &= \log 2^{1-x}\\ \therefore\ (x+2)\log 3 &= (1-x)\log 2\\ \therefore\ x\log 3 + 2\log 3 &= \log 2 - x\log 2\\ \therefore\ x(\log 3 + \log 2) &= \log 2 - 2\log 3\\ \therefore\ x\log 6 &= \log(\tfrac{2}{9})\\ \therefore\ x &= \frac{\log(\frac{2}{9})}{\log 6} \doteqdot -0.839\end{aligned}$$

8 **a**
$$\begin{aligned}&\log 36\\ &= \log(2^2\times 3^2)\\ &= \log 2^2 + \log 3^2\\ &= 2\log 2 + 2\log 3\\ &= 2A + 2B\end{aligned}$$

b
$$\begin{aligned}&\log 54\\ &= \log(2\times 3^3)\\ &= \log 2 + \log 3^3\\ &= \log 2 + 3\log 3\\ &= A + 3B\end{aligned}$$

c
$$\begin{aligned}&\log(8\sqrt{3})\\ &= \log(2^3\times 3^{\frac{1}{2}})\\ &= \log 2^3 + \log 3^{\frac{1}{2}}\\ &= 3\log 2 + \tfrac{1}{2}\log 3\\ &= 3A + \tfrac{1}{2}B\end{aligned}$$

d
$$\begin{aligned}&\log(20.25)\\&=\log(20\tfrac{1}{4})\\&=\log(\tfrac{81}{4})\\&=\log\left(\frac{3^4}{2^2}\right)\\&=\log 3^4-\log 2^2\\&=4\log 3-2\log 2\\&=4B-2A\end{aligned}$$

e
$$\begin{aligned}&\log(0.\bar{8})\\&=\log(\tfrac{8}{9})\\&=\log\left(\frac{2^3}{3^2}\right)\\&=\log 2^3-\log 3^2\\&=3\log 2-2\log 3\\&=3A-2B\end{aligned}$$

9 $P_n=P_0\times 2^{\frac{t}{3}}$

When $n=0$, $P_0=P_0\times 2^0=P_0$. So the initial population was P_0.

a If P_n doubles, $P_n=2P_0$
$$\begin{aligned}\therefore\ \cancel{P_0}2^{\frac{t}{3}}&=2\cancel{P_0}\\\therefore\ 2^{\frac{t}{3}}&=2^1\\\therefore\ \frac{t}{3}&=1\\\therefore\ t&=3\end{aligned}$$

So, it would take 3 years.

b
$$\begin{aligned}\%\text{ increase}&=\left(\frac{P_4-P_0}{P_0}\right)\times 100\%\\&=\left(\frac{P_0\times 2^{\frac{4}{3}}-P_0}{P_0}\right)\times 100\%\\&=(2^{\frac{4}{3}}-1)\times 100\%\\&\doteqdot 151.98\ldots\end{aligned}$$

i.e., the % increase is 152%.

or
$$\begin{aligned}P_4&=P_0\times 2^{\frac{4}{3}}\\&\doteqdot P_0\times 2.52\\&\doteqdot 252\%\text{ of }P_0\end{aligned}$$

i.e., an increase of 152%.

c To increase by 200%,

P_0 becomes $3P_0$
$$\begin{aligned}\therefore\ 3\cancel{P_0}&=\cancel{P_0}\times 2^{\frac{t}{3}}\\\therefore\ 2^{\frac{t}{3}}&=3\\\therefore\ \frac{t}{3}&=\log_2 3\\\therefore\ t&=3\times\frac{\log 3}{\log 2}\doteqdot 4.75\end{aligned}$$

So, it would take 4 years, 9 months.

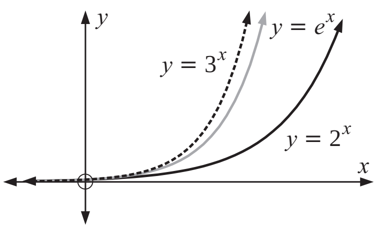
Chapter 5

NATURAL LOGARITHMS

EXERCISE 5A

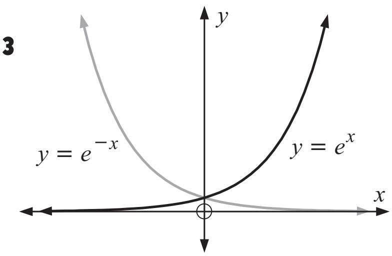
1 $e^1 \doteqdot 2.718\,281\,828\ldots.$

2



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

3



One is the other reflected in the y-axis.

4 When $x = 0$, $y = ae^0 = a \times 1 = a$ $\therefore$ the y-intercept is a

5 **a** The graph of $y = e^x$ is entirely above the x-axis.

So $y > 0$ for all x

i.e., $e^x > 0$ for all x

$\therefore$ $2e^x > 0$ for all x

$\therefore$ y cannot be negative if $y = 2e^x$

b **i** When $x = -20$, $y = 2e^{-20} \doteqdot 4.1 \times 10^{-9}$

ii When $x = 20$, $y = 2e^{20} \doteqdot 9.7 \times 10^{8}$

6 **a** $\doteqdot 7.39$ **b** $\doteqdot 20.1$ **c** $\doteqdot 2.01$ **d** $\doteqdot 1.65$ **e** $\doteqdot 0.368$

7 **a** $\sqrt{e} = e^{\frac{1}{2}}$

b $e\sqrt{e}$
$= e^1 e^{\frac{1}{2}}$
$= e^{\frac{3}{2}}$

c $\dfrac{1}{\sqrt{e}}$
$= \dfrac{1}{e^{\frac{1}{2}}}$
$= e^{-\frac{1}{2}}$

d $\dfrac{1}{e^2} = e^{-2}$

8 **a** $\left(e^{0.36}\right)^{\frac{t}{2}}$
$= e^{0.36 \times \frac{t}{2}}$
$= e^{0.18t}$

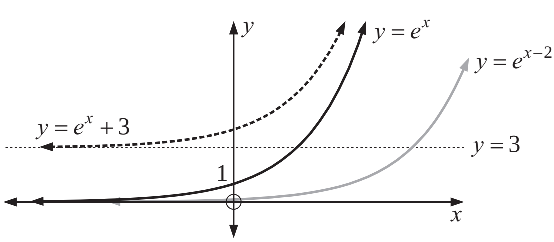
b $\left(e^{0.064}\right)^{\frac{t}{16}}$
$= e^{0.064 \times \frac{t}{16}}$
$= e^{0.004t}$

c $\left(e^{-0.04}\right)^{\frac{t}{8}}$
$= e^{-0.04 \times \frac{t}{8}}$
$= e^{-0.005t}$

d $\left(e^{-0.836}\right)^{\frac{t}{5}}$
$= e^{-0.836 \times \frac{t}{5}}$
$= e^{-0.1672t}$

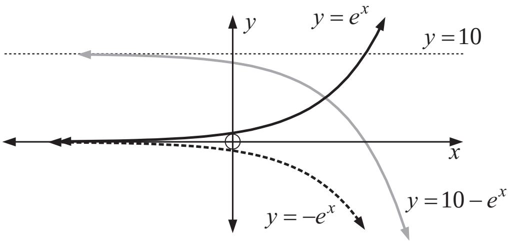
9 **a** 10.074 **b** 0.099 261 **c** 125.09 **d** 0.007 994 5 **e** 41.914 **f** 42.429 **g** 3540.3
h 0.006 342 4

10



Domain of f, g and h is $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$
Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > 3\}$

11



Domain of f, g and h is $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$
Range of g is $\{y : y < 0\}$
Range of h is $\{y : y < 10\}$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

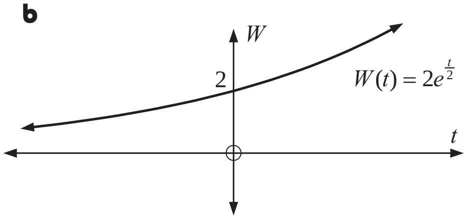
a **i** $W(0) = 2e^0 = 2 \times 1 = 2$ grams

ii $W(\frac{1}{2}) = 2e^{\frac{1}{4}} \doteqdot 2.57$ g

iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}} \doteqdot 4.23$ g

iv $W(6) = 2e^3 \doteqdot 40.2$ g

b

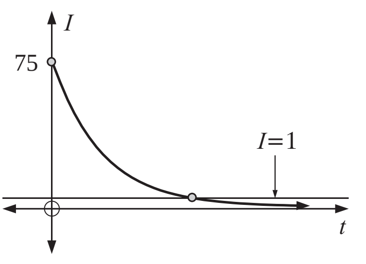


13 $I(t) = 75e^{-0.15t}$

a **i** $I(1) = 75e^{-0.15} = 64.6$ amps

ii $I(10) = 75e^{-1.5} \doteqdot 16.7$ amps

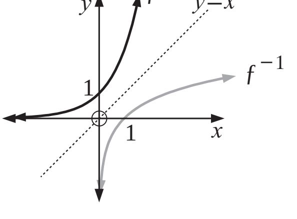
b



c We need to solve $75e^{-0.15t} = 1$.
Using technology, $t \doteqdot 28.8$ sec.

14 **a** $f(x) = e^x$
i.e., $y = e^x$ and has inverse $x = e^y$
$\therefore \; y = \log_e x$
i.e., $f^{-1}(x) = \log_e x$

b



EXERCISE 5B

1 **a** $\ln e^3 = 3 \quad \{\ln e^a = a\}$

b $\ln 1 = \ln e^0 = 0$

c $\ln \sqrt[3]{e} = \ln e^{\frac{1}{3}} = \frac{1}{3}$

d $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$

3 $\ln x$ exists only when $x > 0$. {See the graph of $y = \ln x$.}
So, $\ln(-2)$ and $\ln(0)$ do not exist.

Note: If $\ln(-2) = a$
then $-2 = e^a$
and $e^a = -2$ has no solutions as $e^a > 0$ for all a.

4 **a** 0.693 **b** 2.30 **c** 7.68 **d** −6.39

5
a $\ln e^a = a$

b $\ln(e \times e^a) = \ln e^{1+a} = 1 + a$

c $\ln\left(\dfrac{e^n}{e}\right) = \ln e^{n-1} = n - 1$

d $\ln(e^a \times e^b) = \ln(e^{a+b}) = a + b$

e $\ln(e^a)^b = \ln e^{ab} = ab$

f $\ln\left(\dfrac{e^a}{e^b}\right) = \ln(e^{a-b}) = a - b$

6 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$ **e** $e^{-5.1160}$
f $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$ **i** $e^{-1.8971}$ **j** $e^{-8.8049}$

7
a $\ln x = 3$, $\therefore\ x = e^3$, $\therefore\ x \doteqdot 20.1$

b $\ln x = 1$, $\therefore\ x = e^1$, $\therefore\ x \doteqdot 2.72$

c $\ln x = 0$, $\therefore\ x = e^0$, $\therefore\ x = 1$

d $\ln x = -1$, $\therefore\ x = e^{-1}$, $\therefore\ x \doteqdot 0.368$

e $\ln x = -5$, $\therefore\ x = e^{-5}$, $\therefore\ x \doteqdot 0.006\,74$

f $\ln x \doteqdot 0.835$, $\therefore\ x \doteqdot e^{0.835}$, $\therefore\ x \doteqdot 2.30$

g $\ln x \doteqdot 2.145$, $\therefore\ x \doteqdot e^{2.145}$, $\therefore\ x \doteqdot 8.54$

h $\ln x \doteqdot -3.2971$, $\therefore\ x \doteqdot e^{-3.2971}$, $\therefore\ x \doteqdot 0.0370$

EXERCISE 5C

1
a $\ln 8 + \ln 2 = \ln(8 \times 2) = \ln 16$

b $\ln 8 - \ln 2 = \ln(\frac{8}{2}) = \ln 4$

c $\ln 40 - \ln 5 = \ln(\frac{40}{5}) = \ln 8$

d $\ln 4 + \ln 5 = \ln(4 \times 5) = \ln 20$

e $\ln 5 + \ln(0.4) = \ln(5 \times 0.4) = \ln 2$

f $\ln 2 + \ln 3 + \ln 4 = \ln(2 \times 3 \times 4) = \ln 24$

g $1 + \ln 3 = \ln e^1 + \ln 3 = \ln(e \times 3) = \ln 3e$

h $\ln 4 - 1 = \ln 4 - \ln e^1 = \ln\left(\frac{4}{e}\right)$

i $\ln 5 + \ln 4 - \ln 2 = \ln(5 \times 4) - \ln 2 = \ln(\frac{20}{2}) = \ln 10$

j $2 + \ln 2 = \ln e^2 + \ln 2 = \ln(e^2 \times 2) = \ln(2e^2)$

k $\ln 40 - 2 = \ln 40 - \ln e^2 = \ln\left(\frac{40}{e^2}\right)$

l $\ln 6 - \ln 2 - \ln 3 = \ln(\frac{6}{2}) - \ln 3 = \ln 3 - \ln 3 = \ln(\frac{3}{3}) = \ln 1$

2
a $5\ln 2 + \ln 3 = \ln 2^5 + \ln 3 = \ln(2^5 \times 3) = \ln 96$

b $2\ln 3 + 3\ln 2 = \ln 3^2 + \ln 2^3 = \ln(9 \times 8) = \ln 72$

c $3\ln 4 - \ln 8 = \ln 4^3 - \ln 8 = \ln(\frac{64}{8}) = \ln 8$

d $2\ln 5 - 3\ln 2 = \ln 5^2 - \ln 2^3 = \ln(\frac{25}{8})$

e $\frac{1}{2}\ln 4 + \ln 3 = \ln 4^{\frac{1}{2}} + \ln 3 = \ln 2 + \ln 3 = \ln(2 \times 3) = \ln 6$

f $\frac{1}{3}\ln(\frac{1}{8}) = \frac{1}{3}\ln\left(2^{-3}\right) = \frac{1}{3} \times -3\ln 2 = -\ln 2$ $\left(\text{or } \ln(\frac{1}{2})\right)$

g $-\ln 2 = -1\ln 2 = \ln 2^{-1} = \ln(\frac{1}{2})$

h $-\ln(\frac{1}{3}) = -1\ln(3^{-1}) = \ln(3^{-1})^{-1} = \ln 3$

i $-2\ln(\frac{1}{4}) = \ln(2^{-2})^{-2} = \ln 2^4 = \ln 16$ or $4\ln 2$

3 **a** $\ln 9 = \ln 3^2 = 2\ln 3$

b $\ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2$

c $\ln(\frac{1}{8}) = \ln\left(\frac{1}{2^3}\right) = \ln(2^{-3}) = -3\ln 2$

d $\ln(\frac{1}{5}) = \ln 5^{-1} = -1\ln 5 = -\ln 5$

e $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$

f $\ln\left(\frac{e}{5}\right) = \ln e^1 - \ln 5 = 1 - \ln 5$

g $\ln\sqrt[3]{5} = \ln 5^{\frac{1}{3}} = \frac{1}{3}\ln 5$

h $\ln(\frac{1}{32}) = \ln 2^{-5} = -5\ln 2$

i $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = \ln\left(\frac{1}{2^{\frac{1}{5}}}\right) = \ln 2^{-\frac{1}{5}} = -\frac{1}{5}\ln 2$

4 $\ln\left(\frac{e^2}{8}\right) = \ln e^2 - \ln 8 = 2 - \ln 2^3 = 2 - 3\ln 2$

5 **a**
$$\begin{aligned} \ln D &= \ln x + 1 \\ \therefore\ \ln D - \ln x &= 1 \\ \therefore\ \ln\left(\frac{D}{x}\right) &= 1 \\ \therefore\ \frac{D}{x} &= e^1 \\ \therefore\ D &= ex \end{aligned}$$

b
$$\begin{aligned} \ln F &= -\ln p + 2 \\ \therefore\ \ln F + \ln p &= 2 \\ \therefore\ \ln(Fp) &= 2 \\ \therefore\ Fp &= e^2 \\ \therefore\ F &= \frac{e^2}{p} \end{aligned}$$

c
$$\begin{aligned} \ln P &= \tfrac{1}{2}\ln x \\ \therefore\ \ln P &= \ln x^{\frac{1}{2}} \\ \therefore\ P &= \sqrt{x} \end{aligned}$$

d
$$\begin{aligned} \ln M &= 2\ln y + 3 \\ \therefore\ \ln M - 2\ln y &= 3 \\ \therefore\ \ln\left(\frac{M}{y^2}\right) &= 3 \\ \therefore\ \frac{M}{y^2} &= e^3 \\ \therefore\ M &= e^3y^2 \end{aligned}$$

e
$$\begin{aligned} \ln B &= 3\ln t - 1 \\ \therefore\ \ln B - \ln t^3 &= -1 \\ \therefore\ \ln\left(\frac{B}{t^3}\right) &= -1 \\ \therefore\ \frac{B}{t^3} &= e^{-1} \\ \therefore\ B &= \frac{t^3}{e} \end{aligned}$$

f
$$\begin{aligned} \ln N &= -\tfrac{1}{3}\ln g \\ \therefore\ \ln N &= \ln g^{-\frac{1}{3}} \\ \therefore\ N &= g^{-\frac{1}{3}} \\ \therefore\ N &= \frac{1}{\sqrt[3]{g}} \end{aligned}$$

g
$$\begin{aligned} \ln Q &\doteqdot 3\ln x + 2.159 \\ \therefore\ \ln Q - 3\ln x &\doteqdot 2.159 \\ \therefore\ \ln\left(\frac{Q}{x^3}\right) &\doteqdot 2.159 \\ \therefore\ \frac{Q}{x^3} &\doteqdot e^{2.159} \\ \therefore\ \frac{Q}{x^3} &\doteqdot 8.66 \\ \therefore\ Q &\doteqdot 8.66x^3 \end{aligned}$$

h
$$\begin{aligned} \ln D &\doteqdot 0.4\ln n - 0.6582 \\ \therefore\ \ln D - \ln n^{0.4} &\doteqdot -0.6582 \\ \therefore\ \ln\left(\frac{D}{n^{0.4}}\right) &\doteqdot -0.6582 \\ \therefore\ \frac{D}{n^{0.4}} &\doteqdot e^{-0.6582} \\ \therefore\ \frac{D}{n^{0.4}} &\doteqdot 0.518 \\ \therefore\ D &\doteqdot 0.518n^{0.4} \end{aligned}$$

EXERCISE 5D

1 **a**
$$\begin{aligned} e^x &= 10 \\ \therefore\ x &= \ln 10 \\ \therefore\ x &\doteqdot 2.303 \end{aligned}$$

b
$$\begin{aligned} e^x &= 1000 \\ \therefore\ x &= \ln 1000 \\ \therefore\ x &\doteqdot 6.908 \end{aligned}$$

c
$$\begin{aligned} e^x &= 0.008\,62 \\ \therefore\ x &= \ln(0.008\,62) \\ \therefore\ x &\doteqdot -4.754 \end{aligned}$$

d
$$\begin{aligned} e^{\frac{x}{2}} &= 5 \\ \therefore\ \frac{x}{2} &= \ln 5 \\ \therefore\ x &= 2\ln 5 \\ \therefore\ x &\doteqdot 3.219 \end{aligned}$$

e
$$\begin{aligned} e^{\frac{x}{3}} &= 157.8 \\ \therefore\ \frac{x}{3} &= \ln(157.8) \\ \therefore\ x &= 3\ln(157.8) \\ \therefore\ x &\doteqdot 15.18 \end{aligned}$$

f
$$\begin{aligned} e^{\frac{x}{10}} &= 0.016\,82 \\ \therefore\ \frac{x}{10} &= \ln(0.016\,82) \\ \therefore\ x &= 10\ln(0.016\,82) \\ \therefore\ x &\doteqdot -40.85 \end{aligned}$$

g $20 \times e^{0.06x} = 8.312$
$\therefore \; e^{0.06x} = 0.4156$
$\therefore \; 0.06x = \ln(0.4156)$
$\therefore \; x = \dfrac{\ln(0.4156)}{0.06}$
$\therefore \; x \doteqdot -14.63$

h $50e^{-0.03x} = 0.816$
$\therefore \; e^{-0.03x} = 0.016\,32$
$\therefore \; -0.03x = \ln(0.016\,32)$
$\therefore \; x = \dfrac{\ln(0.016\,32)}{-0.03}$
$\therefore \; x \doteqdot 137.2$

i $41.83e^{0.652x} = 1000$
$\therefore \; e^{0.652x} \doteqdot 23.91$
$\therefore \; 0.652x \doteqdot \ln(23.91)$
$\therefore \; x \doteqdot \dfrac{\ln(23.91)}{0.652}$
$\therefore \; x \doteqdot 4.868$

EXERCISE 5E

1 $M_t = 20e^{0.15t}$

a When $M_t = 25$,
$20e^{0.15t} = 25$
$\therefore \; e^{0.15t} = \frac{25}{20}$
$\therefore \; 0.15t = \ln(\frac{25}{20})$
$\therefore \; t = \dfrac{\ln(1.25)}{0.15}$
$\therefore \; t \doteqdot 1.488$
i.e., after 1 hour 29 min

b When $M_t = 100$,
$20e^{0.15t} = 100$
$\therefore \; e^{0.15t} = 5$
$\therefore \; 0.15t = \ln 5$
$\therefore \; t = \dfrac{\ln 5}{0.15}$
$\therefore \; t \doteqdot 10.73$
i.e., after 10 hours 44 min

2 $T = 100e^{-0.03t}$ ^{o}C

a When $T = 30$,
$100e^{-0.03t} = 30$
$\therefore \; e^{-0.03t} = 0.3$
$\therefore \; -0.03t = \ln(0.3)$
$\therefore \; t = \dfrac{\ln(0.3)}{-0.03}$
$\therefore \; t \doteqdot 40.13$ min

b When $T = 1$,
$100e^{-0.03t} = 1$
$\therefore \; e^{-0.03t} = 0.01$
$\therefore \; -0.03t = \ln(0.01)$
$\therefore \; t = \dfrac{\ln(0.01)}{-0.03}$
$\therefore \; t \doteqdot 153\frac{1}{2}$ min

3 $M_t = 1000e^{-0.04t}$ $\quad \therefore \; M_0 = 1000e^0 = 1000$ g

a For M_t to halve,
$M_t = 500$
$\therefore \; 1000e^{-0.04t} = 500$
$\therefore \; e^{-0.04t} = 0.5$
$\therefore \; -0.04t = \ln(0.5)$
$\therefore \; t = \dfrac{\ln(0.5)}{-0.04}$
$\therefore \; t \doteqdot 17.3$ years

b For $M_t = 25$ g,
$\therefore \; 1000e^{-0.04t} = 25$
$\therefore \; e^{-0.04t} = 0.025$
$\therefore \; -0.04t = \ln(0.025)$
$\therefore \; t = \dfrac{\ln(0.025)}{-0.04}$
$\therefore \; t \doteqdot 92.2$ years

c For $M_t = 1\%$ of M_0
$\therefore \; 1000e^{-0.04t} = 0.01 \times 1000$
$\therefore \; e^{-0.04t} = 0.01$
$\therefore \; -0.04t = \ln(0.01)$
$\therefore \; t = \dfrac{\ln(0.01)}{-0.04}$
$\therefore \; t \doteqdot 115$ years

4 $V = 50\left(1 - e^{-0.2t}\right)$ m/s

So, when $V = 40$,

$$\begin{aligned} 50\left(1 - e^{-0.2t}\right) &= 40 \\ \therefore\quad 1 - e^{-0.2t} &= 0.8 \\ \therefore\quad e^{-0.2t} &= 0.2 \\ \therefore\quad -0.2t &= \ln(0.2) \\ \therefore\quad t &= \frac{\ln(0.2)}{-0.2} \\ \therefore\quad t &\doteqdot 8.05 \text{ sec} \end{aligned}$$

i.e., it would take 8.05 sec

5 $T_m = \left(225 \times e^{-0.17m} - 6\right)$ ^{o}C

So, when $T_m = 0$,

$$\begin{aligned} 225e^{-0.17m} - 6 &= 0 \\ \therefore\quad 225e^{-0.17m} &= 6 \\ \therefore\quad e^{-0.17m} &= \tfrac{6}{225} \\ \therefore\quad -0.17m &= \ln(\tfrac{6}{225}) \\ \therefore\quad m &= \frac{\ln(\frac{6}{225})}{-0.17} \\ \therefore\quad m &\doteqdot 21.3 \end{aligned}$$

i.e., it would take 21.3 minutes

EXERCISE 5F

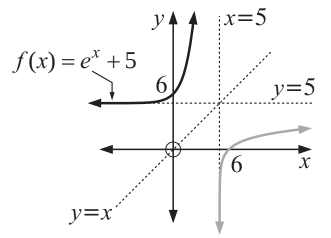
1 **a**

$$\begin{aligned} f(x) &= e^x + 5 \\ \text{i.e.,}\quad y &= e^x + 5 \end{aligned}$$

has inverse function

$$\begin{aligned} x &= e^y + 5 \\ \therefore\quad x - 5 &= e^y \\ \therefore\quad y &= \ln(x-5) \end{aligned}$$

b



c domain of f is $\{x:\ x \in \mathcal{R}\}$, range is $\{y:\ y > 5\}$
domain of f^{-1} is $\{x:\ x > 5\}$, range is $\{y:\ y \in \mathcal{R}\}$

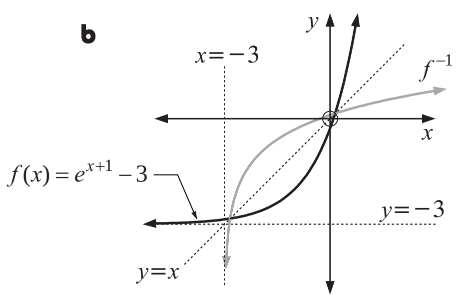
2 **a**

$$\begin{aligned} f(x) &= e^{x+1} - 3 \\ \text{i.e.,}\quad y &= e^{x+1} - 3 \end{aligned}$$

has inverse function

$$\begin{aligned} x &= e^{y+1} - 3 \\ \text{i.e.,}\quad x + 3 &= e^{y+1} \\ \text{i.e.,}\quad y + 1 &= \ln(x+3) \\ \therefore\quad f^{-1}(x) &= \ln(x+3) - 1 \end{aligned}$$

b



c domain of f is $\{x:\ x \in \mathcal{R}\}$, range is $\{y:\ y > -3\}$
domain of f^{-1} is $\{x:\ x > -3\}$, range is $\{y:\ y \in \mathcal{R}\}$

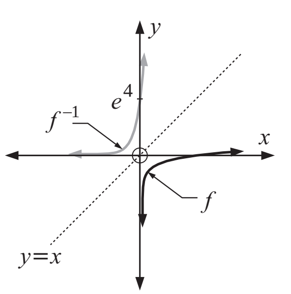
3 **a**

$$\begin{aligned} f(x) &= \ln x - 4 \\ \text{i.e.,}\quad y &= \ln x - 4 \quad \text{and} \end{aligned}$$

has inverse function

$$\begin{aligned} x &= \ln y - 4 \\ \text{i.e.,}\quad x + 4 &= \ln y \\ \therefore\quad y &= e^{x+4} \\ \text{i.e.,}\quad f^{-1}(x) &= e^{x+4} \end{aligned}$$

b



c domain of f is $\{x:\ x > 0\}$, range is $\{y:\ y \in \mathcal{R}\}$
domain of f^{-1} is $\{x:\ x \in \mathcal{R}\}$, range is $\{y:\ y > 0\}$

4 **a** $g(x) = \ln(x-1)+2,\ x>1$

i.e., $y = \ln(x-1)+2$ and

has inverse function

$$x = \ln(y-1)+2$$

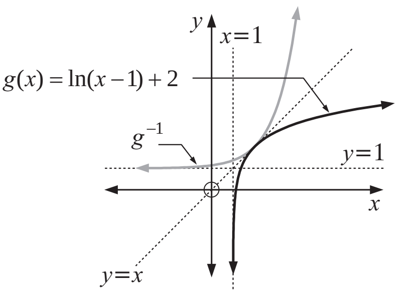
i.e., $\ln(y-1) = x-2$

$\therefore\ y-1 = e^{x-2}$

$\therefore\ y = e^{x-2}+1$

or $g^{-1}(x) = e^{x-2}+1$

b



c domain of g is $\{x:\ x>1\}$, range is $\{y:\ y \in \mathcal{R}\}$

domain of g^{-1} is $\{x:\ x \in \mathcal{R}\}$, range is $\{y:\ y>1\}$

5 $f(x) = e^{2x}$ and $g(x) = 2x-1$

a Now for $f(x) = e^{2x}$, i.e., $y = e^{2x}$

the inverse function is

$x = e^{2y}$

$\therefore\ 2y = \ln x$

$\therefore\ y = \frac{1}{2}\ln x$

i.e., $f^{-1}(x) = \frac{1}{2}\ln x$

$\therefore\ (f^{-1}\circ g)(x) = f^{-1}(g(x))$

$= \frac{1}{2}\ln(g(x))$

$= \frac{1}{2}\ln(2x-1)$

b $(g\circ f)(x)$

$= g(f(x))$

$= 2f(x)-1$

$= 2e^{2x}-1$ i.e., $y = 2e^{2x}-1$

which has inverse function

$x = 2e^{2y}-1$

i.e., $x+1 = 2e^{2y}$

$\therefore\ \dfrac{x+1}{2} = e^{2y}$

$\therefore\ 2y = \ln\left(\dfrac{x+1}{2}\right)$

$\therefore\ y = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

$\therefore\ (g\circ f)^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

6 **a** $y = \ln x$ cuts the x-axis when $y=0$

$\therefore\ \ln x = 0$

$\therefore\ x = e^0 = 1$

So, graph A is that of $y = \ln x$.

Note: x-intercept of $y = \ln(x-2)$

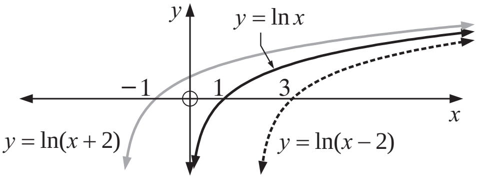
is when $x-2 = e^0 = 1$

i.e., $x = 3$

b The x-intercept of $y = \ln(x+2)$

occurs when $x+2 = e^0 = 1$

$\therefore\ x = -1$



7 Since $y = \ln x^2$ then $y = 2\ln x$ {log law}

i.e., new y-values are $2\times$ old y-values. So, she is correct.

8 $\ln(2x) = \ln 2 + \ln x$

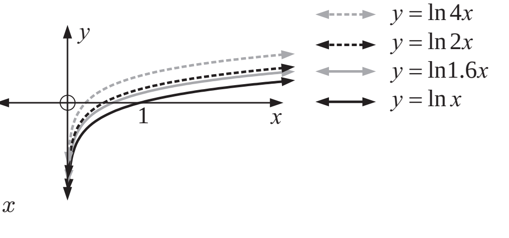
$\ln(4x) = \ln 4 + \ln x$

$\ln(1.6x) = \ln(1.6) + \ln x$

$\therefore\ y = \ln(2x)$ is $\ln 2$ units above $y = \ln x$

$y = \ln(4x)$ is $\ln 4$ units above $y = \ln x$

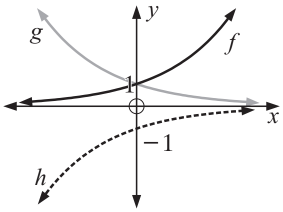
$y = \ln(1.6x)$ is $\ln(1.6)$ units above $y = \ln x$



REVIEW SET 5A

1 **a** $\doteq 54.6$ **b** $\doteq 22.2$ **c** $\doteq 0.0613$ **d** $\doteq 6.07$

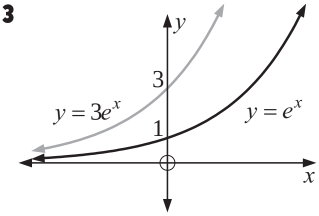
2 **a**



b **i** g is the reflection of f in the y-axis

ii h is the reflection of g in the x-axis

3



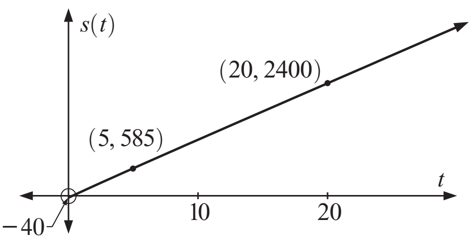
4 $s(t) = 120t - 40e^{-\frac{t}{5}}$ metres

a **i** $s(0) = 0 - 40e^0$
$= -40$ m

ii $s(5) = 600 - 40e^{-1}$
$\doteq 585$ m

iii $s(20) = 2400 - 40e^{-4}$
$\doteq 2399.3$ m

b



5 **a** $\ln e^5 = 5$
$\{\text{as} \quad \ln e^a = a\}$

b $\ln\left(\sqrt{e}\right) = \ln e^{\frac{1}{2}}$
$= \frac{1}{2}$

c $\ln\left(\frac{1}{e}\right) = \ln e^{-1}$
$= -1$

6 **a** $\ln(e^{2x}) = 2x$

b $\ln(e^2 e^x) = \ln(e^{2+x})$
$= 2 + x$

c $\ln\left(\dfrac{e}{e^x}\right) = \ln(e^{1-x})$
$= 1 - x$

7 **a** $\ln x = 5$
$\therefore\ x = e^5$
$\therefore\ x \doteq 148$

b $3\ln x + 2 = 0$
$\therefore\ 3\ln x = -2$
$\therefore\ \ln x = -\frac{2}{3}$
$\therefore\ x = e^{-\frac{2}{3}}$
$\therefore\ x \doteq 0.513$

8 **a** $\ln 6 + \ln 4$
$= \ln(6 \times 4)$
$= \ln 24$

b $\ln 60 - \ln 20$
$= \ln(\frac{60}{20})$
$= \ln 3$

c $\ln 4 + \ln 1$
$= \ln 4 + 0$
$= \ln 4$

d $\ln 200 - \ln 8 + \ln 5$
$= \ln(\frac{200}{8}) + \ln 5$
$= \ln\left(\frac{200}{8} \times 5\right)$
$= \ln 125$

9 **a** $\ln 32 = \ln 2^5$
$= 5\ln 2$

b $\ln 125 = \ln 5^3$
$= 3\ln 5$

c $\ln 729 = \ln 3^6$
$= 6\ln 3$

10 **a** $e^x = 400$
$\therefore\ x = \ln 400$
$\therefore\ x \doteq 5.99$

b $e^{2x+1} = 11$
$\therefore\ 2x + 1 = \ln 11$
$\therefore\ 2x = \ln 11 - 1$
$\therefore\ x = \dfrac{\ln 11 - 1}{2}$
$\therefore\ x \doteq 0.699$

c $25e^{\frac{x}{2}} = 750$
$\therefore\ e^{\frac{x}{2}} = 30$
$\therefore\ \dfrac{x}{2} = \ln 30$
$\therefore\ x = 2\ln 30$
$\therefore\ x \doteq 6.80$

d $e^{2x} = 7e^x - 12$
$\therefore\ e^{2x} - 7e^x + 12 = 0$
$\therefore\ (e^x - 3)(e^x - 4) = 0$
$\therefore\ e^x = 3$ or $e^x = 4$
$\therefore\ x = \ln 3$ or $\ln 4$
$\therefore\ x \doteq 1.10$ or 1.39

REVIEW SET 5B

1 **a** $\dfrac{1}{e^3} = e^{-3}$

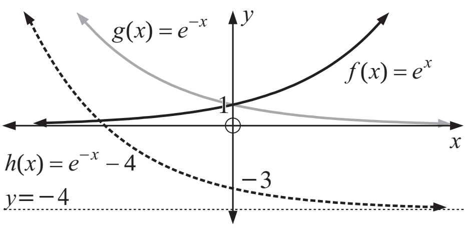
b $\dfrac{\sqrt{e}}{e^2} = \dfrac{e^{\frac{1}{2}}}{e^2} = e^{\frac{1}{2}-2} = e^{-\frac{3}{2}}$

c $e^3\sqrt{e^3} = e^3 \times \left(e^3\right)^{\frac{1}{2}} = e^{3+\frac{3}{2}} = e^{\frac{9}{2}}$

d $\sqrt{10e} = \left(e^{\ln 10}e\right)^{\frac{1}{2}} = e^{\frac{1}{2}(\ln 10+1)}$ $(\doteqdot e^{1.65})$

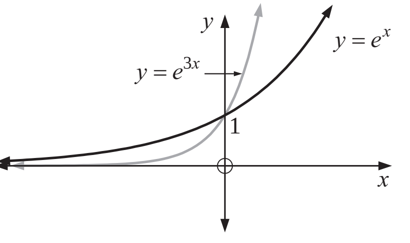
2 **a** $\doteqdot 26.9401$ **b** $\doteqdot 0.109\,447$

3



Each function has domain $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$
Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > -4\}$

4



5 **a** $\ln\left(e\sqrt{e}\right) = \ln\left(e^1 e^{\frac{1}{2}}\right) = \ln e^{1\frac{1}{2}} = 1\frac{1}{2}$

b $\ln\left(\dfrac{1}{e^3}\right) = \ln e^{-3} = -3$

c $\ln\left(\dfrac{e}{\sqrt{e^5}}\right) = \ln\left(\dfrac{e^1}{e^{\frac{5}{2}}}\right) = \ln\left(e^{1-\frac{5}{2}}\right) = \ln e^{-\frac{3}{2}} = -\frac{3}{2}$

6 **a** $20 = e^{\ln 20} \doteqdot e^{3.00}$

b $3000 = e^{\ln 3000} \doteqdot e^{8.01}$

c $0.075 = e^{\ln(0.075)} \doteqdot e^{-2.59}$

7 **a** $4\ln 2 + 2\ln 3 = \ln 2^4 + \ln 3^2 = \ln(16 \times 9) = \ln 144$

b $\frac{1}{2}\ln 9 - \ln 2 = \ln 9^{\frac{1}{2}} - \ln 2 = \ln 3 - \ln 2 = \ln(\frac{3}{2})$

c $2\ln 5 - 1 = \ln 5^2 - \ln e^1 = \ln\left(\dfrac{25}{e}\right)$

d $\frac{1}{4}\ln 81 = \ln\left(3^4\right)^{\frac{1}{4}} = \ln 3^1 = \ln 3$

8 **a**
$$\begin{aligned} \ln P &= 1.5\ln Q + \ln T \\ \therefore\ \ln P &= \ln Q^{\frac{3}{2}} + \ln T \\ &= \ln\left(Q^{\frac{3}{2}}T\right) \\ \therefore\ P &= Q^{\frac{3}{2}}T \end{aligned}$$

b
$$\begin{aligned} \ln M &= 1.2 - 0.5\ln N \\ \therefore\ \ln M + \ln N^{\frac{1}{2}} &= 1.2 \\ \therefore\ \ln\left(M\sqrt{N}\right) &= 1.2 \\ \therefore\ M\sqrt{N} &= e^{1.2} \\ \therefore\ M &\doteqdot \frac{3.32}{\sqrt{N}} \end{aligned}$$

9 **a** $g(x) = 2e^x - 5$ has inverse

function $x = 2e^y - 5$

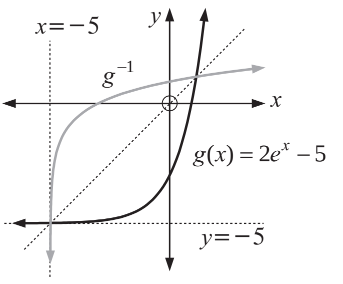
$\therefore\ 2e^y = x + 5$

$\therefore\ e^y = \dfrac{x+5}{2}$

i.e., $y = \ln\left(\dfrac{x+5}{2}\right)$

$\therefore\ g^{-1}(x) = \ln\left(\dfrac{x+5}{2}\right)$

b



c domain of g is $\{x: x \in \mathcal{R}\}$, range is $\{y: y > -5\}$

domain of g^{-1} is $\{x: x > -5\}$, range is $\{y: y \in \mathcal{R}\}$

10 $W_t = 8000 \times e^{-\frac{t}{20}}$ grams

$W_0 = 8000e^0$

$= 8000 \times 1$

$= 8000$ grams

a When $W_t = \frac{1}{2} \times 8000$ grams,

$8000e^{-\frac{t}{20}} = 4000$

$\therefore\ e^{-\frac{t}{20}} = 0.5$

$\therefore\ -\dfrac{t}{20} = \ln(0.5)$

$\therefore\ t = -20\ln(0.5)$

$\therefore\ t \doteqdot 13.9$ weeks

b When $W_t = 1000$,

$8000e^{-\frac{t}{20}} = 1000$

$\therefore\ e^{-\frac{t}{20}} = \frac{1}{8}$

$\therefore\ -\dfrac{t}{20} = \ln(\frac{1}{8})$

$\therefore\ t = -20\ln(\frac{1}{8})$

$\therefore\ t \doteqdot 41.6$ weeks

c When $W_t = 0.1\%$ of W_0

$= \frac{1}{1000}$ of 8000 g

$= 8$ g,

$8000e^{-\frac{t}{20}} = 8$

$\therefore\ e^{-\frac{t}{20}} = 0.001$

$\therefore\ -\dfrac{t}{20} = \ln(0.001)$

$\therefore\ t = -20\ln(0.001)$

$\therefore\ t \doteqdot 138$ weeks

Chapter 6

GRAPHING AND TRANSFORMING FUNCTIONS

EXERCISE 6A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x)+2 = x+2$

c $\frac{1}{2}f(x) = \frac{1}{2}x$

d $2f(x)+3 = 2x+3$

2 $f(x) = x^2$

a $f(3x) = (3x)^2 = 9x^2$

b $f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

c $3f(x) = 3x^2$

d $2f(x-1)+5 = 2(x-1)^2+5 = 2(x^2-2x+1)+5 = 2x^2-4x+7$

3 $f(x) = x^3$

a $f(4x) = (4x)^3 = 64x^3$

b $\frac{1}{2}f(2x) = \frac{1}{2}(2x)^3 = \frac{1}{2}\times 8x^3 = 4x^3$

c $f(x+1) = (x+1)^3 = x^3+3x^2+3x+1$

d $2f(x+1)-3 = 2(x+1)^3-3 = 2(x^3+3x^2+3x+1)-3 = 2x^3+6x^2+6x-1$

4 $f(x) = |x|$

a $f(2x)+3 = |2x|+3$

b $f(-x) = |-x| = |x|$

c $f(x-2) = |x-2|$

d $f(x+1)+2 = |x+1|+2$

5 $f(x) = 2^x$

a $f(2x) = 2^{2x} = 4^x$

b $f(-x)+1 = 2^{-x}+1$

c $f(x-2)+3 = 2^{x-2}+3$

d $2f(x)+3 = 2\times 2^x+3 = 2^{1+x}+3$

6 $f(x) = \frac{1}{x}$

a $f(-x) = \frac{1}{(-x)} = -\frac{1}{x}$

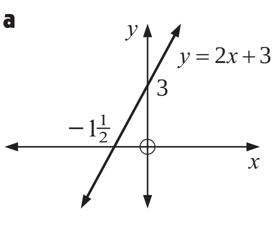
b $f(\frac{1}{2}x) = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$

c $2f(x)+3 = 2\left(\frac{1}{x}\right)+3 = \frac{2}{x}+3$ $\left(\text{or } \frac{2+3x}{x}\right)$

d $3f(x-1)+2 = 3\left(\frac{1}{x-1}\right)+2 = \frac{3}{x-1}+2$ $\left(\text{or } \frac{2x+1}{x-1}\right)$

EXERCISE 6B

1 a



b i When $y=0$, $2x+3 = 0$

$\therefore\ x = -\frac{3}{2}$

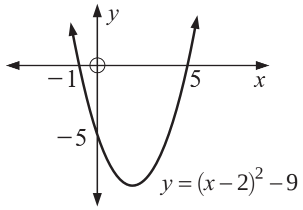
$\therefore$ x-intercept is $-1\frac{1}{2}$

ii When $x=0$, $y = 0+3 = 3$

$\therefore$ y-intercept is 3

iii As $y = 2x+3$, the slope is 2 {the coefficient of x}

2 **a**



b **i** When $y = 0$,

$(x-2)^2 - 9 = 0$

$\therefore\ (x-2)^2 = 9$

$\therefore\ x - 2 = \pm 3$

$\therefore\ x = 2+3$ or $2-3$

$\therefore\ x = 5$ or -1

$\therefore$ x-intercepts are 5 and -1

ii When $x = 0$,

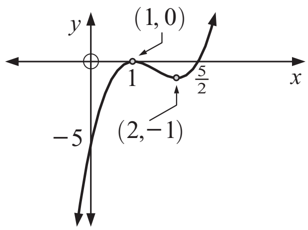
$y = (-2)^2 - 9$

$= 4 - 9$

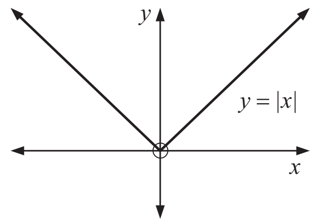
$= -5$

$\therefore$ y-intercept is -5

3 **a**, **b**



4



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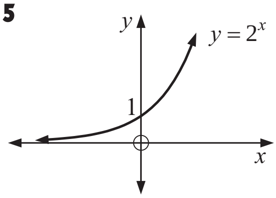
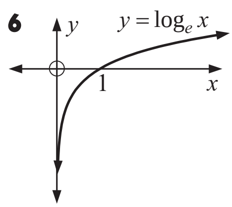
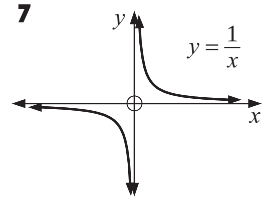


When $x = 0$,
$y = 2^0 = 1$ ✓
$2^x > 0$ for all x as the graph is always above the y-axis. ✓

6

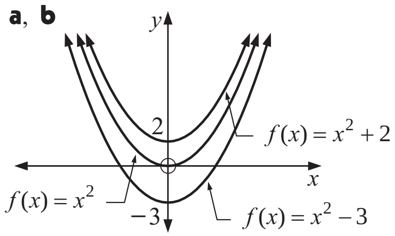


7



EXERCISE 6C.1

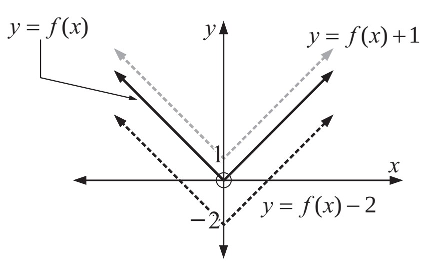
1 **a**, **b**



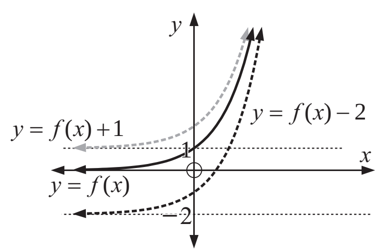
c **i** If $b > 0$, the function is translated vertically upwards through b units.

ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

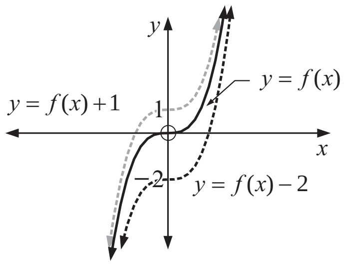
2 **a**



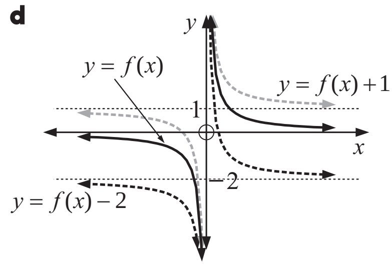
b



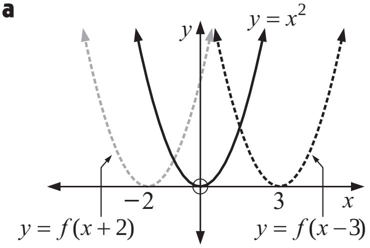
c



d



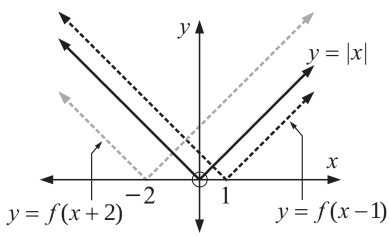
3 **a**



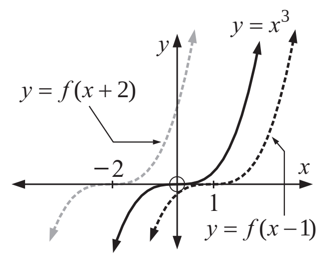
b **i** If $a > 0$, the graph is translated a units right.

ii If $a < 0$, the graph is translated $|a|$ units left.

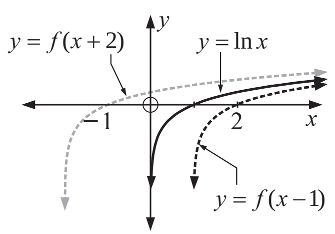
4 **a**



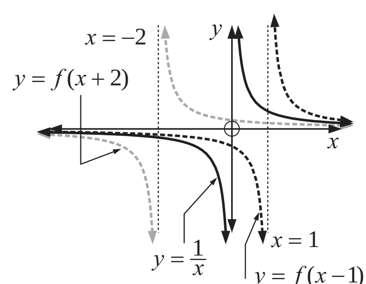
b



c

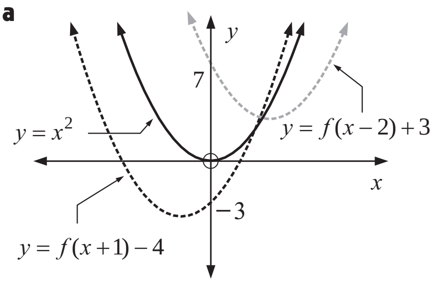


d

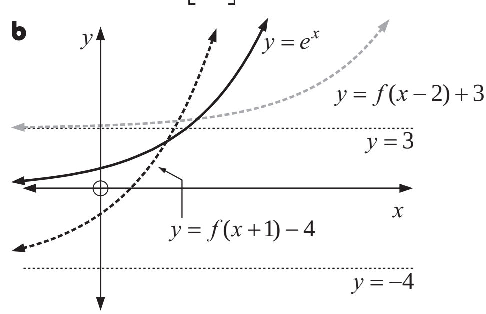


$y = f(x - a)$ is a horizontal translation of $y = f(x)$ through $\begin{bmatrix} a \\ 0 \end{bmatrix}$.

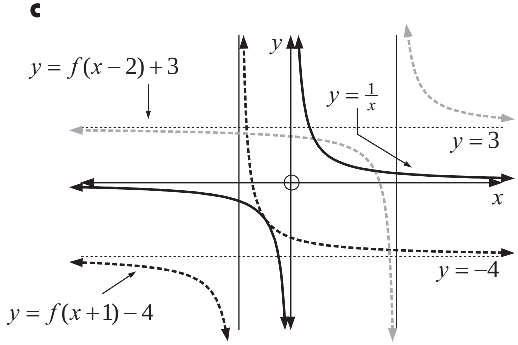
5 **a**



b

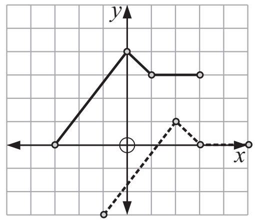


c

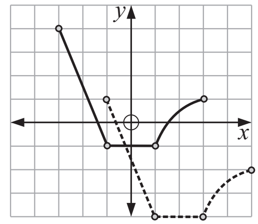


6 A translation of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

a

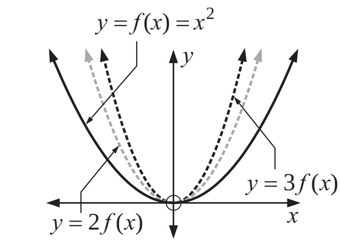


b

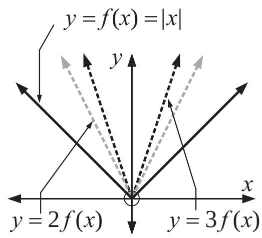


EXERCISE 6C.2

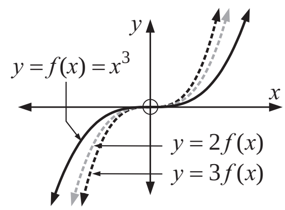
1 **a**



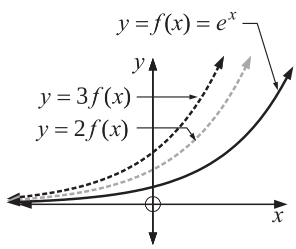
b



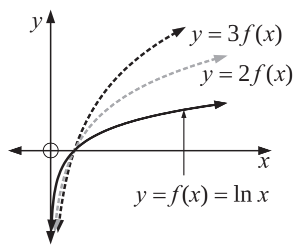
c



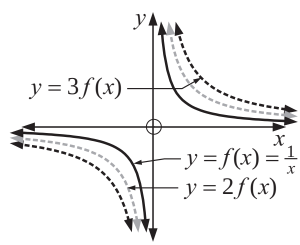
d



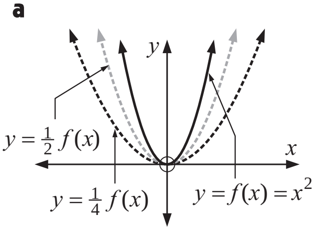
e



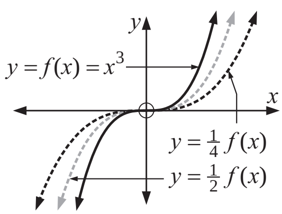
f



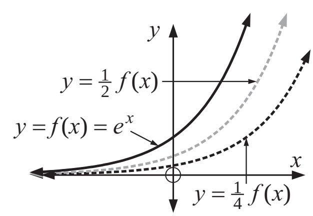
2 **a**



b

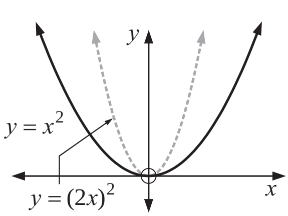


c

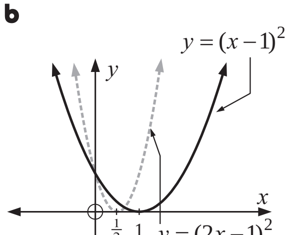


3 p affects the vertical stretching or compression of the graph of $y=f(x)$ by a factor of p.
If $p>1$, stretching occurs. If $0<p<1$, compression occurs.

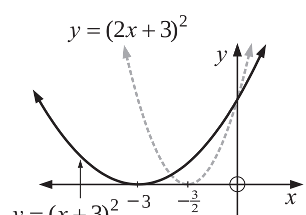
4 **a**



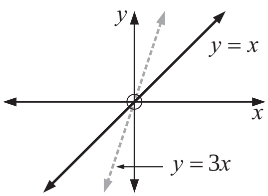
b



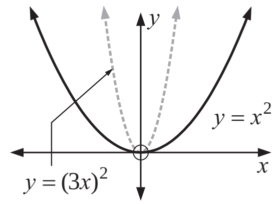
c



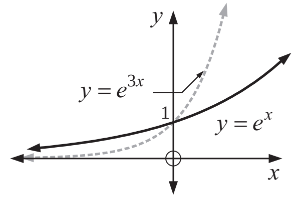
5 **a**



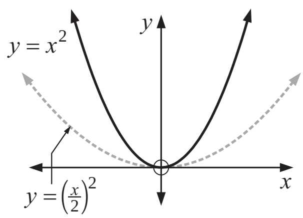
b



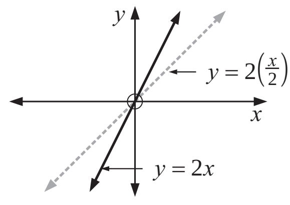
c



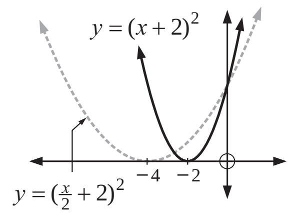
6 **a**



b



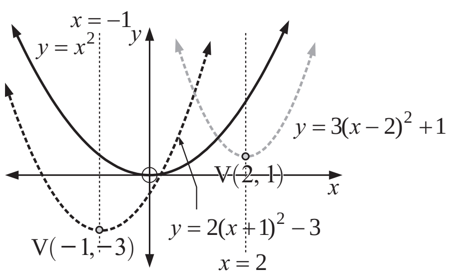
c



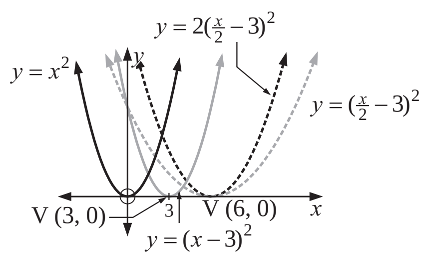
7 k affects the horizontal stretching or compression of $y = f(x)$ by a factor of $\dfrac{1}{k}$.

If $k > 1$, it moves closer to the y-axis. If $0 < k < 1$, it moves further from the y-axis.

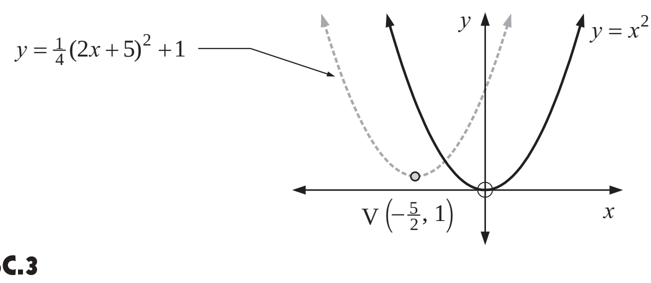
8 **a**



b

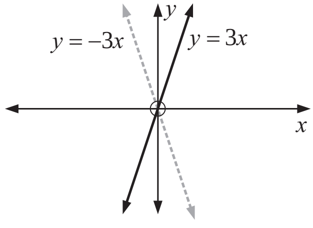


c

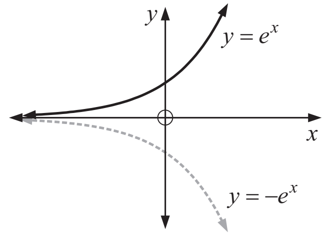


EXERCISE 6C.3

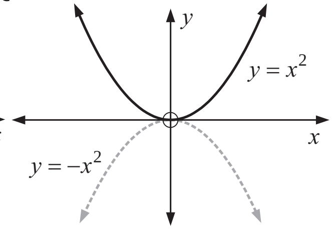
1 **a**



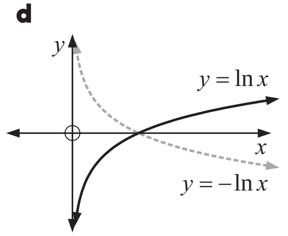
b



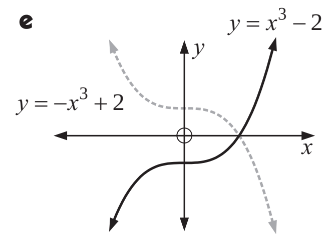
c



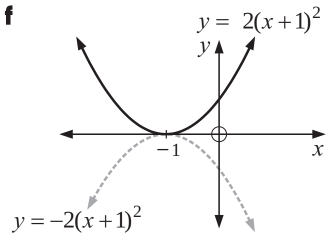
d



e



f



2 $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis.

3 **a** **i** $f(x) = 2x + 1$

$\therefore \ f(-x)$
$= 2(-x) + 1$
$= -2x + 1$

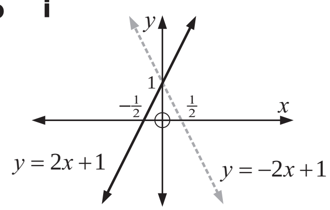
ii $f(x) = x^2 + 2x + 1$

$\therefore \ f(-x)$
$= (-x)^2 + 2(-x) + 1$
$= x^2 - 2x + 1$

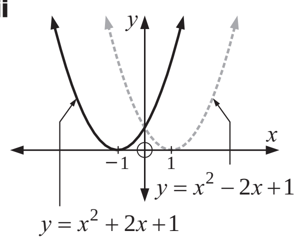
iii $f(x) = |x - 3|$

$\therefore \ f(-x) = |-x - 3|$
$= |-(x + 3)|$
$= |x + 3|$

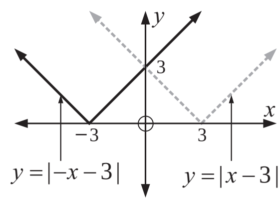
b **i**



ii



iii

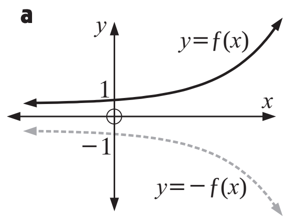


4 $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

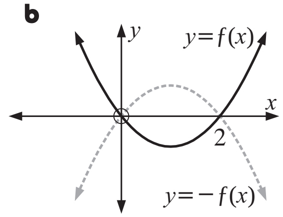
EXERCISE 6D

1 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x-axis.

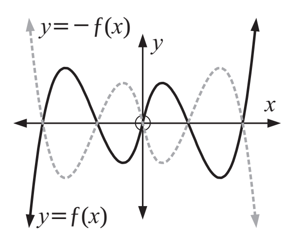
a



b

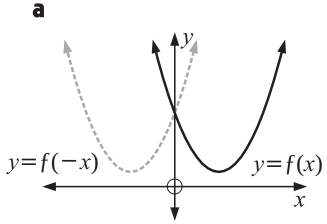


c

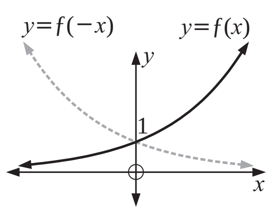


2 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y-axis.

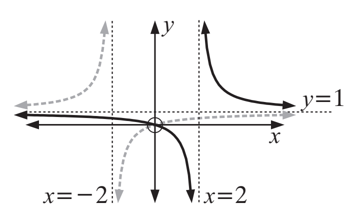
a



b



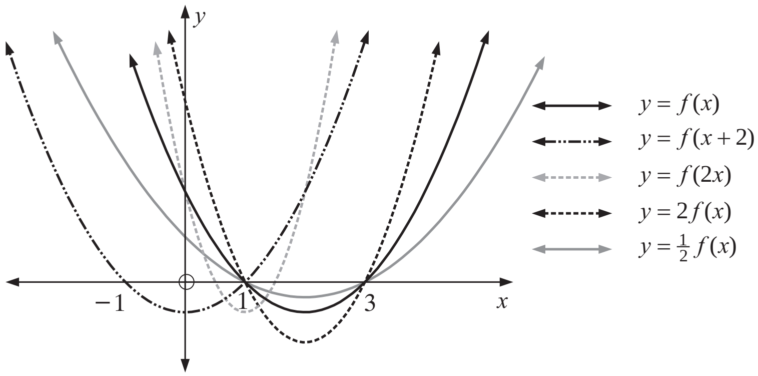
c



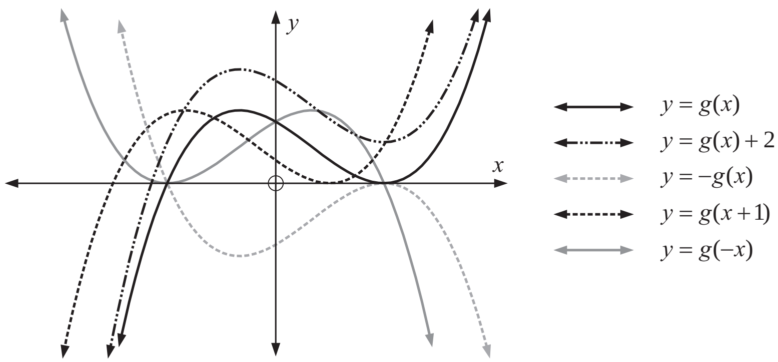
3 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'

$\therefore$ **a** is **A**, **b** is **B**, **c** is **D** and **d** is **C**

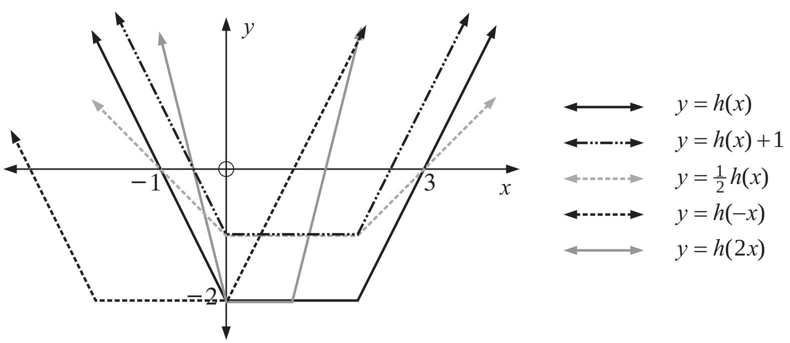
4



5



6



REVIEW SET 6

1 **a** linear **b** exponential **c** reciprocal **d** modulus **e** quadratic **f** logarithmic

2 $f(x) = x^2 - 2x$

a $f(3)$
$= 3^2 - 2(3)$
$= 9 - 6$
$= 3$

b $f(-2)$
$= (-2)^2 - 2(-2)$
$= 4 + 4$
$= 8$

c $f(2x)$
$= (2x)^2 - 2(2x)$
$= 4x^2 - 4x$

d $f(-x)$
$= (-x)^2 - 2(-x)$
$= x^2 + 2x$

e $3f(x) - 2$
$= 3(x^2 - 2x) - 2$
$= 3x^2 - 6x - 2$

3 $f(x) = 5 - x - x^2$

a $f(4)$
$= 5 - 4 - 4^2$
$= 1 - 16$
$= -15$

b $f(-1)$
$= 5 - (-1) - (-1)^2$
$= 5 + 1 - 1$
$= 5$

c $f(x-1)$
$= 5 - (x-1) - (x-1)^2$
$= 5 - x + 1 - [x^2 - 2x + 1]$
$= 6 - x - x^2 + 2x - 1$
$= -x^2 + x + 5$

d $f\left(\frac{x}{2}\right)$
$= 5 - \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$
$= 5 - \frac{x}{2} - \frac{x^2}{4}$

e $2f(x) - f(-x)$
$= 2(5 - x - x^2) - \left[5 - (-x) - (-x)^2\right]$
$= 10 - 2x - 2x^2 - [5 + x - x^2]$
$= 10 - 2x - 2x^2 - 5 - x + x^2$
$= -x^2 - 3x + 5$

4 $f(x) = \frac{4}{x}$

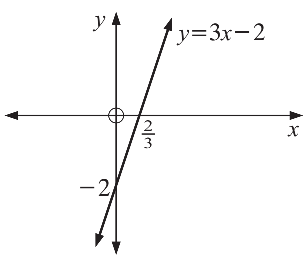
a $f(-4)$
$= \frac{4}{-4}$
$= -1$

b $f(2x)$
$= \frac{4}{2x}$
$= \frac{2}{x}$

c $f\left(\frac{x}{2}\right)$
$= \frac{4}{\frac{x}{2}}$
$= 4 \times \frac{2}{x}$
$= \frac{8}{x}$

d $4f(x+2) - 3$
$= 4\left(\frac{4}{x+2}\right) - 3$
$= \frac{16}{x+2} - 3$
$\left(or \quad \frac{16 - 3(x+2)}{x+2} = \frac{10 - 3x}{x+2} \right)$

5 **a**



b **i** When $y = 0$,
$3x - 2 = 0$
$\therefore \quad x = \frac{2}{3}$
$\therefore \quad x$-intercept is $\frac{2}{3}$

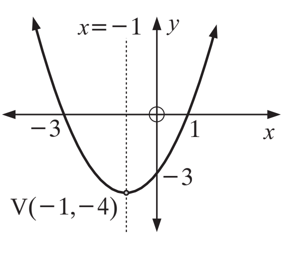
ii When $x = 0$,
$y = 0 - 2 = -2$
$\therefore \quad y$-intercept is -2

iii As $y = 3x - 2$, the slope is 3 {coefficient of x}

c **i** When $x = 0.3$,
$y = 3(0.3) - 2$
$= 0.9 - 2$
$= -1.1$

ii When $y = 0.7$,
$3x - 2 = 0.7$
$\therefore \quad 3x = 2.7$
$\therefore \quad x = 0.9$

6 **a**

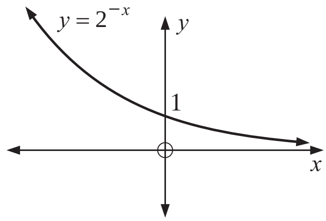


b **i** When $y = 0$,
$(x+1)^2 - 4 = 0$
$\therefore \quad (x+1)^2 = 4$
$\therefore \quad x + 1 = \pm 2$
$\therefore \quad x = 2 - 1$ or $-2 - 1$
i.e., $x = 1$ or -3
$\therefore \quad x$-intercepts are $1, -3$

ii When $x = 0$,
$y = 1^2 - 4$
$= -3$
$\therefore \quad y$-intercept is -3

c Since $y = (x+1)^2 - 4$ then $y + 4 = (x+1)^2$ which is obtained from $y = x^2$ under a translation of $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So, the vertex must be $(-1, -4)$.

7 **a**



b **i** $x \to \infty$ means x is very large and positive.
We see the graph approaching the x-axis
i.e., $y \to 0$ $\therefore$ **true**.

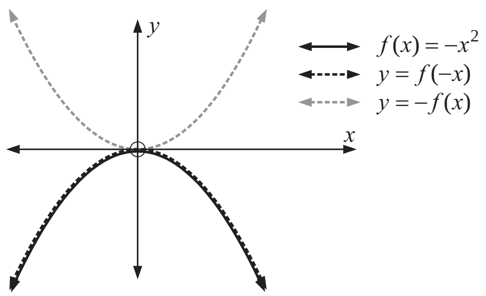
ii $x \to -\infty$ means x is very large and negative.
We see the graph heading for ∞ and positive
$\therefore$ statement is **false**.

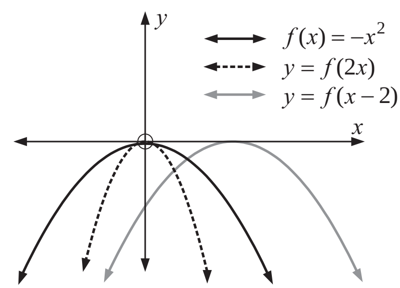
iii The y-intercept is 1, when $x = 0$, $y = 2^0 = 1$ $\therefore$ **false**.

iv The graph is above the x-axis for all x $\therefore$ $2^{-x} > 0$ for all x $\therefore$ **true**.

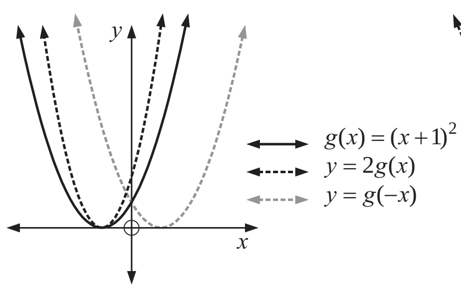
So that you can see the answers more easily in questions **8**, **9** and **11**, they have been drawn on two graphs.

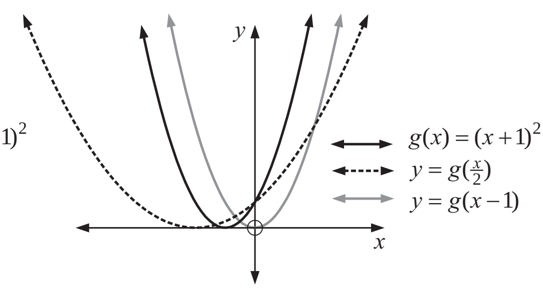
8



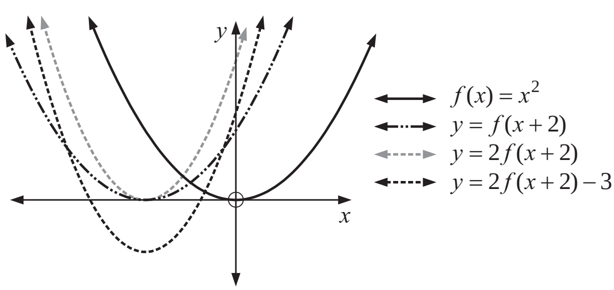


9

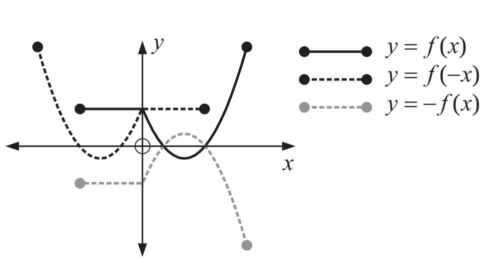


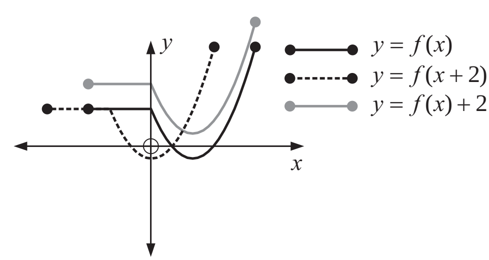


10



11





Chapter 7
COORDINATE GEOMETRY

EXERCISE 7A

1 **a** $\text{AB} = \sqrt{(x_\text{B} - x_\text{A})^2 + (y_\text{B} - y_\text{A})^2}$
$= \sqrt{(4-1)^2 + (5-3)^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$ units

b $\text{OC} = \sqrt{(x_\text{C} - x_\text{O})^2 + (y_\text{C} - y_\text{O})^2}$
$= \sqrt{(3-0)^2 + (-5-0)^2}$
$= \sqrt{9+25}$
$= \sqrt{34}$ units

c $\text{PQ} = \sqrt{(x_\text{Q} - x_\text{P})^2 + (y_\text{Q} - y_\text{P})^2}$
$= \sqrt{(1-5)^2 + (4-2)^2}$
$= \sqrt{16+4}$
$= \sqrt{20}$ units

d $\text{ST} = \sqrt{(x_\text{T} - x_\text{S})^2 + (y_\text{T} - y_\text{S})^2}$
$= \sqrt{(-1-0)^2 + (0 - -3)^2}$
$= \sqrt{1+9}$
$= \sqrt{10}$ units

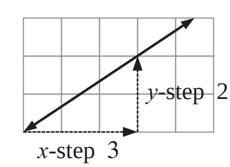
2 **a** $x_\text{M} = \dfrac{x_\text{A} + x_\text{B}}{2} = \dfrac{3+1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A} + y_\text{B}}{2} = \dfrac{6+0}{2} = 3$
$\therefore$ the midpoint is at $(2,\ 3)$

b $x_\text{M} = \dfrac{x_\text{A} + x_\text{B}}{2} = \dfrac{5-1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A} + y_\text{B}}{2} = \dfrac{2-4}{2} = -1$
$\therefore$ the midpoint is at $(2,\ -1)$

c $x_\text{M} = \dfrac{x_\text{A} + x_\text{B}}{2} = \dfrac{7+0}{2} = \frac{7}{2}$ and $y_\text{M} = \dfrac{y_\text{A} + y_\text{B}}{2} = \dfrac{0+3}{2} = \frac{3}{2}$
$\therefore$ the midpoint is at $(\frac{7}{2},\ \frac{3}{2})$

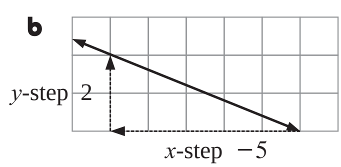
d $x_\text{M} = \dfrac{x_\text{A} + x_\text{B}}{2} = \dfrac{5-1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A} + y_\text{B}}{2} = \dfrac{-2-3}{2} = -\frac{5}{2}$
$\therefore$ the midpoint is at $(2,\ -\frac{5}{2})$

3 **a**



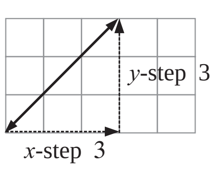
$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{2}{3}$

b



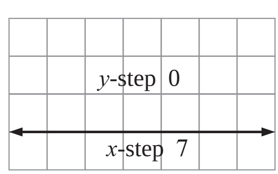
$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{2}{-5}$
$= -\frac{2}{5}$

c



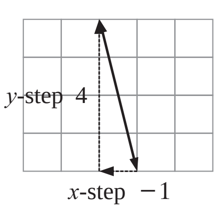
$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{3}{3}$
$= 1$

d



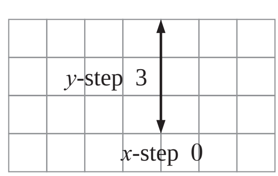
$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{0}{7}$
$= 0$

e



$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{4}{-1}$
$= -4$

f



$\text{slope} = \dfrac{y\text{-step}}{x\text{-step}}$
$= \frac{3}{0}$
which is undefined

4 **a** gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-3}{4-2} = 2$

b gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-2}{5-3} = 3$

c gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-2}{-1-(-1)} = \frac{3}{0}$

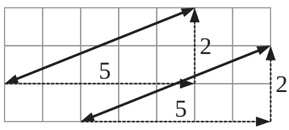
which is undefined

d gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-(-3)}{-1-4} = \frac{0}{-5} = 0$

e gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-0}{-1-0} = -4$

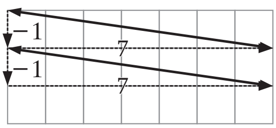
f gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2-(-1)}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$

5 **a**



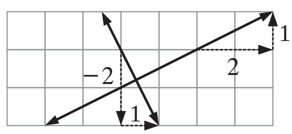
The lines have the same slope ($\frac{2}{5}$), so are parallel.

b



The lines have the same slope ($-\frac{1}{7}$), so are parallel.

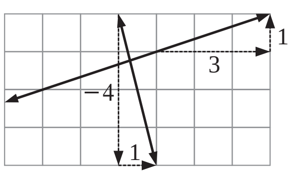
c



The two lines have slopes $\frac{1}{2}$ and $\frac{-2}{1}$, so the product of the slopes is -1.

Hence the lines are perpendicular.

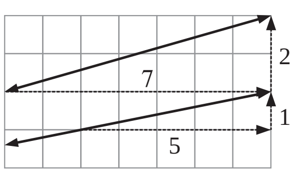
d



The two lines have slopes $\frac{1}{3}$ and $\frac{-4}{1}$, so the product of the slopes is $-\frac{4}{3}$.

Hence the lines are neither parallel nor perpendicular.

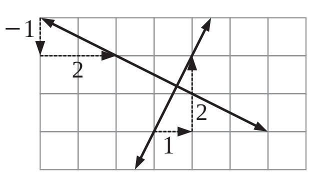
e



The two lines have slopes $\frac{2}{7}$ and $\frac{1}{5}$, so the product of the slopes is $\frac{2}{35}$.

Hence they are neither parallel nor perpendicular.

f



The two lines have slopes $\frac{-1}{2}$ and $\frac{2}{1}$, so the product of the slopes is -1.

Hence the lines are perpendicular.

6 **a** gradient $= -\dfrac{1}{\left(\frac{3}{4}\right)} = -\frac{4}{3}$

b gradient $= -\dfrac{1}{\left(\frac{11}{3}\right)} = -\frac{3}{11}$

c gradient $= -\frac{1}{4}$

d gradient $= -\dfrac{1}{\left(-\frac{1}{3}\right)} = 3$

e gradient $= -\frac{1}{(-5)} = \frac{1}{5}$

f gradient $= -\frac{1}{0}$

which is undefined

7 **a** The equation of the line is

$$\frac{y-1}{x-4} = 2$$
$$\therefore\ y-1 = 2(x-4)$$
$$\therefore\ y-1 = 2x-8$$
$$\therefore\ y = 2x-7$$

b The equation of the line is

$$\frac{y-2}{x-1} = -2$$
$$\therefore\ y-2 = -2(x-1)$$
$$\therefore\ y-2 = -2x+2$$
$$\therefore\ y = -2x+4$$

c The equation of the line is

$$\frac{y-0}{x-5} = 3$$
$$\therefore\ y = 3(x-5)$$
$$\therefore\ y = 3x-15$$

d The equation of the line is

$$\frac{y-7}{x-(-1)} = -3$$
$$\therefore\ y-7 = -3(x+1)$$
$$\therefore\ y-7 = -3x-3$$
$$\therefore\ y = -3x+4$$

e The equation of the line is

$$\frac{y-5}{x-1} = -4$$
$$\therefore\ y-5 = -4(x-1)$$
$$\therefore\ y-5 = -4x+4$$
$$\therefore\ y = -4x+9$$

f The equation of the line is

$$\frac{y-7}{x-2} = 1$$
$$\therefore\ y-7 = x-2$$
$$\therefore\ y = x+5$$

8 **a** The equation of the line is

$$\frac{y-1}{x-2} = \frac{3}{2}$$
$$\therefore\ 2(y-1) = 3(x-2)$$
$$\therefore\ 2y-2 = 3x-6$$
$$\therefore\ 3x-2y = 4$$

b The equation of the line is

$$\frac{y-4}{x-1} = -\frac{3}{2}$$
$$\therefore\ -2(y-4) = 3(x-1)$$
$$\therefore\ -2y+8 = 3x-3$$
$$\therefore\ 3x+2y = 11$$

c The equation of the line is

$$\frac{y-0}{x-4} = \frac{1}{3}$$
$$\therefore\ 3y = x-4$$
$$\therefore\ x-3y = 4$$

d The equation of the line is

$$\frac{y-6}{x-0} = -4$$
$$\therefore\ y-6 = -4x$$
$$\therefore\ 4x+y = 6$$

e The equation of the line is

$$\frac{y-(-3)}{x-(-1)} = 3$$
$$\therefore\ y+3 = 3(x+1)$$
$$\therefore\ y+3 = 3x+3$$
$$\therefore\ 3x-y = 0$$

f The equation of the line is

$$\frac{y-(-2)}{x-4} = -\frac{4}{9}$$
$$\therefore\ -9(y+2) = 4(x-4)$$
$$\therefore\ -9y-18 = 4x-16$$
$$\therefore\ 4x+9y = -2$$

9 **a** The slope of the line is $\frac{2-1}{3-0} = \frac{1}{3}$ $\therefore$ its equation is $\frac{y-1}{x-0} = \frac{1}{3}$

$$\therefore\ 3(y-1) = x$$
$$\therefore\ 3y-3 = x$$
$$\therefore\ x-3y = -3$$

b The slope of the line is $\frac{-1-4}{0-1} = 5$ $\therefore$ its equation is $\frac{y-(-1)}{x-0} = 5$

$$\therefore\ y+1 = 5x$$
$$\therefore\ 5x-y = 1$$

c The slope of the line is $\dfrac{-4-(-1)}{-1-2} = \dfrac{-3}{-3} = 1$ $\quad\therefore$ its equation is $\dfrac{y-(-1)}{x-2} = 1$

$$\therefore\ y+1 = x-2$$
$$\therefore\ x-y = 3$$

d The slope of the line is $\dfrac{2-(-2)}{5-0} = \dfrac{4}{5}$ $\quad\therefore$ its equation is $\dfrac{y-(-2)}{x-0} = \dfrac{4}{5}$

$$\therefore\ 5(y+2) = 4x$$
$$\therefore\ 5y+10 = 4x$$
$$\therefore\ 4x-5y = 10$$

e The slope of the line is $\dfrac{2-0}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$ $\quad\therefore$ its equation is $\dfrac{y-0}{x-(-1)} = \dfrac{1}{2}$

$$\therefore\ 2y = x+1$$
$$\therefore\ x-2y = -1$$

f The slope of the line is $\dfrac{-3-(-1)}{2-(-1)} = \dfrac{-2}{3}$ $\quad\therefore$ its equation is $\dfrac{y-(-1)}{x-(-1)} = -\dfrac{2}{3}$

$$\therefore\ -3(y+1) = 2(x+1)$$
$$\therefore\ -3y-3 = 2x+2$$
$$\therefore\ 2x+3y = -5$$

10 **a** Since both points on the line have y-coordinate -2, it must be horizontal.

$\therefore$ its equation is $y=-2$.

b Since both points on the line have x-coordinate 6, it must be vertical.

$\therefore$ its equation is $x=6$.

c Since both points on the line have x-coordinate -3, it must be vertical.

$\therefore$ its equation is $x=-3$.

11 **a** $2x-3y = 6$

$\therefore\ 3y = 2x-6$

$\therefore\ y = \frac{2}{3}x-2$

$\therefore$ slope $= \frac{2}{3}$, y-intercept $=-2$

When $y=0$, $2x=6$

$\therefore$ x-intercept $=3$

b $4x+5y = 20$

$\therefore\ 5y = -4x+20$

$\therefore\ y = -\frac{4}{5}x+4$

$\therefore$ slope $= -\frac{4}{5}$, y-intercept $=4$

When $y=0$, $4x=20$

$\therefore$ x-intercept $=5$

c $y=-2x+5$

$\therefore$ slope $=-2$, y-intercept $=5$

When $y=0$, $-2x+5 = 0$

$\therefore\ x = \frac{5}{2}$

i.e., the x-intercept $= \frac{5}{2}$

d $x=8$

$\therefore$ the line is vertical

$\therefore$ slope is undefined,

no y-intercept, x-intercept $=8$

e $y=5$

$\therefore$ the line is horizontal

$\therefore$ slope $=0$,

y-intercept $=5$, no x-intercept

f $x+y = 11$

$\therefore\ y = 11-x$

$\therefore$ slope $=-1$, y-intercept $=11$

When $y=0$, $x=11$

$\therefore$ x-intercept $=11$

g $4x + y = 8$
$\therefore\ y = -4x + 8$
$\therefore$ slope $= -4$, y-intercept $= 8$
When $y = 0$, $4x = 8$
$\therefore$ x-intercept $= 2$

h $x - 3y = 12$
$\therefore\ 3y = x - 12$
$\therefore\ y = \frac{1}{3}x - 4$
$\therefore$ slope $= \frac{1}{3}$, y-intercept $= -4$
When $y = 0$, $x = 12$
$\therefore$ x-intercept $= 12$

Summary of results:

	Equation of line	*Slope*	*x-intercept*	*y-intercept*
a	$2x - 3y = 6$	$\frac{2}{3}$	3	-2
b	$4x + 5y = 20$	$-\frac{4}{5}$	5	4
c	$y = -2x + 5$	-2	$\frac{5}{2}$	5
d	$x = 8$	undefined	8	none
e	$y = 5$	0	none	5
f	$x + y = 11$	-1	11	11
g	$4x + y = 8$	-4	2	8
h	$x - 3y = 12$	$\frac{1}{3}$	12	-4

12 **a** Substituting $(3, 4)$ into $3x - 2y = 1$ gives $3(3) - 2(4) = 1$
i.e., $1 = 1$ which is true
$\therefore$ $(3, 4)$ lies on the line

b Substituting $(-2, 5)$ into $5x + 3y = -5$ gives $5(-2) + 3(5) = -5$
i.e., $5 = -5$ which is false
$\therefore$ $(-2, 5)$ does not lie on the line

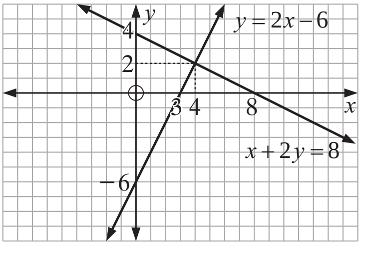
c Substituting $(6, -\frac{1}{2})$ into $3x - 8y = 22$ gives $3(6) - 8(-\frac{1}{2}) = 22$
i.e., $22 = 22$ which is true
$\therefore$ $(6, -\frac{1}{2})$ lies on the line

13 **a** For $x + 2y = 8$,
when $x = 0$, $y = 4$
when $y = 0$, $x = 8$

x	0	8
y	4	0

For $y = 2x - 6$,
when $x = 0$, $y = -6$
when $y = 0$, $x = 3$

x	0	3
y	-6	0



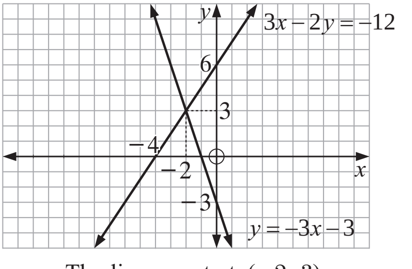
The lines meet at $(4, 2)$.

b For $y = -3x - 3$,
when $x = 0$, $y = -3$
when $y = 0$, $x = -1$

x	0	-1
y	-3	0

For $3x - 2y = -12$,
when $x = 0$, $y = 6$
when $y = 0$, $x = -4$

x	0	-4
y	6	0



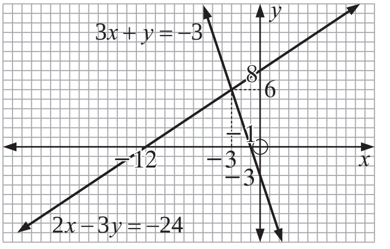
The lines meet at $(-2, 3)$.

c For $3x + y = -3$,
when $x = 0,\quad y = -3$
when $y = 0,\quad x = -1$

x	0	-1
y	-3	0

For $2x - 3y = -24$,
when $x = 0,\quad y = 8$
when $y = 0,\quad x = -12$

x	0	-12
y	8	0



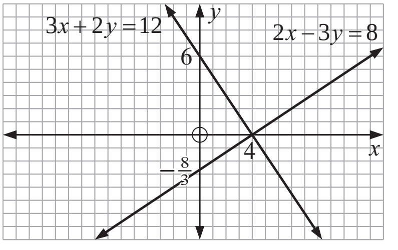
The lines meet at $(-3, 6)$.

d For $2x - 3y = 8$,
when $x = 0,\quad y = -\frac{8}{3}$
when $y = 0,\quad x = 4$

x	0	4
y	$-\frac{8}{3}$	0

For $3x + 2y = 12$,
when $x = 0,\quad y = 6$
when $y = 0,\quad x = 4$

x	0	4
y	6	0



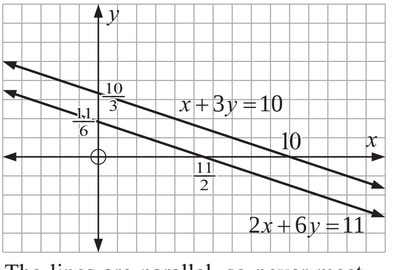
The lines meet at $(4, 0)$.

e For $x + 3y = 10$,
when $x = 0,\quad y = \frac{10}{3}$
when $y = 0,\quad x = 10$

x	0	10
y	$\frac{10}{3}$	0

For $2x + 6y = 11$,
when $x = 0,\quad y = \frac{11}{6}$
when $y = 0,\quad x = \frac{11}{2}$

x	0	$\frac{11}{2}$
y	$\frac{11}{6}$	0



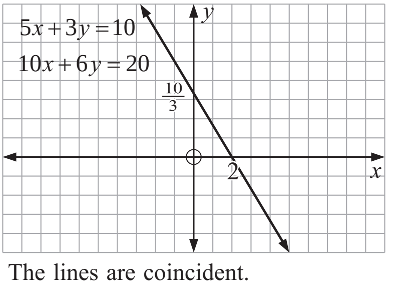
The lines are parallel, so never meet.

f For $5x + 3y = 10$,
when $x = 0,\ y = \frac{10}{3}$
when $y = 0,\ x = 2$

x	0	2
y	$\frac{10}{3}$	0

For $10x + 6y = 20$,
when $x = 0,\ y = \frac{20}{6} = \frac{10}{3}$
when $y = 0,\ x = 2$

x	0	2
y	$\frac{10}{3}$	0



The lines are coincident.

EXERCISE 7B.1

1 **a** Since the line is horizontal, its equation is $y = -4$.
b Since the line is vertical, its equation is $x = 5$.
c Since the line is vertical, its equation is $x = -1$.
d Since the line is horizontal, its equation is $y = 2$.
e The x-axis corresponds to $y = 0$.
f The y-axis corresponds to $x = 0$.

2 **a** The line has equation $\dfrac{y-4}{x-(-1)} = \dfrac{3}{4}$ $\quad \therefore\ 4(y-4) = 3(x+1)$

$\therefore\ 4y - 16 = 3x + 3$

$\therefore\ 3x - 4y = -19$

b The line has slope $= \dfrac{0-(-5)}{7-2} = \dfrac{5}{5} = 1$ $\quad \therefore$ its equation is $\dfrac{y-0}{x-7} = 1$

$\therefore\ y = x - 7$

$\therefore\ x - y = 7$

c $y = 3x - 2$ has slope 3, so this line has slope 3 also.

It passes through $(0, 0)$, so its equation is $\dfrac{y-0}{x-0} = 3$ i.e., $y = 3x$

d Now a line parallel to $2x + 3y = 8$ has equation $2x + 3y = k$, where k is a constant.

Since $(-1, 7)$ lies on the line, $2(-1) + 3(7) = k$

$\therefore\ k = 19$

$\therefore$ the line is $2x + 3y = 19$

e $y = -2x + 5$ has slope -2 $\therefore$ lines perpendicular to it have slope $\frac{1}{2}$.

But this line must pass through $(3, -1)$. $\quad \therefore\ \dfrac{y-(-1)}{x-3} = \dfrac{1}{2}$

$\therefore\ 2(y+1) = x - 3$

$\therefore\ 2y + 2 = x - 3$

$\therefore\ x - 2y = 5$

f If $3x - y = 11$, then $y = 3x - 11$

$\therefore$ this line has slope 3 $\therefore$ lines perpendicular to it have slope $-\frac{1}{3}$.

But this line must pass through $(-2, 5)$. $\quad \therefore\ \dfrac{y-5}{x-(-2)} = -\dfrac{1}{3}$

$\therefore\ -3(y-5) = x + 2$

$\therefore\ -3y + 15 = x + 2$

$\therefore\ x + 3y = 13$

3 **a** Keach Avenue passes through $(5, 11)$ and $(13, 12)$

$\therefore$ its line has slope $\dfrac{12-11}{13-5} = \dfrac{1}{8}$

$\therefore$ its equation is $\dfrac{y-12}{x-13} = \dfrac{1}{8}$ $\quad \therefore\ 8(y-12) = x - 13$

$\therefore\ 8y - 96 = x - 13$

$\therefore\ x - 8y = -83$

b Peacock Street is perpendicular to Keach Avenue, so its line has slope -8.

But this line also passes through $(3, 17)$. $\quad \therefore\ \dfrac{y-17}{x-3} = -8$

$\therefore\ y - 17 = -8(x-3)$

$\therefore\ y - 17 = -8x + 24$

$\therefore\ 8x + y = 41$

c Diagonal Road runs from $(5, 11)$ to $(7, 20)$, but ends at these points.

Hence there is a restricted domain $5 \leqslant x \leqslant 7$.

Its slope is $\dfrac{20-11}{7-5} = \dfrac{9}{2}$

$\therefore$ its equation is $\dfrac{y-11}{x-5} = \dfrac{9}{2}$ $\quad\therefore\ 2(y-11) = 9(x-5)$

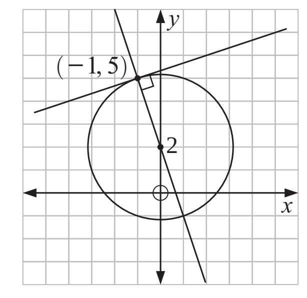
$$\therefore\ 2y-22 = 9x-45$$
$$\therefore\ 9x-2y = 23, \quad 5 \leqslant x \leqslant 7$$

d Plunkit Street has $x = 8$, so it meets Keach Avenue when $8 - 8y = -83$ {using **a**}

$$\therefore\ 8y = 91$$
$$\therefore\ y = \tfrac{91}{8}$$

$\therefore$ they intersect at $(8, \frac{91}{8})$

4 **a**



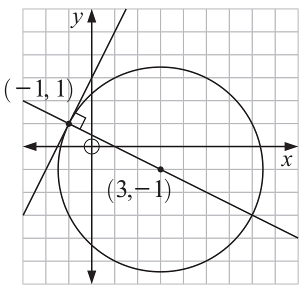
The line through $(-1, 5)$ and $(0, 2)$ has slope $= \dfrac{5-2}{-1-0} = -3$

$\therefore$ the slope of the tangent is $\frac{1}{3}$

$\therefore$ its equation is $\dfrac{y-5}{x-(-1)} = \dfrac{1}{3}$

$$\therefore\ 3(y-5) = x+1$$
$$\therefore\ 3y-15 = x+1$$
$$\therefore\ x-3y = -16$$

b



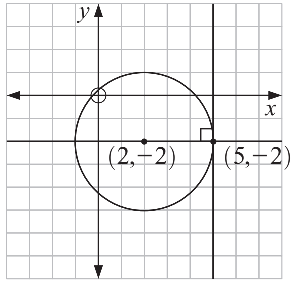
The line through $(-1, 1)$ and $(3, -1)$ has slope $= \dfrac{-1-1}{3-(-1)} = -\dfrac{1}{2}$

$\therefore$ the slope of the tangent is 2

$\therefore$ its equation is $\dfrac{y-1}{x-(-1)} = 2$

$$\therefore\ y-1 = 2(x+1)$$
$$\therefore\ y-1 = 2x+2$$
$$\therefore\ 2x-y = -3$$

c



The line through $(2, -2)$ and $(5, -2)$ has slope $= \dfrac{-2-(-2)}{5-2} = 0$

i.e., it is horizontal

$\therefore$ the tangent must be vertical

$\therefore$ its equation is $x = 5$

EXERCISE 7B.2

1 **a** The equation is $x - 2y = (4) - 2(1)$
i.e., $x - 2y = 2$

b The equation is $2x - 3y = 2(-2) - 3(5)$
i.e., $2x - 3y = -19$

c The equation is $3x - 4y = 3(5) - 4(0)$
i.e., $3x - 4y = 15$

d The equation is $3x - y = 3(3) - (-2)$
i.e., $3x - y = 11$

e The equation is $x + 3y = (1) + 3(4)$
i.e., $x + 3y = 13$

f The equation is $3x + y = 3(0) + (4)$
i.e., $3x + y = 4$

2 **a** slope $= -\frac{2}{3}$ **b** slope $= -\frac{3}{(-7)} = \frac{3}{7}$ **c** slope $= -\frac{6}{(-11)} = \frac{6}{11}$

d slope $= -\frac{5}{6}$ **e** slope $= -\frac{3}{6} = -\frac{1}{2}$ **f** slope $= -\frac{15}{(-5)} = 3$

3 **a** Any line with equation $3x + 5y = ...$ has slope $= -\frac{3}{5}$.

$\therefore$ since any parallel line must also have this slope,
its equation must also have the form $3x + 5y = ...$

b Any line with equation $3x + 5y = ...$ has slope $= -\frac{3}{5}$.

$\therefore$ any perpendicular line has slope $= -\dfrac{1}{(-\frac{3}{5})} = \frac{5}{3}$

$\therefore$ its equation has the form $5x - 3y = ...$

4 **a** The line $3x + 4y = 6$ has slope $= -\frac{3}{4}$

$\therefore$ the new line has slope $= -\frac{3}{4}$ also

$\therefore$ since it passes through $(2, 1)$, its equation is $3x + 4y = 3(2) + 4(1)$
i.e., $3x + 4y = 10$

b The line $5x + 2y = 10$ has slope $= -\frac{5}{2}$

$\therefore$ the new line has slope $= -\dfrac{1}{(-\frac{5}{2})} = \frac{2}{5}$

$\therefore$ since it passes through $(-1, -1)$, its equation is $2x - 5y = 2(-1) - 5(-1)$
i.e., $2x - 5y = 3$

c The line $x - 3y + 6 = 0$ has slope $= -\frac{1}{(-3)} = \frac{1}{3}$

$\therefore$ the new line has slope $= -3$

$\therefore$ since it passes through $(-4, 0)$, its equation is $3x + y = 3(-4) + 0$
i.e., $3x + y = -12$

d The line $x - 3y = 11$ has slope $= -\frac{1}{(-3)} = \frac{1}{3}$

$\therefore$ the new line has slope $= \frac{1}{3}$ also

$\therefore$ since it passes through $(0, 0)$, its equation is $x - 3y = 0$

5 **a** $2x - 3y = 6$ has slope $= -\frac{2}{(-3)} = \frac{2}{3}$

$6x + ky = 4$ has slope $= -\dfrac{6}{k}$

b The lines are parallel if their slopes are equal.

$$\therefore \quad \frac{2}{3} = -\frac{6}{k}$$
$$\therefore \quad k = -6 \times \tfrac{3}{2}$$
$$\therefore \quad k = -9$$

c The lines are perpendicular if the product of their slopes is -1

$$\therefore \quad \tfrac{2}{3} \times \left(-\frac{6}{k}\right) = -1$$
$$\therefore \quad -\frac{4}{k} = -1$$
$$\therefore \quad k = 4$$

EXERCISE 7C

1 **a**
$$\begin{aligned} AB &= \sqrt{(-11-3)^2+(3-8)^2} \\ &= \sqrt{196+25} \\ &= \sqrt{221} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-8-3)^2+(-2-8)^2} \\ &= \sqrt{121+100} \\ &= \sqrt{221} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-8-(-11))^2+(-2-3)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

Since only two sides of the triangle are equal in length, the triangle is isosceles.

b
$$\begin{aligned} PQ &= \sqrt{(0-(-1))^2+(\sqrt{3}-0)^2} \\ &= \sqrt{1+3} \\ &= 2 \text{ units} \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-(-1))^2+0^2} \\ &= \sqrt{4} \\ &= 2 \text{ units} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-0)^2+(0-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= 2 \text{ units} \end{aligned}$$

Since all three sides of the triangle are equal in length, the triangle is equilateral.

c
$$\begin{aligned} KL &= \sqrt{(-2-6)^2+(-3-(-5))^2} \\ &= \sqrt{64+4} \\ &= \sqrt{68} \text{ units} \end{aligned}$$

$$\begin{aligned} KM &= \sqrt{(-1-6)^2+(1-(-5))^2} \\ &= \sqrt{49+36} \\ &= \sqrt{85} \text{ units} \end{aligned}$$

$$\begin{aligned} LM &= \sqrt{(-1-(-2))^2+(1-(-3))^2} \\ &= \sqrt{1+16} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

Now $(\sqrt{68})^2+(\sqrt{17})^2 = (\sqrt{85})^2$

$\therefore$ $KL^2 + LM^2 = KM^2$

$\therefore$ the triangle is right angled at L.

d
$$\begin{aligned} AB &= \sqrt{(3-1)^2+(5-(-1))^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-4-1)^2+(-16-(-1))^2} \\ &= \sqrt{25+225} \\ &= \sqrt{250} \\ &= 5\sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-16-5)^2+(-4-3)^2} \\ &= \sqrt{441+49} \\ &= \sqrt{490} \\ &= 7\sqrt{10} \text{ units} \end{aligned}$$

Now $AB + AC = BC$, so the points are collinear.

e
$$\begin{aligned} AB &= \sqrt{(2-7)^2+(3-5)^2} \\ &= \sqrt{25+4} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(6-7)^2+(-7-5)^2} \\ &= \sqrt{1+144} \\ &= \sqrt{145} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(6-2)^2+(-7-3)^2} \\ &= \sqrt{16+100} \\ &= \sqrt{116 \text{ units}} \end{aligned}$$

Now $(\sqrt{29})^2+(\sqrt{116})^2 = (\sqrt{145})^2$

$\therefore$ $AB^2 + BC^2 = AC^2$

$\therefore$ the triangle is right angled at B.

2 Suppose the outlet is at T(x, 8).

Then AT = BT, so $\sqrt{(x-5)^2+(8-5)^2} = \sqrt{(x-7)^2+(8-10)^2}$

$$\therefore\ (x-5)^2+(8-5)^2 = (x-7)^2+(8-10)^2$$
$$\therefore\ x^2-10x+25+9 = x^2-14x+49+4$$
$$\therefore\ -10x+34 = -14x+53$$
$$\therefore\ 4x = 19$$
$$\therefore\ x = \tfrac{19}{4} \qquad \therefore \text{ the outlet is at } (4\tfrac{3}{4}, 8)$$

3 **a** Suppose C is at $(x, 7)$

Then AC = BC, so $\sqrt{(x-2)^2+(7-3)^2} = \sqrt{(x-5)^2+(7-4)^2}$

$$\therefore\ (x-2)^2+(7-3)^2 = (x-5)^2+(7-4)^2$$
$$\therefore\ x^2-4x+4+16 = x^2-10x+25+9$$
$$\therefore\ -4x+20 = -10x+34$$
$$\therefore\ 6x = 14$$
$$\therefore\ x = \tfrac{7}{3} \qquad \therefore \text{ the pumping station is at } (\tfrac{7}{3}, 7)$$

b The length of each pipe $= \sqrt{(x-2)^2+(7-3)^2}$

$$= \sqrt{(\tfrac{7}{3}-2)^2+16}$$
$$= \sqrt{16\tfrac{1}{9}} \qquad \therefore \text{ the total length } = 2\sqrt{16\tfrac{1}{9}} \doteqdot 8.03 \text{ km}$$

c Now CD is horizontal, so BD is vertical $\therefore$ the x-coordinate of D is 5

$\therefore$ D is at (5, 7)

Then AB + BD $= \sqrt{(5-2)^2+(4-3)^2} + \sqrt{(5-5)^2+(7-4)^2}$

$$= \sqrt{9+1}+\sqrt{0+9}$$
$$= \sqrt{10}+3$$
$$\doteqdot 6.16 \text{ km}$$

Hence yes, it would be much cheaper.

4 **a** $\text{PQ} = \sqrt{(-3-3)^2+(6-a)^2}$

$$\therefore\ \sqrt{(-6)^2+(6-a)^2} = 10$$
$$\therefore\ 36+(6-a)^2 = 100$$
$$\therefore\ (6-a)^2 = 64$$
$$\therefore\ 6-a = \pm 8$$
$$\text{i.e., } 6-a = -8 \quad \text{or} \quad 6-a = 8$$
$$\therefore\ a = 14 \quad \text{or} \quad a = -2$$
$$\text{i.e., } a = -2 \text{ or } 14$$

b $\text{AB} = \sqrt{(-1-a)^2+(5-2)^2}$

$$\therefore\ \sqrt{(-1-a)^2+9} = \sqrt{34}$$
$$\therefore\ (-1-a)^2+9 = 34$$
$$\therefore\ (-1-a)^2 = 25$$
$$\therefore\ -1-a = \pm 5$$
$$\text{i.e., } -1-a = 5 \quad \text{or} \quad -1-a = -5$$
$$\therefore\ a = -6 \quad \text{or} \quad a = 4$$
$$\text{i.e., } a = -6 \text{ or } 4$$

c
$$\begin{aligned} \text{QR} &= \sqrt{(-1-a)^2+(1-2a)^2} \\ \therefore\ \sqrt{(-1-a)^2+(1-2a)^2} &= \sqrt{5} \\ \therefore\ (-1-a)^2+(1-2a)^2 &= 5 \\ \therefore\ 1+2a+a^2+1-4a+4a^2 &= 5 \\ \therefore\ 5a^2-2a-3 &= 0 \\ \therefore\ (5a+3)(a-1) &= 0 \\ \therefore\ a &= -\tfrac{3}{5} \text{ or } 1 \end{aligned}$$

5 Suppose Jason's girlfriend lives at G(x, 8)

Then
$$\begin{aligned} \text{GJ} = \sqrt{(x-4)^2+(8-1)^2} &= 11.73 \\ \therefore\ (x-4)^2+49 &= 11.73^2 \\ \therefore\ (x-4)^2 &= 88.5929 \\ \therefore\ x-4 &\doteqdot \pm 9.412 \\ \therefore\ x &\doteqdot -5.412 \text{ or } 13.412 \end{aligned}$$

i.e., Jason's girlfriend lives at $(-5.412, 8)$ or $(13.41, 8)$

6 **a** Any point on the x-axis can be represented by P(a, 0).

$\therefore$ given that AP = BP,

$$\begin{aligned} \sqrt{(a-(-3))^2+(0-2)^2} &= \sqrt{(a-5)^2+(0-(-1))^2} \\ \therefore\ (a+3)^2+4 &= (a-5)^2+1 \\ \therefore\ a^2+6a+13 &= a^2-10a+26 \\ \therefore\ 16a &= 13 \\ \therefore\ a &= \tfrac{13}{16} \end{aligned}$$

Hence the point is $(\frac{13}{16}, 0)$

b Any point on the y-axis can be represented by P(0, b).

$\therefore$ given that CP = DP,

$$\begin{aligned} \sqrt{(0-5)^2+(b-0)^2} &= \sqrt{(0-(-1))^2+(b-(-3))^2} \\ \therefore\ 25+b^2 &= 1+(b+3)^2 \\ \therefore\ 25+b^2 &= 1+b^2+6b+9 \\ \therefore\ 6b &= 15 \\ \therefore\ b &= \tfrac{5}{2} \end{aligned}$$

Hence the point is $(0, \frac{5}{2})$.

c Let $(a, 3a)$ lie on the line $y = 3x$.

Since this point is 5 units from (2, 1),

$$\begin{aligned} \sqrt{(a-2)^2+(3a-1)^2} &= 5 \\ \therefore\ (a-2)^2+(3a-1)^2 &= 25 \\ \therefore\ a^2-4a+4+9a^2-6a+1 &= 25 \\ \therefore\ 10a^2-10a-20 &= 0 \\ \therefore\ a^2-a-2 &= 0 \\ \therefore\ (a-2)(a+1) &= 0 \\ \therefore\ a &= -1 \text{ or } 2 \end{aligned}$$

$\therefore$ the points are $(-1, -3)$ and $(2, 6)$

EXERCISE 7D.1

1 B is the midpoint of PQ, so it lies at $\left(\dfrac{3+11}{2}, \dfrac{-2-6}{2}\right)$ i.e., B is at $(7, -4)$

A is the midpoint of BP, so it lies at $\left(\dfrac{3+7}{2}, \dfrac{-2-4}{2}\right)$ i.e., A is at $(5, -3)$

C is the midpoint of BQ, so it lies at $\left(\dfrac{7+11}{2}, \dfrac{-4-6}{2}\right)$ i.e., C is at $(9, -5)$

2 B is the midpoint of AC.

$\therefore$ if C is at (a, b), then B is at $\left(\dfrac{-1+a}{2}, \dfrac{4+b}{2}\right)$

But B is at $(1, 1)$ $\quad \therefore \dfrac{-1+a}{2} = 1$ and $\dfrac{4+b}{2} = 1$

$\therefore a - 1 = 2$ and $b + 4 = 2$

$\therefore a = 3$ and $b = -2$ $\quad \therefore$ C is at $(3, -2)$

C is the midpoint of AE.

$\therefore$ if E is at (c, d), then C is at $\left(\dfrac{-1+c}{2}, \dfrac{4+d}{2}\right)$ $\quad \therefore \dfrac{-1+c}{2} = 3$ and $\dfrac{4+d}{2} = -2$

$\therefore c - 1 = 6$ and $d + 4 = -4$

$\therefore c = 7$ and $d = -8$

$\therefore$ E is at $(7, -8)$

D is the midpoint of CE, so it lies at $\left(\dfrac{3+7}{2}, \dfrac{-2-8}{2}\right)$ i.e., D is at $(5, -5)$

3 **a** The midpoint of AC is $\left(\dfrac{-1-3}{2}, \dfrac{-2+2}{2}\right)$, i.e., $(-2, 0)$

The midpoint of BD is $\left(\dfrac{0-4}{2}, \dfrac{1-1}{2}\right)$, i.e., $(-2, 0)$

Hence the diagonals bisect one another.

b AB has slope $\dfrac{1-(-2)}{0-(-1)} = 3$ and CD has slope $\dfrac{-1-2}{-4-(-3)} = \dfrac{-3}{-1} = 3$

Hence AB is parallel to CD.

BC has slope $\dfrac{2-1}{-3-0} = -\dfrac{1}{3}$ and AD has slope $\dfrac{-1-(-2)}{-4-(-1)} = \dfrac{1}{-3}$

Hence BC is parallel to AD.

Hence ABCD is a parallelogram.

4 **a** AB has slope $= \dfrac{-2-6}{-1-3} = \dfrac{-8}{-4} = 2$

AC has slope $= \dfrac{4-6}{7-3} = \dfrac{-2}{4} = -\dfrac{1}{2}$

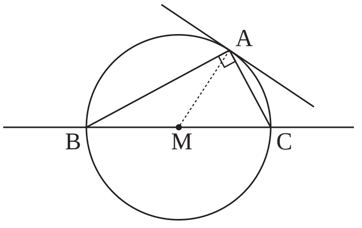
The product of the slopes is -1, so AB and AC are perpendicular.

Hence $\measuredangle$BAC is a right angle.

b The midpoint of BC is $\left(\dfrac{-1+7}{2}, \dfrac{-2+4}{2}\right)$ i.e., $(3, 1)$

c Since triangle ABC is right angled at A, BC must be a diameter of the circle. (This makes $\measuredangle$BAC the angle in the semicircle.)

$\therefore$ the midpoint of BC found in **b** is the centre of the circle.



Now the slope of AM $= \dfrac{1-6}{3-3}$ which is undefined.

i.e., AM is vertical.

Hence the tangent must be horizontal.

$\therefore$ since it passes through A,

its equation is $y = 6$.

5 **a** Since ABCD is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of DB is the same as the midpoint of AC.

Letting D have coordinates (a, b),

$$\frac{a+2}{2} = \frac{3+8}{2} \quad \text{and} \quad \frac{b-1}{2} = \frac{0-2}{2}$$

$$\therefore\ a+2 = 11 \quad \text{and} \quad b-1 = -2$$

$$\therefore\ a = 9 \quad \text{and} \quad b = -1$$

$\therefore$ D is at $(9, -1)$

b Since PQRS is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of PR is the same as the midpoint of QS.

Letting R have coordinates (a, b),

$$\frac{-1+a}{2} = \frac{4-2}{2} \quad \text{and} \quad \frac{4+b}{2} = \frac{0+5}{2}$$

$$\therefore\ a-1 = 2 \quad \text{and} \quad b+4 = 5$$

$$\therefore\ a = 3 \quad \text{and} \quad b = 1$$

$\therefore$ R is at $(3, 1)$

c Since WXYZ is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of WY is the same as the midpoint of XZ.

Letting X have coordinates (a, b),

$$\frac{-1+3}{2} = \frac{a+0}{2} \quad \text{and} \quad \frac{5-2}{2} = \frac{b+4}{2}$$

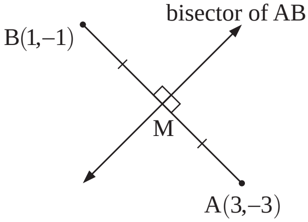
$$\therefore\ 2 = a \quad \text{and} \quad 3 = b+4$$

$$\therefore\ a = 2 \quad \text{and} \quad b = -1$$

$\therefore$ X is at $(2, -1)$

EXERCISE 7D.2

1 **a**



The midpoint M of AB is at $\left(\dfrac{3+1}{2}, \dfrac{-3-1}{2}\right)$,

i.e., at $(2, -2)$

The slope of AB is $\dfrac{-1-(-3)}{1-3} = \dfrac{2}{-2} = -1$

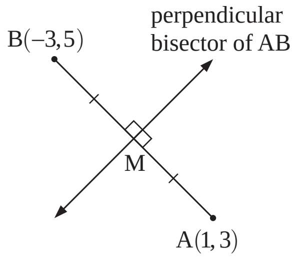
$\therefore$ the slope of the perpendicular is 1

$\therefore$ the equation of the perpendicular bisector is

$$x - y = (2) - (-2)$$

i.e., $x - y = 4$

b



The midpoint M of AB is at $\left(\frac{1-3}{2}, \frac{3+5}{2}\right)$,
i.e., at $(-1, 4)$

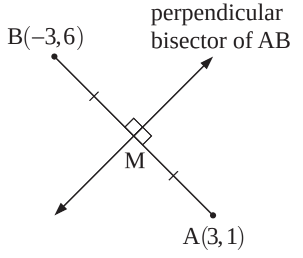
The slope of AB is $\frac{5-3}{-3-1} = -\frac{2}{4} = -\frac{1}{2}$

$\therefore$ the slope of the perpendicular is 2

$\therefore$ the equation of the perpendicular bisector is

$$2x - y = 2(-1) - 4$$
i.e., $2x - y = -6$

c



The midpoint M of AB is at $\left(\frac{3-3}{2}, \frac{1+6}{2}\right)$,
i.e., at $(0, \frac{7}{2})$

The slope of AB is $\frac{6-1}{-3-3} = -\frac{5}{6}$

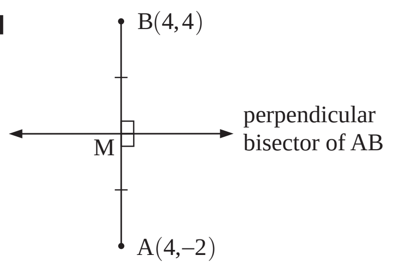
$\therefore$ the slope of the perpendicular is $\frac{6}{5}$

$\therefore$ the equation of the perpendicular bisector is

$$6x - 5y = 6(0) - 5(\tfrac{7}{2})$$
i.e., $6x - 5y = -\frac{35}{2}$

i.e., $12x - 10y = -35$

d



The midpoint M of AB is at $\left(\frac{4+4}{2}, \frac{-2+4}{2}\right)$,
i.e., at $(4, 1)$

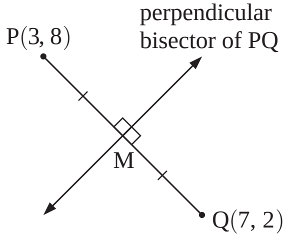
The slope of AB is $\frac{4-(-2)}{4-4}$ which is undefined

$\therefore$ AB is vertical

$\therefore$ its perpendicular bisector is horizontal

$\therefore$ it has equation $y = 1$

2



The midpoint M of PQ is at $\left(\frac{3+7}{2}, \frac{8+2}{2}\right)$,
i.e., at $(5, 5)$

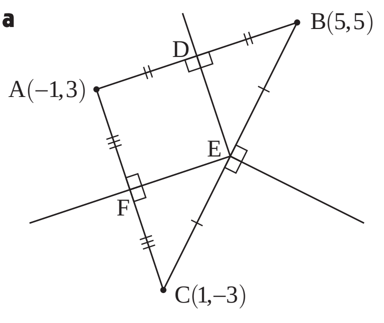
The slope of PQ is $\frac{2-8}{7-3} = \frac{-6}{4} = -\frac{3}{2}$

$\therefore$ the slope of the perpendicular is $\frac{2}{3}$

$\therefore$ the equation of the perpendicular bisector which is to be the boundary between the regions is

$$2x - 3y = 2(5) - 3(5)$$
i.e., $2x - 3y = -5$

3 **a**



Suppose D, E and F are the midpoints of AB, BC and AC respectively.

Then D is at $\left(\frac{-1+5}{2}, \frac{3+5}{2}\right)$, i.e., $(2, 4)$

E is at $\left(\frac{5+1}{2}, \frac{5-3}{2}\right)$, i.e., $(3, 1)$

and F is at $\left(\frac{1-1}{2}, \frac{-3+3}{2}\right)$, i.e., $(0, 0)$

Now AB has slope $\dfrac{5-3}{5-(-1)} = \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore$ the perpendicular bisector through D has slope -3

and hence equation $3x + y = 3(2) + 4$ i.e., $3x + y = 10$

BC has slope $\dfrac{5-(-3)}{5-1} = \dfrac{8}{4} = 2$

$\therefore$ the perpendicular bisector through E has slope $-\frac{1}{2}$,

and hence equation $x + 2y = (3) + 2(1)$ i.e., $x + 2y = 5$

AC has slope $\dfrac{-3-3}{1-(-1)} = \dfrac{-6}{2} = -3$

$\therefore$ the perpendicular bisector through F has slope $\frac{1}{3}$,

and hence equation $x - 3y = 0 - 3(0)$ i.e., $x - 3y = 0$

b The vertex should lie at the intersection of the three edges:
$3x + y = 10$, $x + 2y = 5$ and $x - 3y = 0$.

For $3x + y = 10$,
when $x = 0$, $y = 10$
when $y = 0$, $x = \frac{10}{3}$

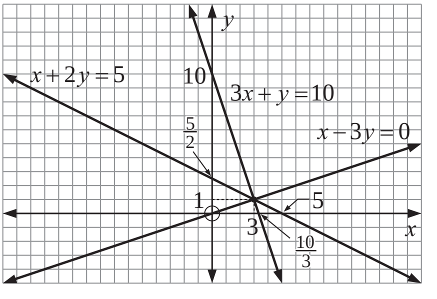
x	0	$\frac{10}{3}$
y	10	0

For $x + 2y = 5$,
when $x = 0$, $y = \frac{5}{2}$
when $y = 0$, $x = 5$

x	0	5
y	$\frac{5}{2}$	0

For $x - 3y = 0$,
when $x = 0$, $y = 0$
when $x = 3$, $y = 1$

x	0	3
y	0	1



All of the graphs meet at (3, 1), so this is the location of the Voronoi vertex.

4 The midpoint of PQ is $\left(\dfrac{5+7}{2}, \dfrac{7+1}{2}\right)$, i.e., (6, 4)

PQ has slope $\dfrac{1-7}{7-5} = \dfrac{-6}{2} = -3$

$\therefore$ its bisector has slope $\frac{1}{3}$ and equation $x - 3y = (6) - 3(4)$
i.e., $x - 3y = -6$ ...... (1)

The midpoint of QR is $\left(\dfrac{7-1}{2}, \dfrac{1+5}{2}\right)$, i.e., (3, 3)

QR has slope $\dfrac{5-1}{-1-7} = \dfrac{4}{-8} = -\dfrac{1}{2}$

$\therefore$ its bisector has slope 2 and equation $2x - y = 2(3) - (3)$
i.e., $2x - y = 3$ ...... (2)

From (1), $x = 3y - 6$

$\therefore$ in (2), $2(3y-6) - y = 3$

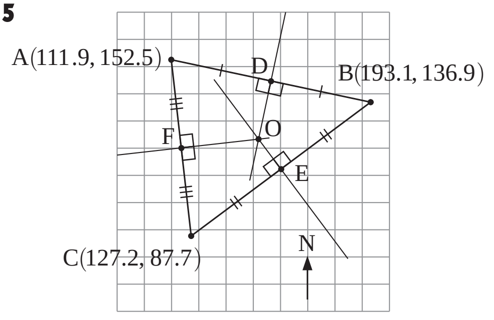
$\therefore$ $6y - 12 - y = 3$

$\therefore$ $5y = 15$

$\therefore$ $y = 3$ and so $x = 9 - 6 = 3$

$\therefore$ the centre of the circle is (3, 3)

5



The oval must be located at the intersection of the perpendicular bisectors between the three towns.

Let D, E and F be the midpoints of AB, BC, and AC respectively.

Then D is at $\left(\dfrac{111.9+193.1}{2}, \dfrac{152.5+136.9}{2}\right)$, i.e., $(152.5, 144.7)$

E is at $\left(\dfrac{127.2+193.1}{2}, \dfrac{87.7+136.9}{2}\right)$, i.e., $(160.15, 112.3)$

F is at $\left(\dfrac{127.2+111.9}{2}, \dfrac{87.7+152.5}{2}\right)$, i.e., $(119.55, 120.1)$

Now we know that the three bisectors meet at a common point, so we need to find the equations of any two of them.

AB has slope $\dfrac{136.9-152.5}{193.1-111.9} = \dfrac{-15.6}{81.2}$, so DO has slope $\dfrac{81.2}{15.6}$

and equation $81.2x - 15.6y = 81.2(152.5) - 15.6(144.7)$

i.e., $81.2x - 15.6y \doteqdot 10\,126$ ...... (1)

BC has slope $\dfrac{87.7-136.9}{127.2-193.1} = \dfrac{-49.2}{-65.9} = \dfrac{49.2}{65.9}$, so EO has slope $-\dfrac{65.9}{49.2}$

and equation $65.9x + 49.2y = 65.9(160.15) + 49.2(112.3)$

i.e., $65.9x + 49.2y \doteqdot 16\,079$ ...... (2)

The location of the oval will be at the intersection of the lines (1) and (2).
We can find this point graphically, or using algebra as follows:

$$\begin{aligned}
49.2 \times (1):&\quad 3995.04x - 767.52y = 498\,199.2\\
15.6 \times (2):&\quad 1028.04x + 767.52y = 250\,832.4\\
\hline
&\quad 5023.08x = 749\,031.6\\
&\quad \therefore\ x \doteqdot 149.12
\end{aligned}$$

Using (2), $49.2y \doteqdot 16\,079 - 65.9 \times 149.12$

$\therefore\ y \doteqdot 127.08$

$\therefore$ the oval should be at $(149.1, 127.1)$

6 **a** P is the midpoint of OA, and lies at $\left(\dfrac{2a+0}{2}, \dfrac{2c+0}{2}\right)$, i.e., (a, c)

Q is the midpoint of AB, and lies at $\left(\dfrac{2a+2b}{2}, \dfrac{2c+0}{2}\right)$, i.e., $(a+b, c)$

Now OA has slope $\dfrac{2c}{2a} = \dfrac{c}{a}$, so the perpendicular bisector through P has slope $-\dfrac{a}{c}$

$\therefore$ it has equation $ax + cy = a(a) + c(c)$,

i.e., $ax + cy = a^2 + c^2$

AB has slope $\dfrac{0-2c}{2b-2a} = -\dfrac{c}{b-a}$, so the perpendicular bisector through Q has slope $\dfrac{b-a}{c}$

$\therefore$ it has equation $(b-a)x - cy = (b-a)(a+b) - c(c)$
$= ba + b^2 - a^2 - ab - c^2$

i.e., $(b-a)x - cy = b^2 - a^2 - c^2$

b From **a**, S lies at the intersection of $ax + cy = a^2 + c^2$ ...... (1)
and $(b-a)x - cy = b^2 - a^2 - c^2$ ...... (2)

Adding (1) and (2) gives $(b-a+a)x + (c-c)y = a^2 + c^2 + b^2 - a^2 - c^2$
$\therefore\ bx = b^2$
$\therefore\ x = b$

c Since R and S have the same x-coordinate, RS is the vertical line $x = b$.
$\therefore$ RS is perpendicular to OB, which is part of the x-axis.

d The perpendicular bisectors of the sides of a triangle intersect at a point.

REVIEW SET 7A

1 PQ $= \sqrt{(-1-(-4))^2 + (3-7)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$ units

2 The midpoint is at $\left(\dfrac{3+7}{2}, \dfrac{5-2}{2}\right)$,
i.e., at $(5, \frac{3}{2})$

3 **a** The equation is $\dfrac{y-(-1)}{x-2} = -3$ $\therefore\ y + 1 = -3(x-2)$
$\therefore\ y + 1 = -3x + 6$
$\therefore\ y = -3x + 5$

b The line has slope $\dfrac{4-(-2)}{-1-3} = \dfrac{6}{-4} = -\dfrac{3}{2}$ $\therefore$ its equation is $\dfrac{y-(-2)}{x-3} = -\dfrac{3}{2}$
$\therefore\ y + 2 = -\frac{3}{2}(x-3)$
$\therefore\ y + 2 = -\frac{3}{2}x + \frac{9}{2}$
$\therefore\ y = -\frac{3}{2}x + \frac{5}{2}$

4 **a** The equation is $2x - 3y = 2(1) - 3(-5)$
i.e., $2x - 3y = 17$

b The line has slope $\dfrac{-5-(-3)}{-4-2} = \dfrac{-2}{-6} = \dfrac{1}{3}$ $\therefore$ its equation is $x - 3y = (2) - 3(-3)$
i.e., $x - 3y = 11$

5 **a** If $y = -\frac{3}{2}x + 7$, when $x = 0$, $y = 7$ i.e., the y-intercept is 7
and when $y = 0$, $\frac{3}{2}x = 7$ $\therefore\ x = \frac{14}{3}$ i.e., the x-intercept is $\frac{14}{3}$

b If $5x - 3y = 12$, when $x = 0$, $y = -4$ i.e., the y-intercept is -4
and when $y = 0$, $5x = 12$ $\therefore\ x = \frac{12}{5}$ i.e., the x-intercept is $\frac{12}{5}$

6 If $(2, -5)$ lies on $3x + 4y = -14$,
then $3(2) + 4(-5) = -14$
$\therefore\ 6 - 20 = -14$ which is true

Hence $(2, -5)$ lies on $3x + 4y = -14$

7 **a** The line $4x - 3y = 6$ has slope $\frac{4}{3}$

$\therefore$ any line parallel to it also has slope $\frac{4}{3}$.

But the line must pass through P(1, −5).

$\therefore$ its equation is $4x - 3y = 4(1) - 3(-5)$

i.e., $4x - 3y = 19$

b The line $y = -4x + 7$ has slope -4

$\therefore$ any line perpendicular to it has slope $\frac{1}{4}$.

But the line must pass through Q(−2, 1).

$\therefore$ its equation is $x - 4y = (-2) - 4(1)$

i.e., $x - 4y = -6$

8 **a** **i** AE has slope $\dfrac{2-2}{7-0} = 0$ so its equation is $y = 2$.

ii CE is perpendicular to AE, so its equation is $x = 0$.

iii BX has slope $\dfrac{7-10}{7-3} = -\dfrac{3}{4}$ $\therefore$ Diagonal Rd has slope $\frac{4}{3}$

$\therefore$ since it passes through (3, 10), its equation is $4x - 3y = 4(3) - 3(10)$

i.e., $4x - 3y = -18$

b **i** Diagonal Rd intersects East Ave when $y = 2$ $\therefore$ $4x - 6 = -18$

$\therefore$ $4x = -12$

$\therefore$ $x = -3$ i.e., at (−3, 2)

ii Diagonal Rd intersects North St when $x = 0$ $\therefore$ $0 - 3y = -18$

$\therefore$ $y = 6$ i.e., at (0, 6)

9 $3x + ky = 7$ has slope $-\dfrac{3}{k}$ $y = 3 - 4x$ has slope -4

a The lines are parallel if the slopes are equal $\therefore$ $-\dfrac{3}{k} = -4$ $\therefore$ $k = \frac{3}{4}$

b The lines are perpendicular if the product of their slopes is −1,

$\therefore$ $\left(-\dfrac{3}{k}\right)(-4) = -1$ $\therefore$ $\dfrac{12}{k} = -1$ $\therefore$ $k = -12$

10 The distance from (0, 4) to (x, y) is three times the distance from (4, 0) to (x, y)

$$\therefore \sqrt{(x-0)^2 + (y-4)^2} = 3\sqrt{(x-4)^2 + (y-0)^2}$$

$$\therefore (x-0)^2 + (y-4)^2 = 9\left((x-4)^2 + (y-0)^2\right)$$

$$\therefore x^2 + y^2 - 8y + 16 = 9x^2 - 72x + 144 + 9y^2$$

$$\therefore 8x^2 + 8y^2 - 72x + 8y + 128 = 0$$

$$\therefore x^2 + y^2 - 9x + y + 16 = 0$$

REVIEW SET 7B

1 **a** $3x + 5y = 7$ has slope $-\frac{3}{5}$

$\therefore$ any line parallel to it must also have slope $-\frac{3}{5}$

But the line required passes through (−1, 3)

$\therefore$ its equation is $3x + 5y = 3(-1) + 5(3)$ i.e., $3x + 5y = 12$

b $2x - 7y = 5$ has slope $\frac{2}{7}$

$\therefore$ any line perpendicular to it must have slope $-\frac{7}{2}$

But the line required passes through (4, 2) $\therefore$ its equation is $7x + 2y = 7(4) + 2(2)$

i.e., $7x + 2y = 32$

2 $5x - 7y = 8$ has slope $\frac{5}{7}$ $\qquad$ $3x + ky = -11$ has slope $-\dfrac{3}{k}$

a If the lines are parallel, their slopes are equal $\quad \therefore \ \dfrac{5}{7} = -\dfrac{3}{k}$

$$\therefore \ k = -\tfrac{21}{5}$$

b If the lines are perpendicular, the product of their slopes is -1.

$$\therefore \ \frac{5}{7}\left(-\frac{3}{k}\right) = -1 \qquad \therefore \ -\frac{15}{7k} = -1$$
$$\therefore \ 7k = 15$$
$$\therefore \ k = \tfrac{15}{7}$$

3 AB has slope $\dfrac{7-2}{1-(-3)} = \dfrac{5}{4}$ $\quad \therefore$ its equation is $\dfrac{y-7}{x-1} = \dfrac{5}{4}$

$$\therefore \ y - 7 = \tfrac{5}{4}(x-1)$$
$$\therefore \ y - 7 = \tfrac{5}{4}x - \tfrac{5}{4}$$
$$\therefore \ y = \tfrac{5}{4}x + \tfrac{23}{4}$$

$\therefore$ this line meets $3x + 2y = 6$ when $\quad 3x + 2\left(\frac{5}{4}x + \frac{23}{4}\right) = 6$

$$\therefore \ 3x + \tfrac{5}{2}x + \tfrac{23}{2} = 6$$
$$\therefore \ \tfrac{11}{2}x = -\tfrac{11}{2}$$
$$\therefore \ x = -1$$
$$\therefore \ y = \tfrac{5}{4}(-1) + \tfrac{23}{4} = \tfrac{18}{4} = \tfrac{9}{2}$$

$\therefore$ the lines intersect at $(-1, \frac{9}{2})$

4
$$\begin{aligned} \text{KL} &= \sqrt{(0-(-5))^2 + (1-(-2))^2} \\ &= \sqrt{25+9} \\ &= \sqrt{34} \text{ units} \end{aligned} \qquad \begin{aligned} \text{KM} &= \sqrt{(3-(-5))^2 + (-4-(-2))^2} \\ &= \sqrt{64+4} \\ &= \sqrt{68} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{LM} &= \sqrt{(3-0)^2 + (-4-1)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

$\therefore$ since LM = KL, the triangle is isosceles and since $\text{KL}^2 + \text{LM}^2 = \text{KM}^2$, the triangle is also right angled at L.

i.e., KLM is a right angled isosceles triangle, with the right angle being at L.

5 Suppose the power outlet is at P$(a, 5)$.

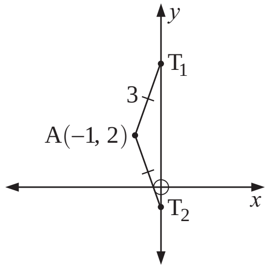
Now AP = BP $\quad \therefore \ \sqrt{(a-3)^2 + (5-2)^2} = \sqrt{(a-5)^2 + (5-7)^2}$

$$\therefore \ (a-3)^2 + (5-2)^2 = (a-5)^2 + (5-7)^2$$
$$\therefore \ a^2 - 6a + 9 + 9 = a^2 - 10a + 25 + 4$$
$$\therefore \ 4a = 11$$
$$\therefore \ a = \tfrac{11}{4} \qquad \therefore \text{ the point is } (\tfrac{11}{4}, 5)$$

6 Now $\sqrt{(2-(-3))^2 + (4-k)^2} = 7 \quad \therefore \ 25 + (4-k)^2 = 49$

$$\therefore \ (4-k)^2 = 24$$
$$\therefore \ 4 - k = \pm\sqrt{24}$$
$$\text{i.e., } \ 4 - k = 2\sqrt{6} \quad \text{or} \quad 4 - k = -2\sqrt{6}$$
$$\therefore \ k = 4 - 2\sqrt{6} \quad \text{or} \quad k = 4 + 2\sqrt{6}$$
$$\therefore \ k = 4 \pm 2\sqrt{6}$$

7 **a**



Suppose T is at $(0, b)$

$$\therefore \quad \sqrt{(-1-0)^2+(2-b)^2} = 3$$
$$\therefore \quad 1+(2-b)^2 = 9$$
$$\therefore \quad (b-2)^2 = 8$$
$$\therefore \quad b-2 = \pm 2\sqrt{2}$$
$$\therefore \quad b = 2 \pm 2\sqrt{2}$$

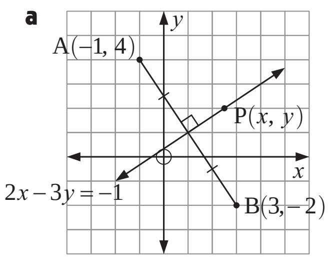
$\therefore$ T$_1$ is $(0,\ 2+2\sqrt{2})$, T$_2$ is $(0,\ 2-2\sqrt{2})$

b slope $AT_1 = \dfrac{(2+2\sqrt{2})-2}{0-(-1)} = \dfrac{2\sqrt{2}}{1} = 2\sqrt{2}$

$\therefore$ the line has equation $2\sqrt{2}x - y = 2\sqrt{2}(-1) - 2$

i.e., $2\sqrt{2}x - y = -2\sqrt{2} - 2$

8 **a**



b The midpoint of AB is $\left(\dfrac{-1+3}{2}, \dfrac{4-2}{2}\right)$, i.e., $(1, 1)$

AB has slope $\dfrac{-2-4}{3-(-1)} = \dfrac{-6}{4} = -\dfrac{3}{2}$

$\therefore$ the perpendicular bisector has slope $\frac{2}{3}$

$\therefore$ its equation is $2x - 3y = 2(1) - 3(1)$

i.e., $2x - 3y = -1$

9 Since ABCD is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of DB is the same as the midpoint of AC.

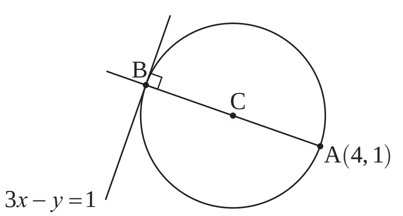
Letting C be at (a, b),

$$\frac{-4+2}{2} = \frac{-5+a}{2} \quad \text{and} \quad \frac{-1+5}{2} = \frac{4+b}{2}$$

$$\therefore \quad -2 = a-5 \quad \text{and} \quad 4 = 4+b$$

$$\therefore \quad a = 3 \quad \text{and} \quad b = 0 \qquad \therefore \text{ C is at } (3, 0)$$

10



Now $3x - y = 1$ has slope 3

$\therefore$ the diameter AB has slope $-\frac{1}{3}$

$\therefore$ since it passes through $(4, 1)$,

its equation is $x + 3y = (4) + 3(1)$

i.e., $x + 3y = 7$

The tangent meets the diameter when $3x - y = 1$ ..... (1) meets $x + 3y = 7$ ..... (2)

Using (1), $y = 3x - 1$ $\therefore$ in (2), $x + 3(3x-1) = 7$

$$\therefore \quad x + 9x - 3 = 7$$
$$\therefore \quad 10x = 10$$
$$\therefore \quad x = 1$$

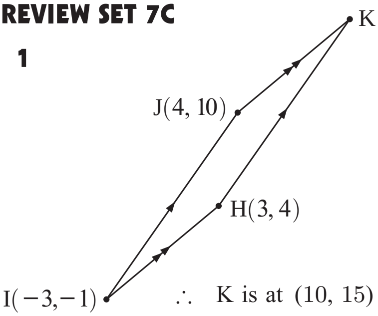
$\therefore$ using (1), $y = 3 \times 1 - 1 = 2$ $\therefore$ B is at $(1, 2)$

The centre of the circle, C, is the midpoint of AB.

$\therefore$ C is at $\left(\dfrac{4+1}{2}, \dfrac{1+2}{2}\right)$ i.e., $\left(\frac{5}{2}, \frac{3}{2}\right)$

REVIEW SET 7C

1



Since HIJK is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of IK is the same as the midpoint of HJ.

Letting K be at (a, b),

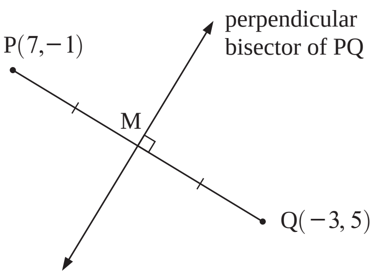
$$\frac{a-3}{2} = \frac{3+4}{2} \quad \text{and} \quad \frac{b-1}{2} = \frac{4+10}{2}$$

$$\therefore \quad a-3 = 7 \quad \text{and} \quad b-1 = 14$$

$$\therefore \quad a = 10 \quad \text{and} \quad b = 15$$

$\therefore$ K is at $(10, 15)$

2



The midpoint M of PQ is at

$\left(\frac{7-3}{2}, \frac{-1+5}{2}\right)$ i.e., at $(2, 2)$

The slope of PQ is $\dfrac{5-(-1)}{-3-7} = \dfrac{6}{-10} = -\dfrac{3}{5}$

$\therefore$ the slope of the perpendicular is $\frac{5}{3}$

$\therefore$ the equation of the perpendicular bisector is

$$5x - 3y = 5(2) - 3(2)$$

i.e., $5x - 3y = 4$

3 If the three points are collinear, then the slope of AB equals the slope of AC.

$$\therefore \quad \frac{8-2}{6-(-3)} = \frac{k-2}{2-(-3)} \quad \text{and so} \quad \frac{6}{9} = \frac{k-2}{5}$$

$$\therefore \quad k-2 = \tfrac{30}{9} = \tfrac{10}{3}$$

$$\therefore \quad k = \tfrac{16}{3}$$

4 Since AB is a diameter, the centre of the circle is the midpoint of AB.

$\therefore$ if A has coordinates (a, b), then $\dfrac{a+6}{2} = -3$ and $\dfrac{b+1}{2} = 4$

$$\therefore \quad a+6 = -6 \quad \text{and} \quad b+1 = 8$$

$$\therefore \quad a = -12 \quad \text{and} \quad b = 7$$

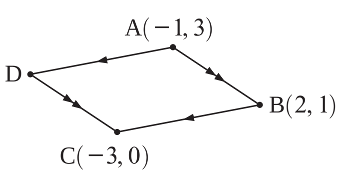
$\therefore$ A is at $(-12, 7)$

5 **a** AB has slope $= \dfrac{1-3}{2-(-1)} = -\dfrac{2}{3}$

$\therefore$ the line through C must also have slope $-\frac{2}{3}$

$\therefore$ its equation is $2x + 3y = 2(-3) + 3(0)$ i.e., $2x + 3y = -6$

b



Since ABCD is a parallelogram, the diagonals bisect each other.

$\therefore$ the midpoint of AC is the same as the midpoint of BD.

Letting D be at (a, b),

$$\frac{-1-3}{2} = \frac{2+a}{2} \quad \text{and} \quad \frac{3+0}{2} = \frac{1+b}{2}$$

$$\therefore \quad -4 = a+2 \quad \text{and} \quad 3 = b+1$$

$$\therefore \quad a = -6 \quad \text{and} \quad b = 2$$

$\therefore$ D is at $(-6, 2)$

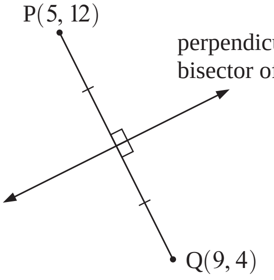
6 Suppose A has x-coordinate a, so its y-coordinate must be $a-1$.

Since it is $2\sqrt{2}$ units from B(4, -1),

$$\begin{aligned} \sqrt{(a-4)^2+((a-1)-(-1))^2} &= 2\sqrt{2} \\ \therefore\ \sqrt{(a-4)^2+a^2} &= 2\sqrt{2} \\ \therefore\ a^2-8a+16+a^2 &= 8 \\ \therefore\ 2a^2-8a+8 &= 0 \\ \therefore\ a^2-4a+4 &= 0 \\ \therefore\ (a-2)^2 &= 0 \\ \therefore\ a &= 2 \end{aligned}$$

Hence A is at (2, 1).

7



The midpoint M of PQ is at $\left(\dfrac{5+9}{2}, \dfrac{12+4}{2}\right)$

i.e., at (7, 8)

The slope of PQ is $\dfrac{4-12}{9-5} = \dfrac{-8}{4} = -2$

$\therefore$ the slope of the perpendicular bisector is $\frac{1}{2}$

$\therefore$ its equation is $x-2y = (7)-2(8)$

i.e., $x-2y = -9$

Hence the equation of the boundary between the two regions is $x-2y=-9$.

8 **a** XY has slope $\dfrac{-3-5}{8-2} = \dfrac{-8}{6} = -\dfrac{4}{3}$

$\therefore$ its equation is $4x+3y = 4(2)+3(5)$

i.e., $4x+3y = 23$ ...... (1)

Now PT is perpendicular to XY, so its slope is $\frac{3}{4}$.

$\therefore$ its equation is $3x-4y = 3(9)-4(4)$

i.e., $3x-4y = 11$ ...... (2)

T lies at the intersection of (1) and (2).

$$\begin{aligned} (1)\times 4: \quad 16x+12y &= 92 \\ (2)\times 3: \quad 9x-12y &= 33 \\ \hline 25x \qquad &= 125 \\ \therefore\ x &= 5 \end{aligned}$$

$\therefore$ using (1), $20+3y = 23$

$\therefore\ 3y = 3$

$\therefore\ y = 1$

Hence T is at (5, 1)

b $$\begin{aligned} \text{PT} &= \sqrt{(9-5)^2+(4-1)^2} \\ &= \sqrt{16+9} \\ &= 5 \text{ units} \\ &\equiv 50 \text{ km} \end{aligned}$$

9 **a** The radius equals the length of AB, so

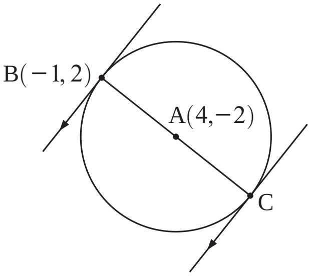
$$\begin{aligned} \text{radius} &= \sqrt{(-1-4)^2+(2-(-2))^2} \\ &= \sqrt{25+16} \\ &= \sqrt{41} \text{ units} \end{aligned}$$

b slope of AB $= \dfrac{2-(-2)}{-1-4} = -\dfrac{4}{5}$

c Using **b**, the slope of the tangent at B is $\frac{5}{4}$.

$\therefore$ its equation is $5x - 4y = 5(-1) - 4(2)$
i.e., $5x - 4y = -13$

d



The other parallel tangent must pass through the point C at the other end of the diameter from B.

Now A is the midpoint of BC, so if C has coordinates (c, d), then

$$\frac{c-1}{2} = 4 \quad \text{and} \quad \frac{d+2}{2} = -2$$

$$\therefore \quad c - 1 = 8 \quad \text{and} \quad d + 2 = -4$$

$$\therefore \quad c = 9 \quad \text{and} \quad d = -6$$

$\therefore$ C is at $(9, -6)$

$\therefore$ since the tangent must have slope $\frac{5}{4}$, its equation is $5x - 4y = 5(9) - 4(-6)$
i.e., $5x - 4y = 69$

10 The distance from $(4, 3)$ to $(7, 6)$ is $\sqrt{(7-4)^2 + (6-3)^2}$
$= \sqrt{9+9}$
$\doteqdot 4.24$ units
$\doteqdot 84.9$ km

Hence the radar will detect the other ship.

Chapter 8

QUADRATIC EQUATIONS AND FUNCTIONS

EXERCISE 8A

1 **a** $y = 3x^2 - 4x + 1$ is of the form $y = ax^2 + bx + c$, where $a = 3$, $b = -4$ and $c = 1$.
Hence it is a quadratic function.

b $y = 5x - 7$ is of the form $y = ax^2 + bx + c$, but requires $a = 0$.
Hence it is not a quadratic function.

c $y = -x^2$ is of the form $y = ax^2 + bx + c$, where $a = -1$, $b = 0$ and $c = 0$.
Hence it is a quadratic function.

d $y = \frac{2}{3}x^2 + 4$ is of the form $y = ax^2 + bx + c$, where $a = \frac{2}{3}$, $b = 0$ and $c = 4$.
Hence it is a quadratic function.

e If $2y + 3x^2 - 5 = 0$, then $2y = -3x^2 + 5$ $\therefore$ $y = -\frac{3}{2}x^2 + \frac{5}{2}$
This is of the form $f(x) = ax^2 + bx + c$, where $a = -\frac{3}{2}$, $b = 0$ and $c = \frac{5}{2}$.
Hence it is a quadratic function.

f $y = 5x^3 + x - 6$ contains the term x^3, so is not of the form $y = ax^2 + bx + c$.
Hence it is not a quadratic function.

2 **a** When $x = 3$, $y = 3^2 + 5 \times 3 - 4$
$= 9 + 15 - 4$
$= 20$

b When $x = -3$, $y = 2 \times (-3)^2 + 9$
$= 2 \times 9 + 9$
$= 27$

c When $x = 1$, $y = -2 \times 1^2 + 3 \times 1 - 5$
$= -2 + 3 - 5$
$= -4$

d When $x = 4$, $y = 4 \times 4^2 - 7 \times 4 + 1$
$= 64 - 28 + 1$
$= 37$

3 **a** $f(2)$
$= 2^2 - 2 \times 2 + 3$
$= 4 - 4 + 3$
$= 3$

b $f(-3)$
$= 4 - (-3)^2$
$= 4 - 9$
$= -5$

c $f(0)$
$= -\frac{1}{4} \times 0^2 + 3 \times 0 - 4$
$= -4$

d $f(2)$
$= \frac{1}{2} \times 2^2 + 3 \times 2$
$= 2 + 6$
$= 8$

4 **a** $f(x) = 5x^2 - 10$
$\therefore$ $f(0) = 5 \times 0 - 10$
$= -10$
$\therefore$ $f(0) \neq 5$
$\therefore$ the function is not satisfied by the ordered pair.

b $y = 2x^2 + 5x - 3$
when $x = 4$,
$y = 2 \times 4^2 + 5 \times 4 - 3$
$= 32 + 20 - 3$
$= 49$
$\therefore$ the function is not satisfied by the ordered pair.

c $y = -2x^2 + 3x$
when $x = -\frac{1}{2}$,
$y = -2 \times (-\frac{1}{2})^2 + 3(-\frac{1}{2})$
$= -2 \times \frac{1}{4} - \frac{3}{2}$
$= -\frac{1}{2} - \frac{3}{2}$
$= -2$
$\therefore$ the function is not satisfied by the ordered pair.

d $y = -7x^2 + 8x + 15$
when $x = -1$,
$y = -7 \times (-1)^2 + 8 \times (-1) + 15$
$= -7 - 8 + 15$
$= 0$
$\therefore$ the function is not satisfied by the ordered pair.

e $f(x) = 3x^2 - 13x + 4$

$\therefore \quad f(2) = 3 \times 2^2 - 13 \times 2 + 4$
$= 12 - 26 + 4$
$= -10$

$\therefore$ the function is satisfied by the ordered pair.

f $f(x) = -3x^2 + x + 2$

$\therefore \quad f(\frac{1}{3}) = -3 \times (\frac{1}{3})^2 + \frac{1}{3} + 2$
$= -\frac{3}{9} + \frac{1}{3} + 2$
$= 2$

$\therefore$ the function is satisfied by the ordered pair.

EXERCISE 8B.1

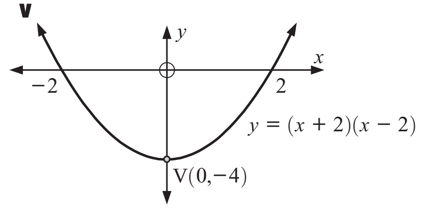
1 **a** **i** $y = (x+2)(x-2)$ has x-intercepts -2 and 2.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 0$.

iii When $x = 0$, $y = 2 \times (-2) = -4$
$\therefore$ the vertex is at $(0, -4)$

iv From **iii**, when $x = 0$, $y = -4$
$\therefore$ the y-intercept is -4.

v



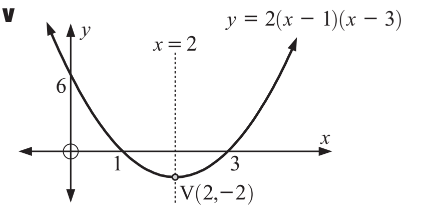
b **i** $y = 2(x-1)(x-3)$ has x-intercepts 1 and 3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = 2 \times 1 \times (-1) = -2$
$\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 2(-1)(-3) = 6$
$\therefore$ the y-intercept is 6.

v



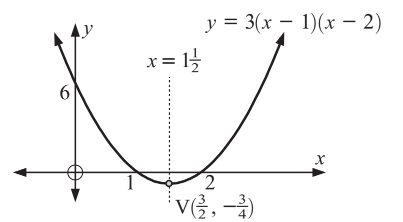
c **i** $y = 3(x-1)(x-2)$ has x-intercepts 1 and 2.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = \frac{3}{2}$.

iii When $x = \frac{3}{2}$, $y = 3 \times \frac{1}{2} \times (-\frac{1}{2}) = -\frac{3}{4}$
$\therefore$ the vertex is at $(\frac{3}{2}, -\frac{3}{4})$

iv When $x = 0$, $y = 3(-1)(-2) = 6$
$\therefore$ the y-intercept is 6.

v



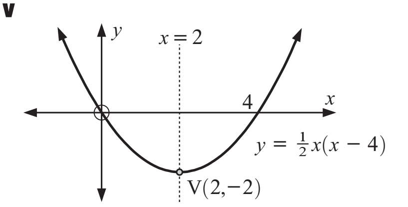
d **i** $y = \frac{1}{2}x(x-4)$ has x-intercepts 0 and 4.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = \frac{1}{2} \times 2 \times (-2) = -2$
$\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 0$ also.
$\therefore$ the y-intercept is 0.

v



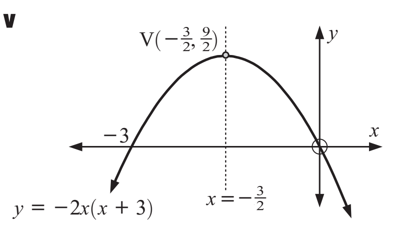
e **i** $y = -2x(x+3)$ has x-intercepts 0 and -3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -\frac{3}{2}$.

iii When $x = -\frac{3}{2}$, $y = -2 \times (-\frac{3}{2}) \times (\frac{3}{2}) = \frac{9}{2}$
$\therefore$ the vertex is at $(-\frac{3}{2}, \frac{9}{2})$

iv When $x = 0$, $y = 0$ also.
$\therefore$ the y-intercept is 0.

v



f **i** $y = -\frac{1}{2}(x+2)(x+3)$ has x-intercepts -2 and -3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -\frac{5}{2}$.

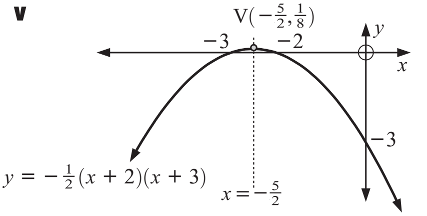
iii When $x = -\frac{5}{2}$, $y = -\frac{1}{2} \times (-\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$

$\therefore$ the vertex is at $(-\frac{5}{2}, \frac{1}{8})$

iv When $x = 0$, $y = -\frac{1}{2} \times 2 \times 3 = -3$

$\therefore$ the y-intercept is -3.

v



2 **a** $y = x(x-2)$ has x-intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 1 \times (-1) = -1$.
Hence the graph is **B**.

b $y = 3x(x-2)$ also has x-intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 3 \times 1 \times (-1) = -3$.
Hence the graph is **A**.

c $y = -x(x-2)$ also has x-intercepts 0 and 2.
$\therefore$ its graph must be the remaining one like this, i.e., **F**.
Check: axis of symmetry is $x = 1$,
so its vertex has $y = -1 \times (-1) = 1$, which is true.

d $y = (x+2)(x-1)$ has x-intercepts -2 and 1.
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{4}$.
Hence the graph is **D**.

e $y = 2(x+2)(x-1)$ also has x-intercepts -2 and 1.
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = 2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{2}$.
Hence the graph is **E**.

f $y = -2(x+2)(x-1)$ also has x-intercepts -2 and 1.
Its graph must be the only one remaining, i.e., **C**.
Check: axis of symmetry is $x = -\frac{1}{2}$,
so its vertex has $y = -2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = \frac{9}{2}$, which is true.

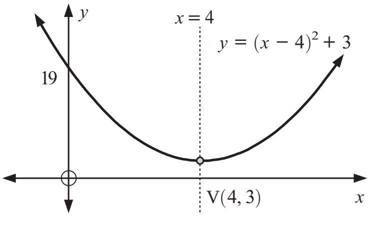
EXERCISE 8B.2

1 **a** **i** $y = (x-4)^2 + 3$ has axis of symmetry $x = 4$.

ii Its vertex is $(4, 3)$.

iii When $x = 0$, $y = (-4)^2 + 3 = 19$
$\therefore$ the y-intercept is 19.

iv

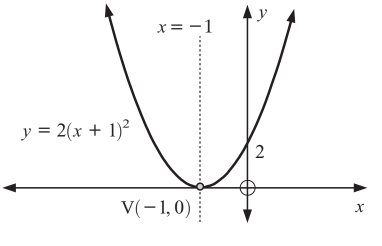


b **i** $y = 2(x+1)^2$ has axis of symmetry $x = -1$.

ii Its vertex is $(-1, 0)$.

iii When $x = 0$, $y = 2 \times 1^2 = 2$
$\therefore$ the y-intercept is 2.

iv

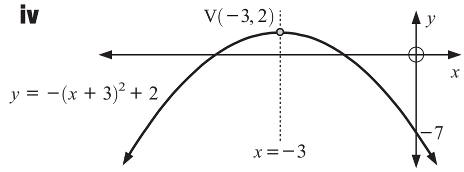


c **i** $y = -(x+3)^2 + 2$ has axis of symmetry $x = -3$.

ii Its vertex is $(-3,\ 2)$.

iii When $x = 0$, $y = -3^2 + 2 = -7$
$\therefore$ the y-intercept is -7.

iv

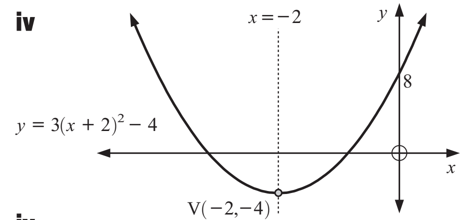


d **i** $y = 3(x+2)^2 - 4$ has axis of symmetry $x = -2$.

ii Its vertex is $(-2,\ -4)$.

iii When $x = 0$, $y = 3 \times 2^2 - 4 = 8$
$\therefore$ the y-intercept is 8.

iv

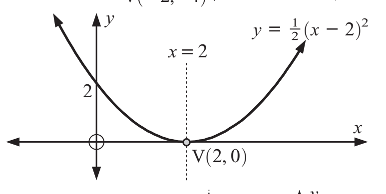


e **i** $y = \frac{1}{2}(x-2)^2$ has axis of symmetry $x = 2$.

ii Its vertex is $(2,\ 0)$.

iii When $x = 0$, $y = \frac{1}{2} \times (-2)^2 = 2$
$\therefore$ the y-intercept is 2.

iv

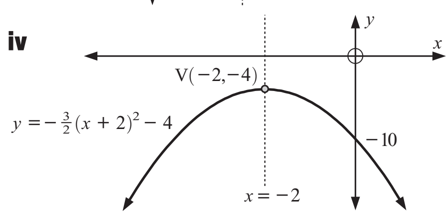


f **i** $y = -\frac{3}{2}(x+2)^2 - 4$ has axis of symmetry $x = -2$.

ii Its vertex is $(-2,\ -4)$.

iii When $x = 0$, $y = -\frac{3}{2} \times 2^2 - 4 = -10$
$\therefore$ the y-intercept is -10.

iv



2 **a** $y = -(x+1)^2 + 3$ has vertex $(-1,\ 3)$, so its graph must be **G**.

b $y = -2(x-3)^2 + 2$ has vertex $(3,\ 2)$, so its graph must be **A**.

c $y = x^2 + 2$ has vertex $(0,\ 2)$, so its graph must be **E**.

d $y = -(x-1)^2 + 1$ has vertex $(1,\ 1)$, so its graph must be **B**.

e $y = (x-2)^2 - 2$ has vertex $(2,\ -2)$, so its graph must be **I**.

f $y = \frac{1}{3}(x+3)^2 - 3$ has vertex $(-3,\ -3)$, so its graph must be **C**.

g $y = -x^2$ has vertex $(0,\ 0)$, so its graph must be **D**.

h $y = -\frac{1}{2}(x-1)^2 + 1$ has vertex $(1,\ 1)$, so its graph must be **F**.

i $y = 2(x+2)^2 - 1$ has vertex $(-2,\ -1)$, so its graph must be **H**.

3 **a** The graph has x-intercepts 0 and 4,
and its axis of symmetry is midway between them, i.e., $x = 2$.

b The graph has x-intercepts -5 and 0,
and its axis of symmetry is midway between them, i.e., $x = -\frac{5}{2}$.

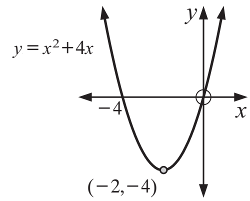
c The graph has x-intercepts -1 and 3,
and its axis of symmetry is midway between them, i.e., $x = 1$.

d The graph touches the x-axis at its vertex $(3,\ 0)$
$\therefore$ its axis of symmetry is $x = 3$.

e The graph touches the x-axis at its vertex $(-4,\ 0)$
$\therefore$ its axis of symmetry is $x = -4$.

f Both $(-8,\ -5)$ and $(0,\ -5)$ lie on the graph. Since they have the same y-coordinate, the axis of symmetry must lie midway between them, i.e., $x = -4$.

4 **a** **i** $y = x^2 + 4x$ has x-intercepts -4 and 0, and y-intercept 0.

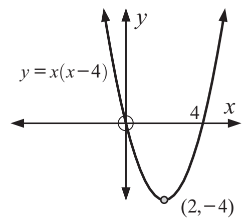


ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -2$.

iii When $x = -2$, $y = (-2)^2 + 4(-2) = -4$

$\therefore$ the vertex is $(-2, -4)$.

b **i** $y = x(x-4)$ has x-intercepts 0 and 4, and y-intercept 0.

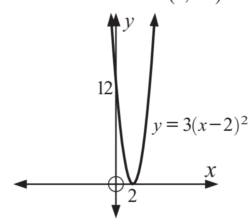


ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = 2(-2) = -4$

$\therefore$ the vertex is $(2, -4)$.

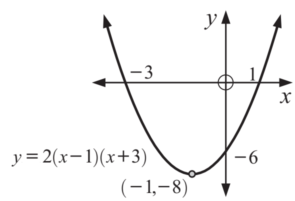
c **i** $y = 3(x-2)^2$ has x-intercept 2 and y-intercept $= 3 \times (-2)^2 = 12$.



ii Since the vertex is when $x = 2$, $x = 2$ is the axis of symmetry.

iii The vertex is $(2, 0)$.

d **i** $y = 2(x-1)(x+3)$ has x-intercepts -3 and 1, and y-intercept -6.

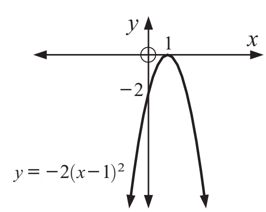


ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -1$.

iii When $x = -1$, $y = 2(-2)(2) = -8$

$\therefore$ the vertex is $(-1, -8)$.

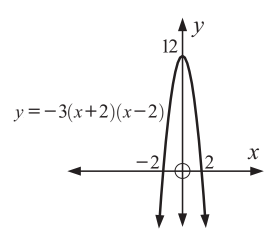
e **i** $y = -2(x-1)^2$ has x-intercept 1 and y-intercept $= -2 \times (-1)^2 = -2$.



ii Since the vertex is when $x = 1$, $x = 1$ is the axis of symmetry.

iii The vertex is $(1, 0)$.

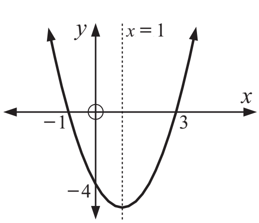
f **i** $y = -3(x+2)(x-2)$ has x-intercepts -2 and 2, and y-intercept $= -3(2)(-2) = 12$.



ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 0$.

iii Since the y-intercept $= 12$, the vertex is $(0, 12)$.

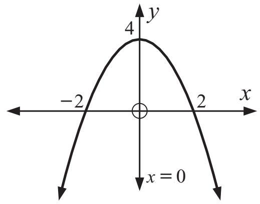
5 **a** **i**



ii The axis of symmetry lies halfway between the x-intercepts,

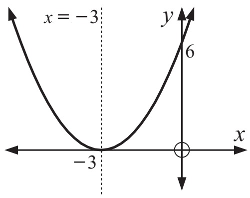
i.e., the axis of symmetry is $x = 1$.

b **i**



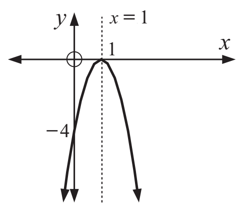
ii The axis of symmetry lies halfway between the x-intercepts,
i.e., the axis of symmetry is $x = 0$.

c **i**



ii Since the vertex is at $x = -3$,
the axis of symmetry is $x = -3$.

d **i**



ii Since the vertex is at $x = 1$,
the axis of symmetry is $x = 1$.

6 **a** The axis of symmetry, $x = 4$, lies midway between the x-intercepts.
$\therefore$ since one x-intercept is 2, the other is 6
$\therefore$ the x-intercepts are 2 and 6.

b The axis of symmetry, $x = -3$, lies midway between the x-intercepts.
$\therefore$ since one x-intercept is -1, the other is -5
$\therefore$ the x-intercepts are -1 and -5.

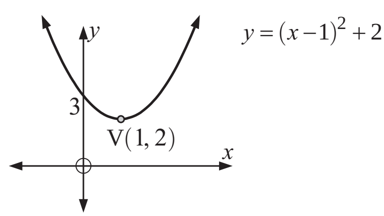
c Since the graph touches the x-axis at 3 and it is a quadratic function,
the only x-intercept is 3 (touching).

EXERCISE 8C

1 **a**

$$y = x^2 - 2x + 3$$
$$\therefore\ y = x^2 - 2x + 1^2 + 3 - 1^2$$
$$\therefore\ y = (x-1)^2 + 2$$

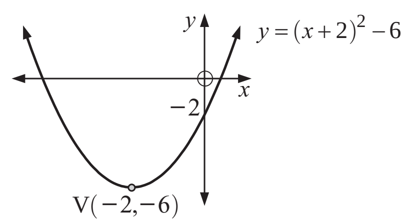
$\therefore$ vertex is $(1, 2)$,
y-intercept is 3



b

$$y = x^2 + 4x - 2$$
$$\therefore\ y = x^2 + 4x + 2^2 - 2 - 2^2$$
$$\therefore\ y = (x+2)^2 - 6$$

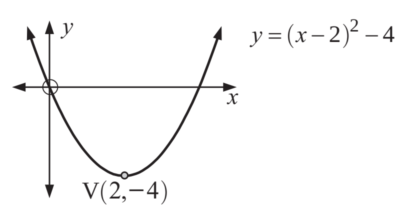
$\therefore$ vertex is $(-2, -6)$,
y-intercept is -2



c

$$y = x^2 - 4x$$
$$\therefore\ y = x^2 - 4x + 2^2 - 2^2$$
$$\therefore\ y = (x-2)^2 - 4$$

$\therefore$ vertex is $(2, -4)$,
y-intercept is 0



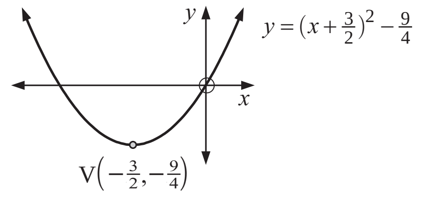
d $y = x^2 + 3x$

$\therefore\ y = x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2$

$\therefore\ y = (x + \frac{3}{2})^2 - \frac{9}{4}$

$\therefore$ vertex is $(-\frac{3}{2}, -\frac{9}{4})$,

y-intercept is 0



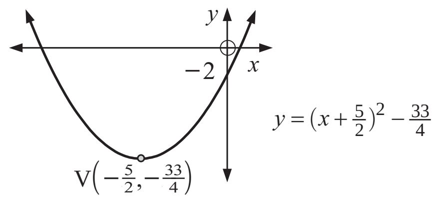
e $y = x^2 + 5x - 2$

$\therefore\ y = x^2 + 5x + (\frac{5}{2})^2 - 2 - (\frac{5}{2})^2$

$\therefore\ y = (x + \frac{5}{2})^2 - \frac{33}{4}$

$\therefore$ vertex is $(-\frac{5}{2}, -\frac{33}{4})$,

y-intercept is -2



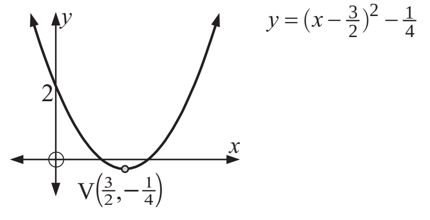
f $y = x^2 - 3x + 2$

$\therefore\ y = x^2 - 3x + (\frac{3}{2})^2 + 2 - (\frac{3}{2})^2$

$\therefore\ y = (x - \frac{3}{2})^2 - \frac{1}{4}$

$\therefore$ vertex is $(\frac{3}{2}, -\frac{1}{4})$,

y-intercept is 2



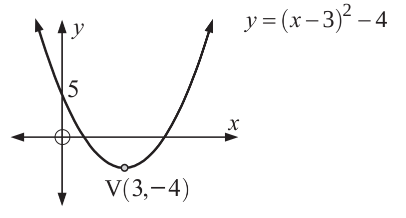
g $y = x^2 - 6x + 5$

$\therefore\ y = x^2 - 6x + 3^2 + 5 - 3^2$

$\therefore\ y = (x - 3)^2 - 4$

$\therefore$ vertex is $(3, -4)$,

y-intercept is 5



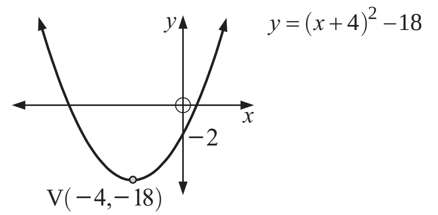
h $y = x^2 + 8x - 2$

$\therefore\ y = x^2 + 8x + 4^2 - 2 - 4^2$

$\therefore\ y = (x + 4)^2 - 18$

$\therefore$ vertex is $(-4, -18)$,

y-intercept is -2



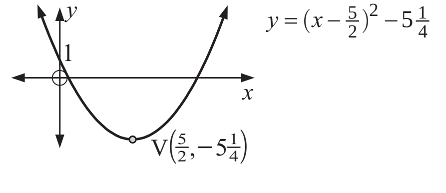
i $y = x^2 - 5x + 1$

$\therefore\ y = x^2 - 5x + (\frac{5}{2})^2 + 1 - (\frac{5}{2})^2$

$\therefore\ y = (x - \frac{5}{2})^2 - \frac{21}{4}$

$\therefore$ vertex is $(\frac{5}{2}, -5\frac{1}{4})$,

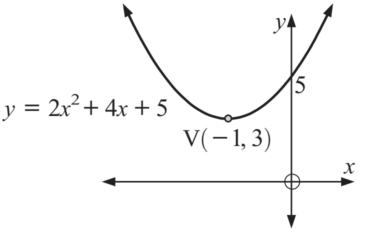
y-intercept is 1



2 **a** **i** $y = 2x^2 + 4x + 5$
$= 2\left[x^2 + 2x + \frac{5}{2}\right]$
$= 2\left[x^2 + 2x + 1^2 - 1^2 + \frac{5}{2}\right]$
$= 2\left[(x+1)^2 + \frac{3}{2}\right]$
$= 2(x+1)^2 + 3$

ii The vertex is $(-1, 3)$.

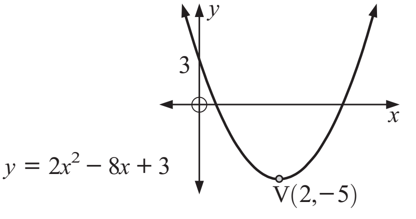
iii When $x = 0$, $y = 5$
$\therefore$ the y-intercept is 5

iv


b **i** $y = 2x^2 - 8x + 3$
$= 2\left[x^2 - 4x + \frac{3}{2}\right]$
$= 2\left[x^2 - 4x + 2^2 - 2^2 + \frac{3}{2}\right]$
$= 2\left[(x-2)^2 - \frac{5}{2}\right]$
$= 2(x-2)^2 - 5$

ii The vertex is $(2, -5)$.

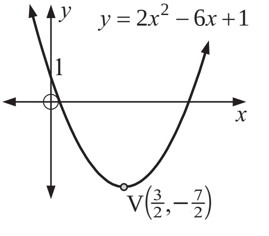
iii When $x = 0$, $y = 3$
$\therefore$ the y-intercept is 3

iv


c **i** $y = 2x^2 - 6x + 1$
$= 2\left[x^2 - 3x + \frac{1}{2}\right]$
$= 2\left[x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + \frac{1}{2}\right]$
$= 2\left[(x - \frac{3}{2})^2 - \frac{7}{4}\right]$
$= 2(x - \frac{3}{2})^2 - \frac{7}{2}$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

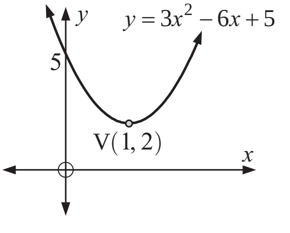
iii When $x = 0$, $y = 1$
$\therefore$ the y-intercept is 1

iv


d **i** $y = 3x^2 - 6x + 5$
$= 3\left[x^2 - 2x + \frac{5}{3}\right]$
$= 3\left[x^2 - 2x + 1^2 - 1^2 + \frac{5}{3}\right]$
$= 3\left[(x-1)^2 + \frac{2}{3}\right]$
$= 3(x-1)^2 + 2$

ii The vertex is $(1, 2)$.

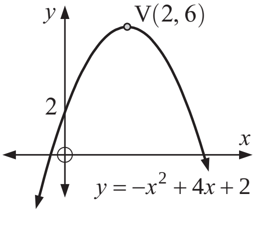
iii When $x = 0$, $y = 5$
$\therefore$ the y-intercept is 5

iv


e **i** $y = -x^2 + 4x + 2$
$= -\left[x^2 - 4x - 2\right]$
$= -\left[x^2 - 4x + 2^2 - 2^2 - 2\right]$
$= -\left[(x-2)^2 - 6\right]$
$= -(x-2)^2 + 6$

ii The vertex is $(2, 6)$.

iii When $x = 0$, $y = 2$
$\therefore$ the y-intercept is 2

iv


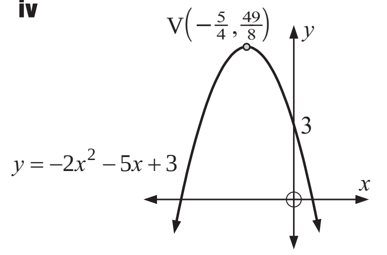
f **i**
$$\begin{aligned} y &= -2x^2 - 5x + 3 \\ &= -2\left[x^2 + \tfrac{5}{2}x - \tfrac{3}{2}\right] \\ &= -2\left[x^2 + \tfrac{5}{2}x + (\tfrac{5}{4})^2 - (\tfrac{5}{4})^2 - \tfrac{3}{2}\right] \\ &= -2\left[(x + \tfrac{5}{4})^2 - \tfrac{25}{16} - \tfrac{24}{16}\right] \\ &= -2\left[(x + \tfrac{5}{4})^2 - \tfrac{49}{16}\right] \\ &= -2(x + \tfrac{5}{4})^2 + \tfrac{49}{8} \end{aligned}$$

ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

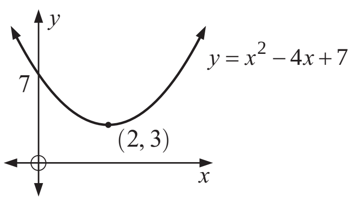
iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



3 **a** Using technology, the graph is



Since the vertex is at (2, 3), the function must be of the form

$y = a(x-2)^2 + 3$ for some a.

$\therefore$ when $x = 0$,

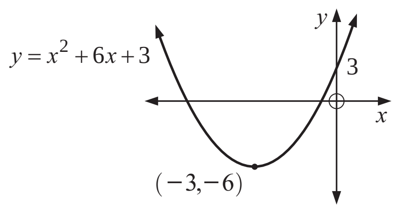
$y = a(-2)^2 + 3 = 4a + 3$

but the y-intercept is 7

$$\begin{aligned} \therefore\ 4a + 3 &= 7 \\ \therefore\ a &= 1 \end{aligned}$$

$\therefore$ the equation is $y = (x-2)^2 + 3$

b Using technology, the graph is



Since the vertex is at $(-3, -6)$, the function must be of the form

$y = a(x+3)^2 - 6$ for some a.

$\therefore$ when $x = 0$,

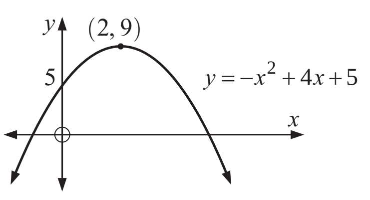
$y = a \times 3^2 - 6 = 9a - 6$

but the y-intercept is 3

$$\begin{aligned} \therefore\ 9a - 6 &= 3 \\ \therefore\ a &= 1 \end{aligned}$$

$\therefore$ the equation is $y = (x+3)^2 - 6$

c Using technology, the graph is



Since the vertex is at (2, 9), the function must be of the form

$y = a(x-2)^2 + 9$ for some a.

$\therefore$ when $x = 0$,

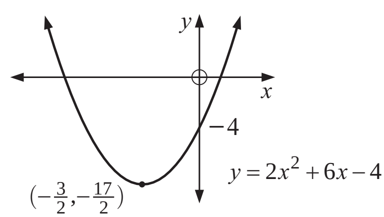
$y = a(-2)^2 + 9 = 4a + 9$

but the y-intercept is 5

$$\begin{aligned} \therefore\ 4a + 9 &= 5 \\ \therefore\ a &= -1 \end{aligned}$$

$\therefore$ the equation is $y = -(x-2)^2 + 9$

d Using technology, the graph is



Since the vertex is at $(-\frac{3}{2}, -\frac{17}{2})$,

the function must be of the form

$y = a(x + \frac{3}{2})^2 - \frac{17}{2}$ for some a.

$\therefore$ when $x = 0$,

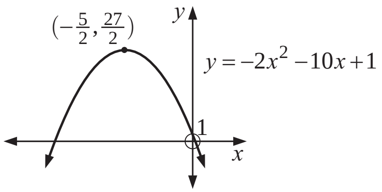
$y = a(\frac{3}{2})^2 - \frac{17}{2} = \frac{9}{4}a - \frac{17}{2}$

but the y-intercept is -4

$$\begin{aligned} \therefore\ \tfrac{9}{4}a - \tfrac{17}{2} &= -4 \\ \therefore\ \tfrac{9}{4}a &= \tfrac{9}{2} \\ \therefore\ a &= \tfrac{4}{9} \times \tfrac{9}{2} = 2 \end{aligned}$$

$\therefore$ the equation is $y = 2(x + \frac{3}{2})^2 - \frac{17}{2}$

e Using technology, the graph is



Since the vertex is at $(-\frac{5}{2}, \frac{27}{2})$,
the function must be of the form
$y = a(x + \frac{5}{2})^2 + \frac{27}{2}$ for some a.
$\therefore$ when $x = 0$,
$y = a(\frac{5}{2})^2 + \frac{27}{2} = \frac{25}{4}a + \frac{27}{2}$
but the y-intercept is 1

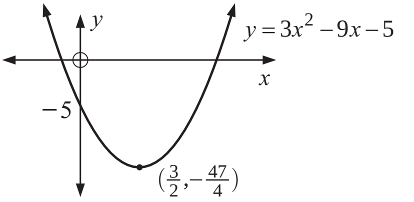
$\therefore\ \frac{25}{4}a + \frac{27}{2} = 1$

$\therefore\ \frac{25}{4}a = -\frac{25}{2}$

$\therefore\ a = -\frac{25}{2} \times \frac{4}{25} = -2$

$\therefore$ the equation is $y = -2(x + \frac{5}{2})^2 + \frac{27}{2}$

f Using technology, the graph is



Since the vertex is at $(\frac{3}{2}, -\frac{47}{4})$,
the function must be of the form
$y = a(x - \frac{3}{2})^2 - \frac{47}{4}$ for some a.
$\therefore$ when $x = 0$,
$y = a(-\frac{3}{2})^2 - \frac{47}{4} = \frac{9}{4}a - \frac{47}{4}$
but the y-intercept is -5

$\therefore\ \frac{9}{4}a - \frac{47}{4} = -5$

$\therefore\ \frac{9}{4}a = \frac{27}{4}$

$\therefore\ a = 3$

$\therefore$ the equation is $y = 3(x - \frac{3}{2})^2 - \frac{47}{4}$

EXERCISE 8D.1

1 **a** $4x^2 + 7x = 0$
$\therefore\ x(4x + 7) = 0$
$\therefore\ x = 0$ or $4x + 7 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $-\frac{7}{4}$

b $6x^2 + 2x = 0$
$\therefore\ 2x(3x + 1) = 0$
$\therefore\ x = 0$ or $3x + 1 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $-\frac{1}{3}$

c $3x^2 - 7x = 0$
$\therefore\ x(3x - 7) = 0$
$\therefore\ x = 0$ or $3x - 7 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $\frac{7}{3}$

d $2x^2 - 11x = 0$
$\therefore\ x(2x - 11) = 0$
$\therefore\ x = 0$ or $2x - 11 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $\frac{11}{2}$

e $3x^2 = 8x$
$\therefore\ 3x^2 - 8x = 0$
$\therefore\ x(3x - 8) = 0$
$\therefore\ x = 0$ or $3x - 8 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $\frac{8}{3}$

f $9x = 6x^2$
$\therefore\ 6x^2 - 9x = 0$
$\therefore\ 3x(2x - 3) = 0$
$\therefore\ x = 0$ or $2x - 3 = 0$ {Null Factor law}
$\therefore\ x = 0$ or $\frac{3}{2}$

g $x^2 - 5x + 6 = 0$
$\therefore\ (x - 2)(x - 3) = 0$
$\therefore\ x - 2 = 0$ or $x - 3 = 0$ {Null Factor law}
$\therefore\ x = 2$ or 3

h $x^2 = 2x + 8$
$\therefore\ x^2 - 2x - 8 = 0$
$\therefore\ (x - 4)(x + 2) = 0$
$\therefore\ x - 4 = 0$ or $x + 2 = 0$ {Null Factor law}
$\therefore\ x = -2$ or 4

i $x^2 + 21 = 10x$
$\therefore\ x^2 - 10x + 21 = 0$
$\therefore\ (x - 3)(x - 7) = 0$
$\therefore\ x - 3 = 0$ or $x - 7 = 0$ {Null Factor law}
$\therefore\ x = 3$ or 7

j $9 + x^2 = 6x$
$\therefore\ x^2 - 6x + 9 = 0$
$\therefore\ (x - 3)^2 = 0$
$\therefore\ x - 3 = 0$
$\therefore\ x = 3$

k $x^2 + x = 12$
$\therefore\ x^2 + x - 12 = 0$
$\therefore\ (x + 4)(x - 3) = 0$
$\therefore\ x + 4 = 0$ or $x - 3 = 0$ {Null Factor law}
$\therefore\ x = -4$ or 3

l $x^2 + 8x = 33$
$\therefore\ x^2 + 8x - 33 = 0$
$\therefore\ (x + 11)(x - 3) = 0$
$\therefore\ x + 11 = 0$ or $x - 3 = 0$ {Null Factor law}
$\therefore\ x = -11$ or 3

2 **a** $9x^2-12x+4=0$
$\therefore\ (3x-2)^2=0$
$\therefore\ x=\frac{2}{3}$

b $2x^2-13x-7=0$
$\therefore\ (2x+1)(x-7)=0$
$\therefore\ x=-\frac{1}{2}$ or 7

c $3x^2=16x+12$
$\therefore\ 3x^2-16x-12=0$
$\therefore\ (3x+2)(x-6)=0$
$\therefore\ x=-\frac{2}{3}$ or 6

d $3x^2+5x=2$
$\therefore\ 3x^2+5x-2=0$
$\therefore\ (3x-1)(x+2)=0$
$\therefore\ x=\frac{1}{3}$ or -2

e $2x^2+3=5x$
$\therefore\ 2x^2-5x+3=0$
$\therefore\ (2x-3)(x-1)=0$
$\therefore\ x=\frac{3}{2}$ or 1

f $3x^2=4x+4$
$\therefore\ 3x^2-4x-4=0$
$\therefore\ (3x+2)(x-2)=0$
$\therefore\ x=-\frac{2}{3}$ or 2

g $3x^2=10x+8$
$\therefore\ 3x^2-10x-8=0$
$\therefore\ (3x+2)(x-4)=0$
$\therefore\ x=-\frac{2}{3}$ or 4

h $4x^2+4x=3$
$\therefore\ 4x^2+4x-3=0$
$\therefore\ (2x+3)(2x-1)=0$
$\therefore\ x=-\frac{3}{2}$ or $\frac{1}{2}$

i $4x^2=11x+3$
$\therefore\ 4x^2-11x-3=0$
$\therefore\ (4x+1)(x-3)=0$
$\therefore\ x=-\frac{1}{4}$ or 3

j $12x^2=11x+15$
$\therefore\ 12x^2-11x-15=0$
$\therefore\ (4x+3)(3x-5)=0$
$\therefore\ x=-\frac{3}{4}$ or $\frac{5}{3}$

k $7x^2+6x=1$
$\therefore\ 7x^2+6x-1=0$
$\therefore\ (7x-1)(x+1)=0$
$\therefore\ x=\frac{1}{7}$ or -1

l $15x^2+2x=56$
$\therefore\ 15x^2+2x-56=0$
$\therefore\ (15x-28)(x+2)=0$
$\therefore\ x=\frac{28}{15}$ or -2

3 **a** $(x+1)^2=2x^2-5x+11$
$\therefore\ x^2+2x+1=2x^2-5x+11$
$\therefore\ x^2-7x+10=0$
$\therefore\ (x-2)(x-5)=0$
$\therefore\ x=2$ or 5

b $(x+2)(1-x)=-4$
$\therefore\ x-x^2+2-2x=-4$
$\therefore\ x^2+x-6=0$
$\therefore\ (x+3)(x-2)=0$
$\therefore\ x=-3$ or 2

c $5-4x^2=3(2x+1)+2$
$\therefore\ 5-4x^2=6x+3+2$
$\therefore\ 4x^2+6x=0$
$\therefore\ 2x(2x+3)=0$
$\therefore\ x=0$ or $-\frac{3}{2}$

d $x+\dfrac{2}{x}=3$
$\therefore\ x^2+2=3x$
$\therefore\ x^2-3x+2=0$
$\therefore\ (x-1)(x-2)=0$
$\therefore\ x=1$ or 2

e $2x-\dfrac{1}{x}=-1$
$\therefore\ 2x^2-1=-x$
$\therefore\ 2x^2+x-1=0$
$\therefore\ (2x-1)(x+1)=0$
$\therefore\ x=\frac{1}{2}$ or -1

f $\dfrac{x+3}{1-x}=-\dfrac{9}{x}$
$\therefore\ x(x+3)=-9(1-x)$
$\therefore\ x^2+3x=-9+9x$
$\therefore\ x^2-6x+9=0$
$\therefore\ (x-3)^2=0$
$\therefore\ x=3$

4 **a** If $y=1$, then $x^2+6x+10=1$
$\therefore\ x^2+6x+9=0$
$\therefore\ (x+3)^2=0$
$\therefore\ x=-3$

b If $y=2$, then $x^2+5x+8=2$
$\therefore\ x^2+5x+6=0$
$\therefore\ (x+3)(x+2)=0$
$\therefore\ x=-3$ or -2

c If $y=-3$, then $x^2-5x+1=-3$
$\therefore\ x^2-5x+4=0$
$\therefore\ (x-4)(x-1)=0$
$\therefore\ x=4$ or 1

d If $y=-3$, then $3x^2=-3$
$\therefore\ x^2=-1$
$\therefore$ there is no solution

5 **a** $f(x) = 5$
$\therefore\ 3x^2 - 2x + 5 = 5$
$\therefore\ 3x^2 - 2x = 0$
$\therefore\ x(3x-2) = 0$
$\therefore\ x = 0$ or $\frac{2}{3}$

b $f(x) = 1$
$\therefore\ x^2 - x - 5 = 1$
$\therefore\ x^2 - x - 6 = 0$
$\therefore\ (x+2)(x-3) = 0$
$\therefore\ x = -2$ or 3

c $f(x) = -4$
$\therefore\ -2x^2 - 13x + 3 = -4$
$\therefore\ 2x^2 + 13x - 7 = 0$
$\therefore\ (2x-1)(x+7) = 0$
$\therefore\ x = \frac{1}{2}$ or -7

d $f(x) = -17$
$\therefore\ 2x^2 - 12x + 1 = -17$
$\therefore\ 2x^2 - 12x + 18 = 0$
$\therefore\ 2(x^2 - 6x + 9) = 0$
$\therefore\ 2(x-3)^2 = 0$
$\therefore\ x = 3$

6 **a** **i** $h(1) = 30 - 5$
$= 25$ m

ii $h(5) = 30 \times 5 - 5 \times 5^2$
$= 150 - 125$
$= 25$ m

iii $h(3) = 30 \times 3 - 5 \times 3^2$
$= 90 - 45$
$= 45$ m

b **i** When $h = 40$ m,
$30t - 5t^2 = 40$
$\therefore\ 5t^2 - 30t + 40 = 0$
$\therefore\ 5(t^2 - 6t + 8) = 0$
$\therefore\ 5(t-4)(t-2) = 0$
$\therefore\ t = 4$ or 2 seconds

ii When $h = 0$ m,
$30t - 5t^2 = 0$
$\therefore\ 5t(6-t) = 0$
$\therefore\ t = 0$ or 6 seconds

c We get two answers in each case because the object is projected up, and gravity brings it back down.

7 **a** **i** $P(0) = -\frac{1}{4} \times 0 + 16 \times 0 - 30$
$= -\$30$

ii $P(10) = -\frac{1}{4} \times 10^2 + 16 \times 10 - 30$
$= -25 + 160 - 30$
$= \$105$

b If $P(x) = 57$, then $-\frac{1}{4}x^2 + 16x - 30 = 57$
$\therefore\ \frac{1}{4}x^2 - 16x + 87 = 0$
$\therefore\ x^2 - 64x + 348 = 0$
$\therefore\ (x-6)(x-58) = 0$
$\therefore\ x = 6$ or 58

i.e., either 6 or 58 cakes are made.

EXERCISE 8D.2

1 **a** $(x+5)^2 = 2$
$\therefore\ x + 5 = \pm\sqrt{2}$
$\therefore\ x = -5 \pm \sqrt{2}$

b $(x+6)^2 = 11$
$\therefore\ x + 6 = \pm\sqrt{11}$
$\therefore\ x = -6 \pm \sqrt{11}$

c $(x-4)^2 = 8$
$\therefore\ x - 4 = \pm\sqrt{8}$
$\therefore\ x = 4 \pm 2\sqrt{2}$

d $(x-8)^2 = 7$
$\therefore\ x - 8 = \pm\sqrt{7}$
$\therefore\ x = 8 \pm \sqrt{7}$

e $2(x+3)^2 = 10$
$\therefore\ (x+3)^2 = 5$
$\therefore\ x + 3 = \pm\sqrt{5}$
$\therefore\ x = -3 \pm \sqrt{5}$

f $3(x-2)^2 = 18$
$\therefore\ (x-2)^2 = 6$
$\therefore\ x - 2 = \pm\sqrt{6}$
$\therefore\ x = 2 \pm \sqrt{6}$

g $(x+1)^2 + 1 = 11$
$\therefore\ (x+1)^2 = 10$
$\therefore\ x + 1 = \pm\sqrt{10}$
$\therefore\ x = -1 \pm \sqrt{10}$

h $(2x+1)^2 = 3$
$\therefore\ 2x + 1 = \pm\sqrt{3}$
$\therefore\ 2x = -1 \pm \sqrt{3}$
$\therefore\ x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$

2 **a**
$$\begin{aligned} x^2-4x+1 &= 0 \\ \therefore\ x^2-4x &= -1 \\ \therefore\ x^2-4x+(-2)^2 &= -1+(-2)^2 \\ \therefore\ (x-2)^2 &= 3 \\ \therefore\ x-2 &= \pm\sqrt{3} \\ \therefore\ x &= 2\pm\sqrt{3} \end{aligned}$$

b
$$\begin{aligned} x^2+6x+2 &= 0 \\ \therefore\ x^2+6x &= -2 \\ \therefore\ x^2+6x+3^2 &= -2+3^2 \\ \therefore\ (x+3)^2 &= 7 \\ \therefore\ x+3 &= \pm\sqrt{7} \\ \therefore\ x &= -3\pm\sqrt{7} \end{aligned}$$

c
$$\begin{aligned} x^2-14x+46 &= 0 \\ \therefore\ x^2-14x &= -46 \\ \therefore\ x^2-14x+(-7)^2 &= -46+(-7)^2 \\ \therefore\ (x-7)^2 &= 3 \\ \therefore\ x-7 &= \pm\sqrt{3} \\ \therefore\ x &= 7\pm\sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2 &= 4x+3 \\ \therefore\ x^2-4x &= 3 \\ \therefore\ x^2-4x+(-2)^2 &= 3+(-2)^2 \\ \therefore\ (x-2)^2 &= 7 \\ \therefore\ x-2 &= \pm\sqrt{7} \\ \therefore\ x &= 2\pm\sqrt{7} \end{aligned}$$

e
$$\begin{aligned} x^2+6x+7 &= 0 \\ \therefore\ x^2+6x &= -7 \\ \therefore\ x^2+6x+3^2 &= -7+3^2 \\ \therefore\ (x+3)^2 &= 2 \\ \therefore\ x+3 &= \pm\sqrt{2} \\ \therefore\ x &= -3\pm\sqrt{2} \end{aligned}$$

f
$$\begin{aligned} x^2 &= 2x+6 \\ \therefore\ x^2-2x &= 6 \\ \therefore\ x^2-2x+(-1)^2 &= 6+(-1)^2 \\ \therefore\ (x-1)^2 &= 7 \\ \therefore\ x-1 &= \pm\sqrt{7} \\ \therefore\ x &= 1\pm\sqrt{7} \end{aligned}$$

g
$$\begin{aligned} x^2+6x &= 2 \\ \therefore\ x^2+6x+3^2 &= 2+3^2 \\ \therefore\ (x+3)^2 &= 11 \\ \therefore\ x+3 &= \pm\sqrt{11} \\ \therefore\ x &= -3\pm\sqrt{11} \end{aligned}$$

h
$$\begin{aligned} x^2+10 &= 8x \\ \therefore\ x^2-8x &= -10 \\ \therefore\ x^2-8x+(-4)^2 &= -10+(-4)^2 \\ \therefore\ (x-4)^2 &= 6 \\ \therefore\ x-4 &= \pm\sqrt{6} \\ \therefore\ x &= 4\pm\sqrt{6} \end{aligned}$$

i
$$\begin{aligned} x^2+6x &= -11 \\ \therefore\ x^2+6x+3^2 &= -11+3^2 \\ \therefore\ (x+3)^2 &= -2 \end{aligned}$$

$\therefore$ x has no real solutions, since the perfect square cannot be negative.

3 **a**
$$\begin{aligned} 2x^2+4x+1 &= 0 \\ \therefore\ x^2+2x+\tfrac{1}{2} &= 0 \\ \therefore\ x^2+2x &= -\tfrac{1}{2} \\ \therefore\ x^2+2x+1^2 &= -\tfrac{1}{2}+1^2 \\ \therefore\ (x+1)^2 &= \tfrac{1}{2} \\ \therefore\ x+1 &= \pm\tfrac{1}{\sqrt{2}} \\ \therefore\ x &= -1\pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

b
$$\begin{aligned} 2x^2-10x+3 &= 0 \\ \therefore\ x^2-5x+\tfrac{3}{2} &= 0 \\ \therefore\ x^2-5x &= -\tfrac{3}{2} \\ \therefore\ x^2-5x+(-\tfrac{5}{2})^2 &= -\tfrac{3}{2}+(-\tfrac{5}{2})^2 \\ \therefore\ (x-\tfrac{5}{2})^2 &= -\tfrac{3}{2}+\tfrac{25}{4} \\ \therefore\ (x-\tfrac{5}{2})^2 &= \tfrac{19}{4} \\ \therefore\ x-\tfrac{5}{2} &= \pm\tfrac{\sqrt{19}}{2} \\ \therefore\ x &= \tfrac{5}{2}\pm\tfrac{\sqrt{19}}{2} \end{aligned}$$

c
$$\begin{aligned} 3x^2+12x+5 &= 0 \\ \therefore\ x^2+4x+\tfrac{5}{3} &= 0 \\ \therefore\ x^2+4x &= -\tfrac{5}{3} \\ \therefore\ x^2+4x+2^2 &= -\tfrac{5}{3}+2^2 \\ \therefore\ (x+2)^2 &= \tfrac{7}{3} \\ \therefore\ x+2 &= \pm\sqrt{\tfrac{7}{3}} \\ \therefore\ x &= -2\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

d
$$\begin{aligned} 3x^2 &= 6x+4 \\ \therefore\ x^2 &= 2x+\tfrac{4}{3} \\ \therefore\ x^2-2x &= \tfrac{4}{3} \\ \therefore\ x^2-2x+(-1)^2 &= \tfrac{4}{3}+(-1)^2 \\ \therefore\ (x-1)^2 &= \tfrac{7}{3} \\ \therefore\ x-1 &= \pm\sqrt{\tfrac{7}{3}} \\ \therefore\ x &= 1\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

e
$$\begin{aligned} 5x^2 - 15x + 2 &= 0 \\ \therefore\ x^2 - 3x + \tfrac{2}{5} &= 0 \\ \therefore\ x^2 - 3x &= -\tfrac{2}{5} \\ \therefore\ x^2 - 3x + (-\tfrac{3}{2})^2 &= -\tfrac{2}{5} + (-\tfrac{3}{2})^2 \\ \therefore\ (x - \tfrac{3}{2})^2 &= -\tfrac{2}{5} + \tfrac{9}{4} = \tfrac{37}{20} \\ \therefore\ x - \tfrac{3}{2} &= \pm\sqrt{\tfrac{37}{20}} \\ \therefore\ x &= \tfrac{3}{2} \pm \sqrt{\tfrac{37}{20}} \end{aligned}$$

f
$$\begin{aligned} 4x^2 + 4x &= 5 \\ \therefore\ x^2 + x &= \tfrac{5}{4} \\ \therefore\ x^2 + x + (\tfrac{1}{2})^2 &= \tfrac{5}{4} + (\tfrac{1}{2})^2 \\ \therefore\ (x + \tfrac{1}{2})^2 &= \tfrac{6}{4} \\ \therefore\ x + \tfrac{1}{2} &= \pm\tfrac{\sqrt{6}}{2} \\ \therefore\ x &= -\tfrac{1}{2} \pm \tfrac{\sqrt{6}}{2} \end{aligned}$$

EXERCISE 8E

1 **a** $x^2 - 4x - 3 = 0$
has $a = 1,\quad b = -4,\quad c = -3$
$$\begin{aligned} \therefore\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{4 \pm \sqrt{28}}{2} \\ &= \frac{4 \pm 2\sqrt{7}}{2} \\ &= 2 \pm \sqrt{7} \end{aligned}$$

b $x^2 + 6x + 7 = 0$
has $a = 1,\quad b = 6,\quad c = 7$
$$\begin{aligned} \therefore\ x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{8}}{2} \\ &= \frac{-6 \pm 2\sqrt{2}}{2} \\ &= -3 \pm \sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^2 + 1 &= 4x \\ \therefore\ x^2 - 4x + 1 &= 0 \end{aligned}$$
which has $a = 1,\quad b = -4,\quad c = 1$
$$\begin{aligned} \therefore\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{4 \pm \sqrt{12}}{2} \\ &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= 2 \pm \sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2 + 4x &= 1 \\ \therefore\ x^2 + 4x - 1 &= 0 \end{aligned}$$
which has $a = 1,\quad b = 4,\quad c = -1$
$$\begin{aligned} \therefore\ x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= \frac{-4 \pm 2\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5} \end{aligned}$$

e $x^2 - 4x + 2 = 0$
has $a = 1,\quad b = -4,\quad c = 2$
$$\begin{aligned} \therefore\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{8}}{2} \\ &= \frac{4 \pm 2\sqrt{2}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

f $2x^2 - 2x - 3 = 0$
has $a = 2,\quad b = -2,\quad c = -3$
$$\begin{aligned} \therefore\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{2 \pm \sqrt{28}}{4} \\ &= \frac{2 \pm 2\sqrt{7}}{4} \\ &= \tfrac{1}{2} \pm \tfrac{\sqrt{7}}{2} \end{aligned}$$

g $x^2 - 2\sqrt{2}x + 2 = 0$
has $a = 1,\quad b = -2\sqrt{2},\quad c = 2$
$$\begin{aligned} \therefore\ x &= \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2(1)} \\ &= \frac{2\sqrt{2} \pm \sqrt{8 - 8}}{2} \\ &= \sqrt{2} \end{aligned}$$

h
$$\begin{aligned}(3x+1)^2 &= -2x\\ \therefore\ 9x^2+6x+1 &= -2x\\ \therefore\ 9x^2+8x+1 &= 0\end{aligned}$$
which has $a=9,\quad b=8,\quad c=1$
$$\begin{aligned}\therefore\ x &= \frac{-8\pm\sqrt{8^2-4(9)(1)}}{2(9)}\\ &= \frac{-8\pm\sqrt{28}}{18}\\ &= \frac{-8\pm2\sqrt{7}}{18} \quad\text{or}\quad -\tfrac{4}{9}\pm\tfrac{\sqrt{7}}{9}\end{aligned}$$

i
$$\begin{aligned}(x+3)(2x+1) &= 9\\ \therefore\ 2x^2+x+6x+3 &= 9\\ \therefore\ 2x^2+7x-6 &= 0\end{aligned}$$
which has $a=2,\quad b=7,\quad c=-6$
$$\begin{aligned}\therefore\ x &= \frac{-7\pm\sqrt{7^2-4(2)(-6)}}{2(2)}\\ &= \frac{-7\pm\sqrt{49+48}}{4}\\ &= -\tfrac{7}{4}\pm\tfrac{\sqrt{97}}{4}\end{aligned}$$

2 **a**
$$\begin{aligned}(x+2)(x-1) &= 2-3x\\ \therefore\ x^2-x+2x-2 &= 2-3x\\ \therefore\ x^2+4x-4 &= 0\end{aligned}$$
which has $a=1,\quad b=4,\quad c=-4$
$$\begin{aligned}\therefore\ x &= \frac{-4\pm\sqrt{4^2-4(1)(-4)}}{2(1)}\\ &= \frac{-4\pm\sqrt{32}}{2}\\ &= \frac{-4\pm4\sqrt{2}}{2}\\ &= -2\pm2\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}(2x+1)^2 &= 3-x\\ \therefore\ 4x^2+4x+1 &= 3-x\\ \therefore\ 4x^2+5x-2 &= 0\end{aligned}$$
which has $a=4,\quad b=5,\quad c=-2$
$$\begin{aligned}\therefore\ x &= \frac{-5\pm\sqrt{5^2-4(4)(-2)}}{2(4)}\\ &= \frac{-5\pm\sqrt{25+32}}{8}\\ &= -\tfrac{5}{8}\pm\tfrac{\sqrt{57}}{8}\end{aligned}$$

c
$$\begin{aligned}(x-2)^2 &= 1+x\\ \therefore\ x^2-4x+4 &= 1+x\\ \therefore\ x^2-5x+3 &= 0\end{aligned}$$
which has $a=1,\quad b=-5,\quad c=3$
$$\begin{aligned}\therefore\ x &= \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)}\\ &= \frac{5\pm\sqrt{25-12}}{2}\\ &= \tfrac{5}{2}\pm\tfrac{\sqrt{13}}{2}\end{aligned}$$

d
$$\begin{aligned}\frac{x-1}{2-x} &= 2x+1\\ \therefore\ x-1 &= (2x+1)(2-x)\\ \therefore\ x-1 &= 4x-2x^2+2-x\\ \therefore\ 2x^2-2x-3 &= 0\end{aligned}$$
which has $a=2,\quad b=-2,\quad c=-3$
$$\begin{aligned}\therefore\ x &= \frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)}\\ &= \frac{2\pm\sqrt{28}}{4}\\ &= \frac{2\pm2\sqrt{7}}{4} \quad\text{or}\quad \tfrac{1}{2}\pm\tfrac{\sqrt{7}}{2}\end{aligned}$$

e
$$\begin{aligned}x-\frac{1}{x} &= 1\\ \therefore\ x^2-1 &= x\\ \therefore\ x^2-x-1 &= 0\end{aligned}$$
which has $a=1,\quad b=-1,\quad c=-1$
$$\begin{aligned}\therefore\ x &= \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}\\ &= \frac{1\pm\sqrt{1+4}}{2}\\ &= \tfrac{1}{2}\pm\tfrac{\sqrt{5}}{2}\end{aligned}$$

f
$$\begin{aligned}2x-\frac{1}{x} &= 3\\ \therefore\ 2x^2-1 &= 3x\\ \therefore\ 2x^2-3x-1 &= 0\end{aligned}$$
which has $a=2,\quad b=-3,\quad c=-1$
$$\begin{aligned}\therefore\ x &= \frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-1)}}{2(2)}\\ &= \frac{3\pm\sqrt{9+8}}{4}\\ &= \tfrac{3}{4}\pm\tfrac{\sqrt{17}}{4}\end{aligned}$$

EXERCISE 8F

1 **a** $x = -3.414$ or -0.586 **b** $x = 0.317$ or -6.317 **c** $x = 2.77$ or -1.27
d $x = -1.08$ or 3.41 **e** $x = -0.892$ or 3.64 **f** $x = 1.34$ or -2.54

2 **a** $x = -4.83$ or 0.828 **b** $x = -1.57$ or 0.319 **c** $x = 0.697$ or 4.30
d $x = -0.823$ or 1.82 **e** $x = -0.618$ or 1.62 **f** $x = -0.281$ or 1.78

EXERCISE 8G

1 Let the smaller of the integers be x.
Since they differ by 12,
the other integer is $(x + 12)$.

$\therefore$ the sum of their squares is

$$x^2 + (x+12)^2 = 74$$
$$\therefore\ x^2 + x^2 + 24x + 144 = 74$$
$$\therefore\ 2x^2 + 24x + 70 = 0$$
$$\therefore\ x^2 + 12x + 35 = 0$$
$$\therefore\ (x+7)(x+5) = 0$$
$$\therefore\ x = -7 \text{ or } -5$$

$\therefore$ the larger integer is 5 or 7

i.e., the integers are -7 and 5, or -5 and 7

2 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

They have sum $x + \dfrac{1}{x} = 5\frac{1}{5}$

$$\therefore\ x^2 + 1 = \tfrac{26}{5}x$$
$$\therefore\ x^2 - \tfrac{26}{5}x + 1 = 0$$
$$\therefore\ 5x^2 - 26x + 5 = 0$$
$$\therefore\ (5x-1)(x-5) = 0$$
$$\therefore\ x = \tfrac{1}{5} \text{ or } 5$$

i.e., the number is either $\frac{1}{5}$ or 5

3 Let the number be x so its square is x^2.

$\therefore$ the sum is $x + x^2 = 210$

$$\therefore\ x^2 + x - 210 = 0$$
$$\therefore\ (x+15)(x-14) = 0$$
$$\therefore\ x = -15 \text{ or } 14$$

But x is a natural number, so $x > 0$,

$\therefore$ the number is 14.

4 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 360$

$$\therefore\ x^2 + 2x - 360 = 0$$
$$\therefore\ (x+20)(x-18) = 0$$
$$\therefore\ x = -20 \text{ or } 18$$

$\therefore$ the numbers are -20 and -18,
or 18 and 20.

5 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 255$

$$\therefore\ x^2 + 2x - 255 = 0$$
$$\therefore\ (x+17)(x-15) = 0$$
$$\therefore\ x = -17 \text{ or } 15$$

$\therefore$ the numbers are -17 and -15,
or 15 and 17.

6 If the polygon has n sides, then

$$\frac{n}{2}(n-3) = 90$$
$$\therefore\ \tfrac{1}{2}n^2 - \tfrac{3}{2}n = 90$$
$$\therefore\ n^2 - 3n - 180 = 0$$
$$\therefore\ (n-15)(n+12) = 0$$
$$\therefore\ n = -12 \text{ or } 15$$

$\therefore$ the polygon has 15 sides. $\{$as $n > 0\}$

7 **a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm^2.

The box has height $(x+1)$ cm,
so the area of each of the side faces is $x(x+1)$ cm.

$\therefore$ the total surface area is

$$\begin{aligned} A &= 2x^2 + 4x(x+1) \\ &= 2x^2 + 4x^2 + 4x \\ &= 6x^2 + 4x \text{ cm}^2 \end{aligned}$$

b

$$6x^2 + 4x = 240$$
$$\therefore\ 3x^2 + 2x - 120 = 0$$
$$\therefore\ (3x+20)(x-6) = 0$$
$$\therefore\ x = -\tfrac{20}{3} \text{ or } 6$$

but $x > 0$, so $x = 6$ cm

$\therefore$ the box is 6 cm $\times$ 6 cm $\times$ 7 cm

8 If the width of the rectangle is w cm, then its length is $(w+4)$ cm.

$\therefore$ the area is $w(w+4) = 26$

$\therefore\ w^2 + 4w - 26 = 0$

which has $a = 1,\ \ b = 4,\ \ c = -26$

$$\therefore\ w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{120}}{2}$$

$$= \frac{-4 \pm 2\sqrt{30}}{2}$$

$$= -2 \pm \sqrt{30}$$

But $w > 0$, so $w = -2 + \sqrt{30}$
$\doteqdot 3.477$ cm

i.e., the width is approximately 3.48 cm.

9 Suppose the tin plate was x cm $\times$ x cm. When 3 cm $\times$ 3 cm squares are cut from the corners, the base of the open box formed is $(x-6)$ cm $\times$ $(x-6)$ cm.

The open box has height 3 cm, so its volume is $3 \times (x-6) \times (x-6) = 80$

$\therefore\ 3(x^2 - 12x + 36) = 80$

$\therefore\ 3x^2 - 36x + 108 = 80$

$\therefore\ 3x^2 - 36x + 28 = 0$

which has $a = 3,\ \ b = -36,\ \ c = 28$

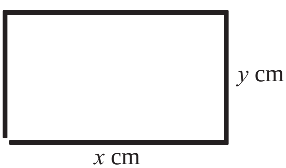
$$\therefore\ x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 0,$$

$$x = 6 + \frac{\sqrt{960}}{6} \doteqdot 11.16 \text{ cm}$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

10



Suppose one side of the rectangle has length x cm and the other has length y cm.
The perimeter is $(2x + 2y)$ cm,

so $2x + 2y = 20$

$\therefore\ 2y = 20 - 2x$

$\therefore\ y = 10 - x$

The area of the rectangle is therefore $x(10-x)$ cm^2.

$\therefore$ if the area is 30 cm^2, then

$x(10-x) = 30$

$\therefore\ 10x - x^2 = 30$

$\therefore\ x^2 - 10x + 30 = 0$

which has $a = 1,\ \ b = -10,\ \ c = 30$

$$\therefore\ x = \frac{-(-10)\sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - 120}}{2}$$

$$= \frac{10 \pm \sqrt{-120}}{2}$$

$\therefore$ x has no real solutions, so it is not possible.

11 The smaller rectangle is similar to the original rectangle.

$$\therefore\ \frac{AB}{AD} = \frac{BC}{BY}$$

But AD = BC and BY = AB $-$ AY
= AB $-$ AD

$$\therefore\ \frac{AB}{AD} = \frac{AD}{AB - AD}$$

Suppose AB = x units, and AD = BC = 1 unit

$$\therefore\ \frac{x}{1} = \frac{1}{x-1}$$

$\therefore\ x(x-1) = 1$

$\therefore\ x^2 - x - 1 = 0$

which has $a = 1,\ \ b = -1,\ \ c = -1$

$$\therefore\ x = \frac{-(-1)\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

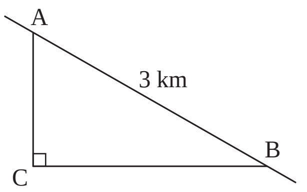
$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore\ x = \frac{1 + \sqrt{5}}{2}, \quad \text{since } x > 0$$

But $\dfrac{AB}{AD} = x$, which is the golden ratio

$\therefore$ the golden ratio is $\dfrac{1+\sqrt{5}}{2}$

12



Suppose AC is x hundred metres, so BC is $(x+4)$ hundred metres.

$$\begin{aligned} \text{Now} \quad AC^2 + BC^2 &= AB^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x^2 + (x+4)^2 &= 30^2 \\ \therefore \quad x^2 + x^2 + 8x + 16 &= 900 \\ \therefore \quad 2x^2 + 8x - 884 &= 0 \\ \therefore \quad x^2 + 4x - 442 &= 0 \end{aligned}$$

which has $a = 1$, $b = 4$, $c = -442$

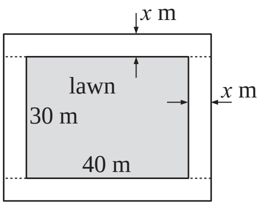
$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-442)}}{2(1)} = \frac{-4 \pm \sqrt{1784}}{2}$$

$\therefore$ since $x > 0$, $x = \dfrac{-4 \pm \sqrt{1784}}{2} \doteqdot 19.12$

$\therefore$ AC $\doteqdot$ 19.12 hundred metres and BC $\doteqdot$ 23.12 hundred metres

$\therefore$ since the paddock is triangular, its area is $\frac{1}{2} \times 19.12 \times 23.12 \doteqdot 221$ hectares.

13



Suppose the concrete has width x m around the lawn. We divide the concrete up into four regions as shown.

The smaller regions have area $30x$ m^2, whilst the larger regions have area $x(40+2x)$ m^2.

Now the total area of concrete is one quarter the area of the lawn.

$$\begin{aligned} \therefore \quad 2 \times 30x + 2 \times x(40+2x) &= \tfrac{1}{4} \times 30 \times 40 \\ \therefore \quad 60x + 80x + 4x^2 &= 300 \\ \therefore \quad 4x^2 + 140x &= 300 \\ \therefore \quad x^2 + 35x - 75 &= 0 \end{aligned}$$

which has $a = 1$, $b = 35$, $c = -75$

$$\therefore \quad x = \frac{-35 \pm \sqrt{35^2 - 4(1)(-75)}}{2(1)} = \frac{-35 \pm \sqrt{1525}}{2}$$

But $x > 0$,

so $x = \dfrac{-35 + \sqrt{1525}}{2} \doteqdot 2.026$ m

$\therefore$ the path is about 2.03 m wide.

14 Suppose Hassan's speed is h kmph.

We know that $\text{speed} = \dfrac{\text{distance}}{\text{time}}$,

so $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$\therefore$ if it takes Hassan t hours, $t = \dfrac{40}{h}$ .... (1)

Now Chuong says he will drive home at speed $(h+40)$ kmph and arrive in time $(t - \frac{1}{3})$ hrs.

$\therefore \quad t - \frac{1}{3} = \dfrac{40}{h+40}$

i.e., $t = \dfrac{40}{h+40} + \frac{1}{3}$ ..... (2)

$\therefore$ using (1) and (2), $\dfrac{40}{h} = \dfrac{40}{h+40} + \frac{1}{3}$

$$\begin{aligned} \therefore \quad 40(h+40) &= 40h + \tfrac{1}{3}h(h+40) \\ \therefore \quad 40h + 1600 &= 40h + \tfrac{1}{3}h^2 + \tfrac{40}{3}h \\ \therefore \quad h^2 + 40h - 4800 &= 0 \end{aligned}$$

which has $a = 1$, $b = 40$, $c = -4800$

$$\therefore \quad h = \frac{-40 \pm \sqrt{40^2 - 4(1)(-4800)}}{2(1)} = \frac{-40 \pm \sqrt{1600 + 19\,200}}{2} = \frac{-40 \pm \sqrt{20\,800}}{2}$$

But $h > 0$, so $h = \dfrac{-40 + \sqrt{20\,800}}{2} \doteqdot 52.1$ kmph

i.e., Hassan's speed is approximately 52.1 kmph.

15 Suppose the speed of the plane is x kmph.

We know speed $= \dfrac{\text{distance}}{\text{time}}$, so time $= \dfrac{\text{distance}}{\text{speed}}$

Using the information given, $\dfrac{1000}{x} = \dfrac{1000}{x-120} - \frac{1}{2}$

$$\therefore\ 1000(x-120) = 1000x - \tfrac{1}{2}x(x-120)$$
$$\therefore\ 1000x - 120\,000 = 1000x - \tfrac{1}{2}x^2 + 60x$$
$$\therefore\ x^2 - 120x - 240\,000 = 0$$

which has $a = 1$, $b = -120$, $c = -240\,000$

$$\therefore\ x = \frac{-(-120) \pm \sqrt{(-120)^2 - 4(1)(-240\,000)}}{2(1)}$$
$$= \frac{120 \pm \sqrt{974\,400}}{2}$$

But $x > 0$, so $x = \dfrac{120 + \sqrt{974\,400}}{2} \doteqdot 553.6$ kmph,

$\therefore$ the plane has speed approximately 554 kmph.

16 Suppose the express train travels at x kmph (on average).

We know speed $= \dfrac{\text{distance}}{\text{time}}$, so time $= \dfrac{\text{distance}}{\text{speed}}$

$\therefore$ it takes the express train $\dfrac{105}{x}$ hours and the normal train $\dfrac{105}{x-10}$ hours.

$$\therefore\ \frac{105}{x} + \tfrac{1}{2} = \frac{105}{x-10}$$
$$\therefore\ 105(x-10) + \tfrac{1}{2}x(x-10) = 105x$$
$$\therefore\ 105x - 1050 + \tfrac{1}{2}x^2 - 5x = 105x$$
$$\therefore\ x^2 - 10x - 2100 = 0$$

which has $a = 1$, $b = -10$, $c = -2100$

$$\therefore\ x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-2100)}}{2(1)} = \frac{10 \pm \sqrt{8500}}{2}$$

But $x > 0$, so $x = \dfrac{10 + \sqrt{8500}}{2} \doteqdot 51.1$ kmph

$\therefore$ the express train travels on average at about 51.1 kmph.

17 Suppose n elderly citizens ended up going on the trip, so the cost per person was $\$\dfrac{160}{n}$.

If the original number of elderly citizens had gone, there would have been $(n+8)$,

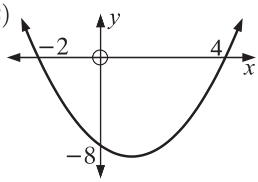
and the cost per person would have been $\$\dfrac{160}{n+8}$.

$$\text{Hence } \frac{160}{n} = \frac{160}{n+8} + 1$$
$$\therefore\ 160(n+8) = 160n + n(n+8)$$
$$\therefore\ 160n + 1280 = 160n + n^2 + 8n$$
$$\therefore\ n^2 + 8n - 1280 = 0$$
$$\therefore\ (n-32)(n+40) = 0$$

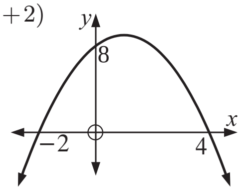
$\therefore$ since $n > 0$, $n = 32$, i.e., 32 elderly citizens went on the trip.

EXERCISE 8H

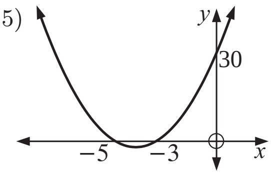
1 **a** $y = (x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept -8



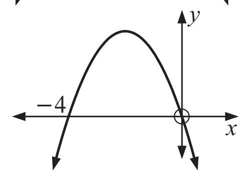
b $y = -(x-4)(x+2)$ has x-intercepts -2 and 4 and y-intercept 8



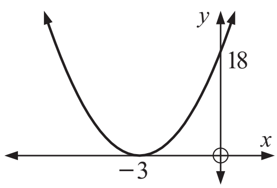
c $y = 2(x+3)(x+5)$ has x-intercepts -5 and -3 and y-intercept 30



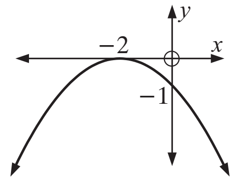
d $y = -3x(x+4)$ has x-intercepts 0 and -4 and y-intercept 0



e $y = 2(x+3)^2$ has x-intercept -3 and y-intercept 18



f $y = -\frac{1}{4}(x+2)^2$ has x-intercept -2 and y-intercept -1



2 **a** The average of the x-intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

b The average of the x-intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

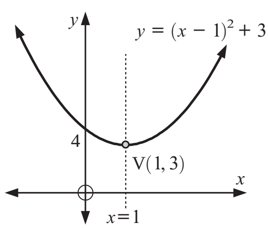
c The average of the x-intercepts is -4 $\therefore$ the axis of symmetry is $x = -4$.

d The average of the x-intercepts is -2 $\therefore$ the axis of symmetry is $x = -2$.

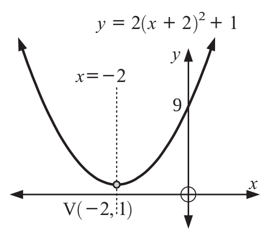
e The only x-intercept is -3, so the axis of symmetry is $x = -3$.

f The only x-intercept is -2, so the axis of symmetry is $x = -2$.

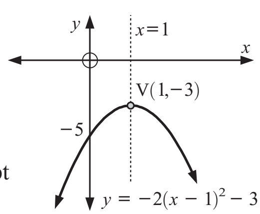
3 **a** The vertex is $(1, 3)$. The axis of symmetry is $x = 1$. The y-intercept is 4.



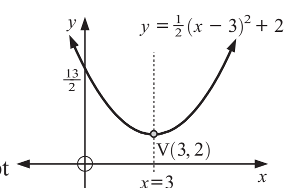
b The vertex is $(-2, 1)$. The axis of symmetry is $x = -2$. The y-intercept is 9.



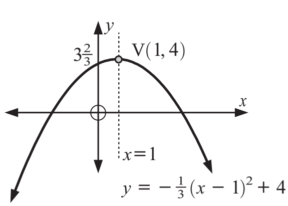
c The vertex is $(1, -3)$. The axis of symmetry is $x = 1$. The y-intercept is -5.



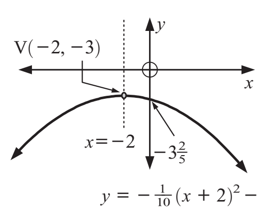
d The vertex is $(3, 2)$. The axis of symmetry is $x = 3$. The y-intercept is $\frac{13}{2}$.



e The vertex is $(1, 4)$. The axis of symmetry is $x = 1$. The y-intercept is $3\frac{2}{3}$.



f The vertex is $(-2, -3)$. The axis of symmetry is $x = -2$. The y-intercept is $-3\frac{2}{5}$.



4 **a** $y = x^2 - 4x + 2$ has $a = 1$, $b = -4$, $c = 2$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(1)} = 2$

$\therefore$ the axis of symmetry is $x = 2$

When $x = 2$,
$y = 2^2 - 4 \times 2 + 2 = -2$

$\therefore$ the vertex is at $(2, -2)$.

b $y = x^2 + 2x - 3$ has $a = 1$, $b = 2$, $c = -3$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{2}{2(1)} = -1$

$\therefore$ the axis of symmetry is $x = -1$

When $x = -1$,
$y = (-1)^2 + 2(-1) - 3 = -4$

$\therefore$ the vertex is at $(-1, -4)$.

c $y = 2x^2 + 4$ has $a = 2$, $b = 0$, $c = 4$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{0}{2(2)} = 0$

$\therefore$ the axis of symmetry is $x = 0$

When $x = 0$, $y = 4$

$\therefore$ the vertex is at $(0, 4)$.

d $y = -3x^2 + 1$ has $a = -3$, $b = 0$, $c = 1$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{0}{2(-3)} = 0$

$\therefore$ the axis of symmetry is $x = 0$

When $x = 0$, $y = 1$

$\therefore$ the vertex is at $(0, 1)$.

e $y = 2x^2 + 8x - 7$ has $a = 2$, $b = 8$, $c = -7$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{8}{2(2)} = -2$

$\therefore$ the axis of symmetry is $x = -2$

When $x = -2$,
$y = 2(-2)^2 + 8(-2) - 7 = -15$

$\therefore$ the vertex is at $(-2, -15)$.

f $y = -x^2 - 4x - 9$ has $a = -1$, $b = -4$, $c = -9$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{(-4)}{2(-1)} = -2$

$\therefore$ the axis of symmetry is $x = -2$

When $x = -2$, $y = -(-2)^2 - 4(-2) - 9$
$= -4 + 8 - 9$
$= -5$

$\therefore$ the vertex is at $(-2, -5)$.

g $y = 2x^2 + 6x - 1$ has $a = 2$, $b = 6$, $c = -1$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{6}{2(2)} = -\dfrac{3}{2}$

$\therefore$ the axis of symmetry is $x = -\frac{3}{2}$

When $x = -\frac{3}{2}$,
$y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 1$
$= \frac{9}{2} - 9 - 1$
$= -\frac{11}{2}$

$\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

h $y = 2x^2 - 10x + 3$ has $a = 2$, $b = -10$, $c = 3$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{(-10)}{2(2)} = \dfrac{5}{2}$

$\therefore$ the axis of symmetry is $x = \frac{5}{2}$

When $x = \frac{5}{2}$, $y = 2(\frac{5}{2})^2 - 10(\frac{5}{2}) + 3$
$= \frac{25}{2} - \frac{50}{2} + 3$
$= -\frac{19}{2}$

$\therefore$ the vertex is at $(\frac{5}{2}, -\frac{19}{2})$.

i $y = -\frac{1}{2}x^2 + x - 5$ has $a = -\frac{1}{2}$, $b = 1$, $c = -5$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{1}{2(-\frac{1}{2})} = 1$

$\therefore$ the axis of symmetry is $x = 1$

When $x = 1$,
$y = -\frac{1}{2}(1)^2 + 1 - 5 = -\frac{9}{2}$

$\therefore$ the vertex is at $(1, -\frac{9}{2})$.

j $y = -2x^2 + 8x - 2$ has $a = -2$, $b = 8$, $c = -2$

$\therefore\ -\dfrac{b}{2a} = -\dfrac{8}{2(-2)} = 2$

$\therefore$ the axis of symmetry is $x = 2$

When $x = 2$, $y = -2(2)^2 + 8(2) - 2$
$= -8 + 16 - 2$
$= 6$

$\therefore$ the vertex is at $(2, 6)$.

5 **a** When $y = 0$, $x^2 - 9 = 0$

$\therefore \ (x+3)(x-3) = 0$

$\therefore \ x = \pm 3$

$\therefore$ the x-intercepts are ± 3

b When $y = 0$, $2x^2 - 6 = 0$

$\therefore \ x^2 - 3 = 0$

$\therefore \ (x+\sqrt{3})(x-\sqrt{3}) = 0$

$\therefore \ x = \pm\sqrt{3}$

$\therefore$ the x-intercepts are $\pm\sqrt{3}$

c When $y = 0$, $x^2 + 7x + 10 = 0$

$\therefore \ (x+5)(x+2) = 0$

$\therefore \ x = -5$ or -2

$\therefore$ the x-intercepts are -5 and -2

d When $y = 0$, $x^2 + x - 12 = 0$

$\therefore \ (x+4)(x-3) = 0$

$\therefore \ x = -4$ or 3

$\therefore$ the x-intercepts are -4 and 3

e When $y = 0$, $4x - x^2 = 0$

$\therefore \ x(4-x) = 0$

$\therefore \ x = 0$ or 4

$\therefore$ the x-intercepts are 0 and 4

f When $y = 0$, $-x^2 - 6x - 8 = 0$

$\therefore \ x^2 + 6x + 8 = 0$

$\therefore \ (x+4)(x+2) = 0$

$\therefore \ x = -4$ or -2

$\therefore$ the x-intercepts are -4 and -2

g When $y = 0$, $-2x^2 - 4x - 2 = 0$

$\therefore \ x^2 + 2x + 1 = 0$

$\therefore \ (x+1)^2 = 0$

$\therefore \ x = -1$

$\therefore$ the x-intercept is -1 (touching)

h When $y = 0$, $4x^2 - 24x + 36 = 0$

$\therefore \ x^2 - 6x + 9 = 0$

$\therefore \ (x-3)^2 = 0$

$\therefore \ x = 3$

$\therefore$ the x-intercept is 3 (touching)

i When $y = 0$, $x^2 - 4x + 1 = 0$

$a = 1$, $b = -4$ and $c = 1$

$$\therefore \ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

$\therefore$ the x-intercepts are $2 \pm \sqrt{3}$

j When $y = 0$, $x^2 + 4x - 3 = 0$

$a = 1$, $b = 4$ and $c = -3$

$$\therefore \ x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{28}}{2}$$

$$= \frac{-4 \pm 2\sqrt{7}}{2}$$

$$= -2 \pm \sqrt{7}$$

$\therefore$ the x-intercepts are $-2 \pm \sqrt{7}$

k When $y = 0$, $x^2 - 6x - 2 = 0$

$a = 1$, $b = -6$ and $c = -2$

$$\therefore \ x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{44}}{2}$$

$$= \frac{6 \pm 2\sqrt{11}}{2}$$

$$= 3 \pm \sqrt{11}$$

$\therefore$ the x-intercepts are $3 \pm \sqrt{11}$

l When $y = 0$, $x^2 + 8x + 11 = 0$

$a = 1$, $b = 8$ and $c = 11$

$$\therefore \ x = \frac{-8 \pm \sqrt{8^2 - 4(1)(11)}}{2(1)}$$

$$= \frac{-8 \pm \sqrt{20}}{2}$$

$$= \frac{-8 \pm 2\sqrt{5}}{2}$$

$$= -4 \pm \sqrt{5}$$

$\therefore$ the x-intercepts are $-4 \pm \sqrt{5}$

6 **a** **i** $y = x^2 - 2x + 5$

has $a = 1, \quad b = -2, \quad c = 5$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{(-2)}{2(1)} = 1$

$\therefore$ the axis of symmetry is $x = 1$

ii When $x = 1$,

$$\begin{aligned} y &= 1^2 - 2(1) + 5 \\ &= 1 - 2 + 5 \\ &= 4 \end{aligned}$$

$\therefore$ the vertex is at $(1, 4)$

iii When $x = 0, \quad y = 5$,

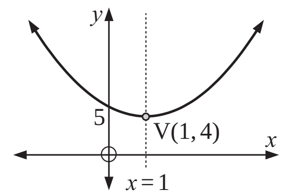
so the y-intercept is 5

When $y = 0, \quad x^2 - 2x + 5 = 0$

$$\begin{aligned} \therefore \quad x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 - 20}}{2} \end{aligned}$$

This has no real solutions,
so there are no x-intercepts.

iv



b **i** $y = x^2 + 4x - 1$

has $a = 1, \quad b = 4, \quad c = -1$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{4}{2(1)} = -2$

$\therefore$ the axis of symmetry is $x = -2$

ii When $x = -2$,

$$\begin{aligned} y &= (-2)^2 + 4(-2) - 1 \\ &= 4 - 8 - 1 \\ &= -5 \end{aligned}$$

$\therefore$ the vertex is at $(-2, -5)$

iii When $x = 0, \quad y = -1$,

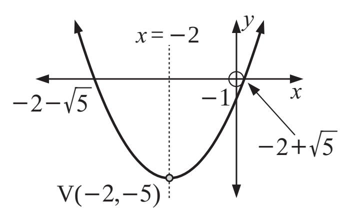
so the y-intercept is -1.

When $y = 0, \quad x^2 + 4x - 1 = 0$

$$\begin{aligned} \therefore \quad x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= \frac{-4 \pm 2\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5} \end{aligned}$$

$\therefore$ the x-intercepts are $-2 \pm \sqrt{5}$

iv



c **i** $y = 2x^2 - 5x + 2$

has $a = 2, \quad b = -5, \quad c = 2$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{(-5)}{2(2)} = \dfrac{5}{4}$

$\therefore$ the axis of symmetry is $x = \frac{5}{4}$

ii When $x = \frac{5}{4}$,

$$\begin{aligned} y &= 2(\tfrac{5}{4})^2 - 5(\tfrac{5}{4}) + 2 \\ &= \tfrac{50}{16} - \tfrac{25}{4} + 2 \\ &= -\tfrac{9}{8} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iii When $x = 0, \quad y = 2$,

so the y-intercept is 2.

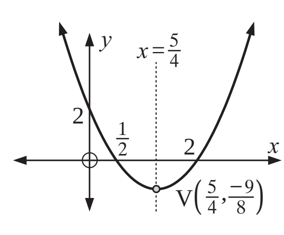
When $y = 0, \quad 2x^2 - 5x + 2 = 0$

$\therefore \quad (2x - 1)(x - 2) = 0$

$\therefore \quad x = \frac{1}{2}$ or 2

$\therefore$ the x-intercepts are $\frac{1}{2}$ and 2

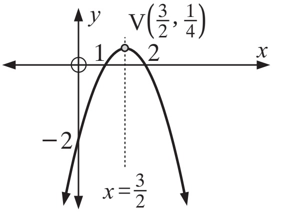
iv



d **i** $y = -x^2 + 3x - 2$
has $a = -1,\quad b = 3,\quad c = -2$
$\therefore\quad -\dfrac{b}{2a} = -\dfrac{3}{2(-1)} = \dfrac{3}{2}$
$\therefore$ the axis of symmetry is $x = \frac{3}{2}$

ii When $x = \frac{3}{2}$,
$y = -(\frac{3}{2})^2 + 3(\frac{3}{2}) - 2$
$= -\frac{9}{4} + \frac{9}{2} - 2$
$= \frac{1}{4}$
$\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$

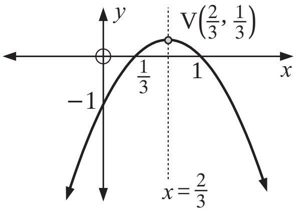
iii When $x = 0,\quad y = -2$,
so the y-intercept is -2.
When $y = 0,\quad -x^2 + 3x - 2 = 0$
$\therefore\quad x^2 - 3x + 2 = 0$
$\therefore\quad (x-1)(x-2) = 0$
$\therefore\quad x = 1$ or 2
$\therefore$ the x-intercepts are 1 and 2

iv


e **i** $y = -3x^2 + 4x - 1$
has $a = -3,\quad b = 4,\quad c = -1$
$\therefore\quad -\dfrac{b}{2a} = -\dfrac{4}{2(-3)} = \dfrac{2}{3}$
$\therefore$ the axis of symmetry is $x = \frac{2}{3}$

ii When $x = \frac{2}{3}$,
$y = -3(\frac{2}{3})^2 + 4(\frac{2}{3}) - 1$
$= -\frac{4}{3} + \frac{8}{3} - 1$
$= \frac{1}{3}$
$\therefore$ the vertex is at $(\frac{2}{3}, \frac{1}{3})$

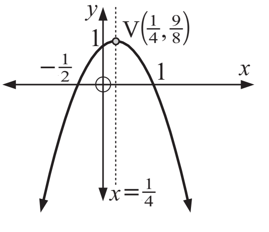
iii When $x = 0,\quad y = -1$,
so the y-intercept is -1.
When $y = 0,\quad -3x^2 + 4x - 1 = 0$
$\therefore\quad 3x^2 - 4x + 1 = 0$
$\therefore\quad (3x-1)(x-1) = 0$
$\therefore\quad x = \frac{1}{3}$ or 1
$\therefore$ the x-intercepts are $\frac{1}{3}$ and 1

iv


f **i** $y = -2x^2 + x + 1$
has $a = -2,\quad b = 1,\quad c = 1$
$\therefore\quad -\dfrac{b}{2a} = -\dfrac{1}{2(-2)} = \dfrac{1}{4}$
$\therefore$ the axis of symmetry is $x = \frac{1}{4}$

ii When $x = \frac{1}{4}$,
$y = -2(\frac{1}{4})^2 + \frac{1}{4} + 1$
$= -\frac{1}{8} + \frac{1}{4} + 1$
$= \frac{9}{8}$
$\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$

iii When $x = 0,\quad y = 1$,
so the y-intercept is 1.
When $y = 0,\quad -2x^2 + x + 1 = 0$
$\therefore\quad 2x^2 - x - 1 = 0$
$\therefore\quad (2x+1)(x-1) = 0$
$\therefore\quad x = -\frac{1}{2}$ or 1
$\therefore$ the x-intercepts are $-\frac{1}{2}$ and 1

iv


g **i** $y = 6x - x^2$
has $a = -1, \quad b = 6, \quad c = 0$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{6}{2(-1)} = 3$

$\therefore$ the axis of symmetry is $x = 3$

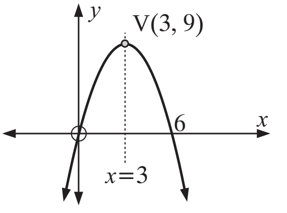
ii When $x = 3$,
$y = 6 \times 3 - 3^2$
$= 9$

$\therefore$ the vertex is at $(3, 9)$

iii When $x = 0, \quad y = 0$,
so the y-intercept is 0.
When $y = 0, \quad 6x - x^2 = 0$
$\therefore \quad x(6 - x) = 0$
$\therefore \quad x = 0$ or 6

$\therefore$ the x-intercepts are 0 and 6

iv



h **i** $y = -x^2 - 6x - 8$
has $a = -1, \quad b = -6, \quad c = -8$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{(-6)}{2(-1)} = -3$

$\therefore$ the axis of symmetry is $x = -3$

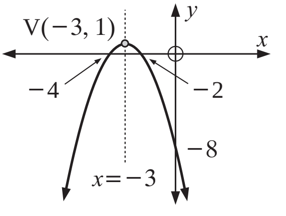
ii When $x = -3$,
$y = -(-3)^2 - 6(-3) - 8$
$= -9 + 18 - 8$
$= 1$

$\therefore$ the vertex is at $(-3, 1)$

iii When $x = 0, \quad y = -8$,
so the y-intercept is -8.
When $y = 0, \quad -x^2 - 6x - 8 = 0$
$\therefore \quad x^2 + 6x + 8 = 0$
$\therefore \quad (x + 4)(x + 2) = 0$
$\therefore \quad x = -4$ or -2

$\therefore$ the x-intercepts are -4 and -2

iv



i **i** $y = -\frac{1}{4}x^2 + 2x + 1$
has $a = -\frac{1}{4}, \quad b = 2, \quad c = 1$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{2}{2(-\frac{1}{4})} = 4$

$\therefore$ the axis of symmetry is $x = 4$

ii When $x = 4$,
$y = -\frac{1}{4}(4)^2 + 2(4) + 1$
$= -4 + 8 + 1$
$= 5$

$\therefore$ the vertex is at $(4, 5)$

iii When $x = 0, \quad y = 1$,
so the y-intercept is 1.
When $y = 0, \quad -\frac{1}{4}x^2 + 2x + 1 = 0$
$\therefore \quad x^2 - 8x - 4 = 0$

which has $a = 1, \quad b = -8, \quad c = -4$

$$\therefore \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)}$$

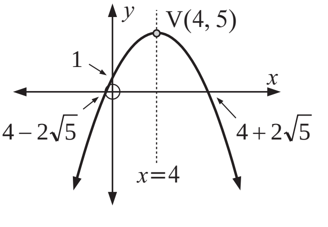
$$= \frac{8 \pm \sqrt{80}}{2}$$

$$= \frac{8 \pm 4\sqrt{5}}{2}$$

$$= 4 \pm 2\sqrt{5}$$

$\therefore$ the x-intercepts are $4 \pm 2\sqrt{5}$

iv



EXERCISE 8I.1

1 **a** $x^2 + 7x - 2 = 0$
has $a = 1, \quad b = 7, \quad c = -2$

$$\therefore \quad \Delta = b^2 - 4ac = 7^2 - 4(1)(-2) = 57$$

Since $\Delta > 0$, there are 2 distinct real solutions.

b $x^2 + 4\sqrt{2}x + 8 = 0$
has $a = 1, \quad b = 4\sqrt{2}, \quad c = 8$

$$\therefore \quad \Delta = b^2 - 4ac = (4\sqrt{2})^2 - 4(1)(8) = 32 - 32 = 0$$

$\therefore$ there is one repeated real root.

c $2x^2 + 3x - 1 = 0$
has $a = 2, \quad b = 3, \quad c = -1$

$$\therefore \quad \Delta = b^2 - 4ac = 3^2 - 4(2)(-1) = 17$$

Since $\Delta > 0$, there are 2 distinct real solutions.

d $6x^2 + 5x - 4 = 0$
has $a = 6, \quad b = 5, \quad c = -4$

$$\therefore \quad \Delta = b^2 - 4ac = 5^2 - 4(6)(-4) = 121$$

Since $\Delta > 0$, there are 2 distinct real solutions.

e $x^2 + x + 6 = 0$
has $a = 1, \quad b = 1, \quad c = 6$

$$\therefore \quad \Delta = b^2 - 4ac = 1^2 - 4(1)(6) = -23$$

Since $\Delta < 0$, there are no real roots.

f $9x^2 + 6x + 1 = 0$
has $a = 9, \quad b = 6, \quad c = 1$

$$\therefore \quad \Delta = b^2 - 4ac = 6^2 - 4(9)(1) = 0$$

$\therefore$ there is one repeated real root.

2 **a** $2x^2 + 7x - 4 = 0$
has $a = 2, \quad b = 7, \quad c = -4$

$$\therefore \quad \Delta = b^2 - 4ac = 7^2 - 4(2)(-4) = 81$$

$\therefore \quad \sqrt{\Delta} = 9$, so the equation has rational roots.

b $3x^2 - 7x - 6 = 0$
has $a = 3, \quad b = -7, \quad c = -6$

$$\therefore \quad \Delta = b^2 - 4ac = (-7)^2 - 4(3)(-6) = 121$$

$\therefore \quad \sqrt{\Delta} = 11$, so the equation has rational roots.

c $2x^2 + 6x + 1 = 0$
has $a = 2, \quad b = 6, \quad c = 1$

$$\therefore \quad \Delta = b^2 - 4ac = 6^2 - 4(2)(1) = 28$$

$\therefore \quad \sqrt{\Delta} = \sqrt{28} = 2\sqrt{7}$, so the equation does not have rational roots.

d $6x^2 + 19x + 10 = 0$
has $a = 6, \quad b = 19, \quad c = 10$

$$\therefore \quad \Delta = b^2 - 4ac = 19^2 - 4(6)(10) = 121$$

$\therefore \quad \sqrt{\Delta} = 11$, so the equation has rational roots.

e $4x^2 - 3x + 3 = 0$
has $a = 4, \quad b = -3, \quad c = 3$

$$\therefore \quad \Delta = b^2 - 4ac = (-3)^2 - 4(4)3 = -39$$

Since $\Delta < 0$, the equation does not have rational roots.

f $8x^2 - 10x - 3 = 0$
has $a = 8, \quad b = -10, \quad c = -3$

$$\therefore \quad \Delta = b^2 - 4ac = (-10)^2 - 4(8)(-3) = 196$$

$\therefore \quad \sqrt{\Delta} = 14$, so the equation has rational roots.

3 **a** $x^2+3x+m=0$ has $a=1,\ b=3,\ c=m$

$\therefore\ \Delta = b^2-4ac = 3^2-4(1)m = 9-4m$

i For a repeated root, $\Delta = 0$ $\therefore\ 9-4m=0$ $\therefore\ 4m=9$ $\therefore\ m=\frac{9}{4}$

ii For two distinct real roots, $\Delta > 0$ $\therefore\ 9-4m>0$ $\therefore\ 4m<9$ $\therefore\ m<\frac{9}{4}$

iii For no real roots, $\Delta < 0$ $\therefore\ 9-4m<0$ $\therefore\ 4m>9$ $\therefore\ m>\frac{9}{4}$

b $x^2-5x+m=0$ has $a=1,\ b=-5,\ c=m$

$\therefore\ \Delta = b^2-4ac = (-5)^2-4(1)m = 25-4m$

i For a repeated root, $\Delta = 0$ $\therefore\ 25-4m=0$ $\therefore\ 4m=25$ $\therefore\ m=\frac{25}{4}$

ii For two distinct real roots, $\Delta > 0$ $\therefore\ 25-4m>0$ $\therefore\ 4m<25$ $\therefore\ m<\frac{25}{4}$

iii For no real roots, $\Delta < 0$ $\therefore\ 25-4m<0$ $\therefore\ 4m>25$ $\therefore\ m>\frac{25}{4}$

c $mx^2-x+1=0$ has $a=m,\ b=-1,\ c=1$

$\therefore\ \Delta = b^2-4ac = (-1)^2-4m(1) = 1-4m$

i For a repeated root, $\Delta = 0$ $\therefore\ 1-4m=0$ $\therefore\ 4m=1$ $\therefore\ m=\frac{1}{4}$

ii For two distinct real roots, $\Delta > 0$ $\therefore\ 1-4m>0$ $\therefore\ 4m<1$ $\therefore\ m<\frac{1}{4}$

iii For no real roots, $\Delta < 0$ $\therefore\ 1-4m<0$ $\therefore\ 4m>1$ $\therefore\ m>\frac{1}{4}$

d $mx^2+2x+3=0$ has $a=m,\ b=2,\ c=3$

$\therefore\ \Delta = b^2-4ac = 2^2-4m(3) = 4-12m$

i For a repeated root, $\Delta = 0$ $\therefore\ 4-12m=0$ $\therefore\ 12m=4$ $\therefore\ m=\frac{1}{3}$

ii For two distinct real roots, $\Delta > 0$ $\therefore\ 4-12m>0$ $\therefore\ 12m<4$ $\therefore\ m<\frac{1}{3}$

iii For no real roots, $\Delta < 0$ $\therefore\ 4-12m<0$ $\therefore\ 12m>4$ $\therefore\ m>\frac{1}{3}$

e $2x^2+7x+m=0$ has $a=2,\ b=7,\ c=m$

$\therefore\ \Delta = b^2-4ac = 7^2-4(2)m = 49-8m$

i For a repeated root, $\Delta = 0$ $\therefore\ 49-8m=0$ $\therefore\ 8m=49$ $\therefore\ m=\frac{49}{8}$

ii For two distinct real roots, $\Delta > 0$ $\therefore\ 49-8m>0$ $\therefore\ 8m<49$ $\therefore\ m<\frac{49}{8}$

iii For no real roots, $\Delta < 0$ $\therefore\ 49-8m<0$ $\therefore\ 8m>49$ $\therefore\ m>\frac{49}{8}$

f $mx^2-5x+4=0$ has $a=m,\ b=-5,\ c=4$

$\therefore\ \Delta = b^2-4ac = (-5)^2-4m(4) = 25-16m$

i For a repeated root,
$$\Delta = 0$$
$$\therefore\ 25 - 16m = 0$$
$$\therefore\ 16m = 25$$
$$\therefore\ m = \tfrac{25}{16}$$

ii For two distinct real roots,
$$\Delta > 0$$
$$\therefore\ 25 - 16m > 0$$
$$\therefore\ 16m < 25$$
$$\therefore\ m < \tfrac{25}{16}$$

iii For no real roots,
$$\Delta < 0$$
$$\therefore\ 25 - 16m < 0$$
$$\therefore\ 16m > 25$$
$$\therefore\ m > \tfrac{25}{16}$$

EXERCISE 8I.2

1 **a** $y = x^2 + 7x - 2$
has $a = 1,\quad b = 7,\quad c = -2$
$$\therefore\ \Delta = b^2 - 4ac = 7^2 - 4(1)(-2) = 57$$
Since $\Delta > 0$, the graph cuts the x-axis twice.

b $y = x^2 + 4\sqrt{2}x + 8$
has $a = 1,\quad b = 4\sqrt{2},\quad c = 8$
$$\therefore\ \Delta = b^2 - 4ac = (4\sqrt{2})^2 - 4(1)(8) = 0$$
$\therefore$ the graph touches the x-axis.

c $y = -2x^2 + 3x + 1$
has $a = -2,\quad b = 3,\quad c = 1$
$$\therefore\ \Delta = b^2 - 4ac = 3^2 - 4(-2)(1) = 17$$
Since $\Delta > 0$, the graph cuts the x-axis twice.

d $y = 6x^2 + 5x - 4$
has $a = 6,\quad b = 5,\quad c = -4$
$$\therefore\ \Delta = b^2 - 4ac = 5^2 - 4(6)(-4) = 121$$
Since $\Delta > 0$, the graph cuts the x-axis twice.

e $y = -x^2 + x + 6$
has $a = -1,\quad b = 1,\quad c = 6$
$$\therefore\ \Delta = b^2 - 4ac = 1^2 - 4(-1)(6) = 25$$
Since $\Delta > 0$, the graph cuts the x-axis twice.

f $y = 9x^2 + 6x + 1$
has $a = 9,\quad b = 6,\quad c = 1$
$$\therefore\ \Delta = b^2 - 4ac = 6^2 - 4(9)(1) = 0$$
$\therefore$ the graph touches the x-axis.

2 **a** $x^2 - 3x + 6$
has $a = 1,\quad b = -3,\quad c = 6$
$$\therefore\ \Delta = b^2 - 4ac = (-3)^2 - 4(1)(6) = -15$$
$\therefore$ since $a > 0$ and $\Delta < 0$,
$x^2 - 3x + 6 > 0$ for all x.

b $4x - x^2 - 6$
has $a = -1,\quad b = 4,\quad c = -6$
$$\therefore\ \Delta = b^2 - 4ac = 4^2 - 4(-1)(-6) = -8$$
$\therefore$ since $a < 0$ and $\Delta < 0$,
$4x - x^2 - 6 < 0$ for all x.

c $2x^2 - 4x + 7$
has $a = 2,\quad b = -4,\quad c = 7$
$$\therefore\ \Delta = b^2 - 4ac = (-4)^2 - 4(2)(7) = -40$$
$\therefore$ since $a > 0$ and $\Delta < 0$,
$2x^2 - 4x + 7 > 0$ for all x.
$\therefore$ it is positive definite.

d $-2x^2 + 3x - 4$
has $a = -2,\quad b = 3,\quad c = -4$
$$\therefore\ \Delta = b^2 - 4ac = 3^2 - 4(-2)(-4) = -23$$
$\therefore$ since $a < 0$ and $\Delta < 0$,
$-2x^2 + 3x - 4 < 0$ for all x.
$\therefore$ it is negative definite.

3 $3x^2 + kx - 1$ has $a = 3,\quad b = k,\quad c = -1$

$$\therefore \quad \Delta = b^2 - 4ac = k^2 - 4(3)(-1) = k^2 + 12$$

Now $k^2 \geqslant 0$ for all k

$\therefore\ k^2 + 12 > 0$ for all k

$\therefore\ \Delta > 0$ for all k

$\therefore\ 3x^2 + kx - 1$ has two real distinct roots for all k.

$\therefore$ it can never be positive definite.

4 $2x^2 + kx + 2$ has $a = 2,\quad b = k,\quad c = 2$

$$\therefore \quad \Delta = b^2 - 4ac = k^2 - 4(2)(2) = k^2 - 16$$

Now $2x^2 + kx + 2$ has $a > 0$.

$\therefore$ it is positive definite provided $k^2 - 16 < 0$

$\therefore\ k^2 < 16$

$\therefore\ -4 < k < 4$

EXERCISE 8J

1 **a** The x-intercepts are 1 and 2.

$\therefore\ y = a(x-1)(x-2)$ for some $a \neq 0$.

But the y-intercept is 4.

$\therefore\ a(-1)(-2) = 4$

$\therefore\ 2a = 4$

$\therefore\ a = 2$

$\therefore\ y = 2(x-1)(x-2)$

b The graph touches the x-axis when $x = 2$.

$\therefore\ y = a(x-2)^2$ for some $a \neq 0$.

But the y-intercept is 8.

$\therefore\ a(-2)^2 = 8$

$\therefore\ 4a = 8$

$\therefore\ a = 2$

$\therefore\ y = 2(x-2)^2$

c The x-intercepts are 1 and 3.

$\therefore\ y = a(x-1)(x-3)$ for some $a \neq 0$.

But the y-intercept is 3.

$\therefore\ a(-1)(-3) = 3$

$\therefore\ 3a = 3$

$\therefore\ a = 1$

$\therefore\ y = (x-1)(x-3)$

d The x-intercepts are -1 and 3.

$\therefore\ y = a(x+1)(x-3)$ for some $a \neq 0$.

But the y-intercept is 3.

$\therefore\ a(1)(-3) = 3$

$\therefore\ a = -1$

$\therefore\ y = -(x+1)(x-3)$

e The graph touches the x-axis when $x = 1$.

$\therefore\ y = a(x-1)^2$ for some $a \neq 0$.

But the y-intercept is -3.

$\therefore\ a(-1)^2 = -3$

$\therefore\ a = -3$

$\therefore\ y = -3(x-1)^2$

f The x-intercepts are -2 and 3.

$\therefore\ y = a(x+2)(x-3)$ for some $a \neq 0$.

But the y-intercept is 12.

$\therefore\ a(2)(-3) = 12$

$\therefore\ -6a = 12$

$\therefore\ a = -2$

$\therefore\ y = -2(x+2)(x-3)$

2 **a** $y = 2(x-1)(x-4)$ has x-intercepts 1 and 4, and y-intercept 8

$\therefore$ its graph is **C**

b $y = -(x+1)(x-4)$ has x-intercepts -1 and 4, and y-intercept 4

$\therefore$ its graph is **E**

c $y = (x-1)(x-4)$ has x-intercepts 1 and 4, and y-intercept 4

$\therefore$ its graph is **B**

d $y = (x+1)(x-4)$ has x-intercepts -1 and 4, and y-intercept -4

$\therefore$ its graph is **F**

e $y = 2(x+4)(x-1)$ has x-intercepts -4 and 1, and y-intercept -8

$\therefore$ its graph is **G**

f $y = -3(x+4)(x-1)$ has x-intercepts -4 and 1, and y-intercept 12

$\therefore$ its graph is **H**

g $y = -(x-1)(x-4)$
has x-intercepts 1 and 4,
and y-intercept -4
$\therefore$ its graph is **A**

h $y = -3(x-1)(x-4)$
has x-intercepts 1 and 4,
and y-intercept -12
$\therefore$ its graph is **D**

3 **a** As the axis of symmetry is $x = 3$,
the other x-intercept is 4.
$\therefore$ $y = a(x-2)(x-4)$ for some $a \neq 0$.
But the y-intercept $= 12$

$$\therefore\ a(-2)(-4) = 12$$
$$\therefore\ 8a = 12$$
$$\therefore\ a = \tfrac{12}{8} = \tfrac{3}{2}$$
$$\therefore\ y = \tfrac{3}{2}(x-2)(x-4)$$

b As the axis of symmetry is $x = -1$,
the other x-intercept is 2.
$\therefore$ $y = a(x+4)(x-2)$ for some $a \neq 0$.
But the y-intercept $= 4$

$$\therefore\ a(4)(-2) = 4$$
$$\therefore\ -8a = 4$$
$$\therefore\ a = -\tfrac{1}{2}$$
$$\therefore\ y = -\tfrac{1}{2}(x+4)(x-2)$$

c The graph touches the x-axis at $x = -3$,
$\therefore$ $y = a(x+3)^2$ for some $a \neq 0$.
But the y-intercept $= -12$

$$\therefore\ a(3)^2 = -12$$
$$\therefore\ 9a = -12$$
$$\therefore\ a = -\tfrac{12}{9} = -\tfrac{4}{3} \qquad \therefore\ y = -\tfrac{4}{3}(x+3)^2$$

4 **a** Since the x-intercepts are 5 and 1, the equation is $y = a(x-5)(x-1)$ for some $a \neq 0$.
But when $x = 2$, $y = -9$

$$\therefore\ -9 = a(2-5)(2-1)$$
$$\therefore\ -9 = a(-3)(1)$$
$$\therefore\ -3a = -9$$
$$\therefore\ a = 3$$

$\therefore$ the equation is

$$y = 3(x-5)(x-1)$$
i.e., $y = 3(x^2 - 6x + 5)$
i.e., $y = 3x^2 - 18x + 15$

b Since the x-intercepts are 2 and $-\frac{1}{2}$, the equation is $y = a(x-2)(x+\frac{1}{2})$ for some $a \neq 0$.
But when $x = 3$, $y = -14$

$$\therefore\ -14 = a(3-2)(3+\tfrac{1}{2})$$
$$\therefore\ -14 = a(1)(\tfrac{7}{2})$$
$$\therefore\ \tfrac{7}{2}a = -14$$
$$\therefore\ a = -4$$

$\therefore$ the equation is

$$y = -4(x-2)(x+\tfrac{1}{2})$$
i.e., $y = -4(x^2 - \frac{3}{2}x - 1)$
i.e., $y = -4x^2 + 6x + 4$

c Since the graph touches the x-axis at 3, its equation is $y = a(x-3)^2$, for some $a \neq 0$.
But when $x = -2$, $y = -25$

$$\therefore\ -25 = a(-2-3)^2$$
$$\therefore\ -25 = 25a$$
$$\therefore\ a = -1$$

$\therefore$ the equation is

$$y = -(x-3)^2$$
i.e., $y = -(x^2 - 6x + 9)$
i.e., $y = -x^2 + 6x - 9$

d Since the graph touches the x-axis at -2, its equation is $y = a(x+2)^2$, for some $a \neq 0$.
But when $x = -1$, $y = 4$

$$\therefore\ 4 = a(-1+2)^2$$
$$\therefore\ 4 = a$$

$\therefore$ the equation is

$$y = 4(x+2)^2$$
i.e., $y = 4(x^2 + 4x + 4)$
i.e., $y = 4x^2 + 16x + 16$

e Since the graph cuts the x-axis at 3 and has axis of symmetry $x = 2$, it must also cut the x-axis at 1.

$\therefore$ the x-intercepts are 3 and 1, and the equation is $y = a(x-3)(x-1)$ for some $a \neq 0$.

But when $x = 5$, $y = 12$

$$\therefore\ 12 = a(5-3)(5-1)$$
$$\therefore\ 12 = a(2)(4)$$
$$\therefore\ 8a = 12$$
$$\therefore\ a = \tfrac{3}{2}$$

$\therefore$ the equation is

$$y = \tfrac{3}{2}(x-3)(x-1)$$

i.e., $y = \frac{3}{2}(x^2 - 4x + 3)$

i.e., $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$

f Since the graph cuts the x-axis at 5 and has axis of symmetry $x = 1$, it must also cut the x-axis at -3.

$\therefore$ the x-intercepts are 5 and -3, and the equation is $y = a(x-5)(x+3)$ for some $a \neq 0$.

But when $x = 2$, $y = 5$

$$\therefore\ 5 = a(2-5)(2+3)$$
$$\therefore\ 5 = a(-3)(5)$$
$$\therefore\ -3a = 1$$
$$\therefore\ a = -\tfrac{1}{3}$$

$\therefore$ the equation is

$$y = -\tfrac{1}{3}(x-5)(x+3)$$

i.e., $y = -\frac{1}{3}(x^2 - 2x - 15)$

i.e., $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

5 **a** The vertex is $(2, 4)$, so the quadratic has equation $y = a(x-2)^2 + 4$ for some $a \neq 0$.

But the graph passes through the origin

$$\therefore\ 0 = a(0-2)^2 + 4$$
$$\therefore\ 4a + 4 = 0$$
$$\therefore\ a = -1$$

$\therefore$ the equation is $y = -(x-2)^2 + 4$

b The vertex is $(2, -1)$, so the quadratic has equation $y = a(x-2)^2 - 1$ for some $a \neq 0$.

But the graph passes through $(0, 7)$

$$\therefore\ 7 = a(0-2)^2 - 1$$
$$\therefore\ 7 = 4a - 1$$
$$\therefore\ 4a = 8$$
$$\therefore\ a = 2$$

$\therefore$ the equation is $y = 2(x-2)^2 - 1$

c The vertex is $(3, 8)$, so the quadratic has equation $y = a(x-3)^2 + 8$ for some $a \neq 0$.

But the graph passes through $(1, 0)$

$$\therefore\ 0 = a(1-3)^2 + 8$$
$$\therefore\ 0 = 4a + 8$$
$$\therefore\ a = -2$$

$\therefore$ the equation is $y = -2(x-3)^2 + 8$

d The vertex is $(4, -6)$, so the quadratic has equation $y = a(x-4)^2 - 6$ for some $a \neq 0$.

But the graph passes through $(7, 0)$

$$\therefore\ 0 = a(7-4)^2 - 6$$
$$\therefore\ 9a - 6 = 0$$
$$\therefore\ a = \tfrac{2}{3}$$

$\therefore$ the equation is $y = \frac{2}{3}(x-4)^2 - 6$

e The vertex is $(2, 3)$, so the quadratic has equation $y = a(x-2)^2 + 3$ for some $a \neq 0$.

But the graph passes through $(3, 1)$

$$\therefore\ 1 = a(3-2)^2 + 3$$
$$\therefore\ 1 = a + 3$$
$$\therefore\ a = -2$$

$\therefore$ the equation is $y = -2(x-2)^2 + 3$

f The vertex is $(\frac{1}{2}, -\frac{3}{2})$, so the quadratic has equation $y = a(x - \frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.

But the graph passes through $(\frac{3}{2}, \frac{1}{2})$

$$\therefore\ \tfrac{1}{2} = a(\tfrac{3}{2} - \tfrac{1}{2})^2 - \tfrac{3}{2}$$
$$\therefore\ \tfrac{1}{2} = a - \tfrac{3}{2}$$
$$\therefore\ a = 2$$

$\therefore$ the equation is $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 8K

1 **a** $y = x^2 - 2x + 8$ meets $y = x + 6$

when $x^2 - 2x + 8 = x + 6$

$\therefore\ x^2 - 3x + 2 = 0$

$\therefore\ (x-1)(x-2) = 0$

$\therefore\ x = 1$ or 2

Substituting into $y = x + 6$,

when $x = 1$, $y = 7$

and when $x = 2$, $y = 8$

$\therefore$ the graphs meet at (1, 7) and (2, 8)

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$

when $-x^2 + 3x + 9 = 2x - 3$

$\therefore\ x^2 - x - 12 = 0$

$\therefore\ (x-4)(x+3) = 0$

$\therefore\ x = 4$ or -3

Substituting into $y = 2x - 3$,

when $x = -3$, $y = 2(-3) - 3 = -9$

and when $x = 4$, $y = 2(4) - 3 = 5$

$\therefore$ the graphs meet at $(-3, -9)$ and (4, 5)

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$

when $x^2 - 4x + 3 = 2x - 6$

$\therefore\ x^2 - 6x + 9 = 0$

$\therefore\ (x-3)^2 = 0$

$\therefore\ x = 3$

Substituting into $y = 2x - 6$,

when $x = 3$, $y = 0$

$\therefore$ the graphs touch at (3, 0)

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$

when $-x^2 + 4x - 7 = 5x - 4$

$\therefore\ x^2 + x + 3 = 0$

which has $a = 1$, $b = 1$, $c = 3$

$$\therefore\ x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2} = \frac{-1 \pm \sqrt{-11}}{2}$$

$\therefore$ there are no real solutions

$\therefore$ the graphs do not meet.

2 **a** (0.59, 5.59) and (3.41, 8.41) **b** $(3, -4)$ touching **c** graphs do not meet

d $(-2.56, -18.81)$ and (1.56, 1.81)

3 **a** $y = x^2$ meets $y = x + 2$

when $x^2 = x + 2$

$\therefore\ x^2 - x - 2 = 0$

$\therefore\ (x+1)(x-2) = 0$

$\therefore\ x = -1$ or 2

Substituting into $y = x + 2$,

when $x = -1$, $y = 1$

and when $x = 2$, $y = 4$

$\therefore$ the graphs meet at $(-1, 1)$ and (2, 4).

b $y = x^2 + 2x - 3$ meets $y = x - 1$

when $x^2 + 2x - 3 = x - 1$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x-1)(x+2) = 0$

$\therefore\ x = 1$ or -2

Substituting into $y = x - 1$,

when $x = 1$, $y = 0$

and when $x = -2$, $y = -3$

$\therefore$ the graphs meet at (1, 0) and $(-2, -3)$.

c $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$

when $2x^2 - x + 3 = 2 + x + x^2$

$\therefore\ x^2 - 2x + 1 = 0$

$\therefore\ (x-1)^2 = 0$

$\therefore\ x = 1$

Substituting into $y = 2 + x + x^2$,

when $x = 1$, $y = 2 + 1 + 1 = 4$

$\therefore$ the graphs meet at (1, 4)

d Now if $xy = 4$, then $y = \dfrac{4}{x}$

$xy = 4$ meets $y = x + 3$

when $\dfrac{4}{x} = x + 3$

$\therefore\ 4 = x^2 + 3x$

$\therefore\ x^2 + 3x - 4 = 0$

$\therefore\ (x+4)(x-1) = 0$

$\therefore\ x = -4$ or 1

Substituting into $y = x + 3$,

when $x = -4$, $y = -1$

and when $x = 1$, $y = 4$

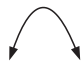
$\therefore$ the graphs meet at $(-4, -1)$ and (1, 4)

EXERCISE 8L

1 **a** $H(t) = 36t - 2t^2$

has $a = -2$ and $b = 36$

Since $a < 0$,
the graph is



The maximum height reached occurs

when $t = -\dfrac{b}{2a} = -\dfrac{36}{2(-2)} = 9$

i.e., the maximum height is reached after 9 seconds.

b $H(9) = 36 \times 9 - 2 \times 9^2$
$= 324 - 162$
$= 162$

$\therefore$ the maximum height reached is 162 m.

c The ball hits the ground when $H(t) = 0$

$\therefore$ $36t - 2t^2 = 0$

$\therefore$ $2t(18 - t) = 0$

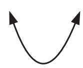
$\therefore$ $t = 0$ or 18

$\therefore$ the ball hits the ground after 18 seconds.

2 **a** $C(x) = x^2 - 24x + 244$

has $a = 1$, $b = -24$ and $c = 244$

Since $a > 0$,
the graph is



The minimum cost occurs

when $x = -\dfrac{b}{2a} = -\dfrac{(-24)}{2(1)} = 12$

i.e., the minimum cost is when twelve skateboards are produced.

b $C(12) = 12^2 - 24(12) + 244$
$= 144 - 288 + 244$
$= 100$

$\therefore$ the minimum cost is \$100.

c $C(0) = 0 - 24(0) + 244$
$= 244$

$\therefore$ if no skateboards are made, there is a fixed cost of \$244.

3 **a** The driver applies the brakes when $t = 0$.

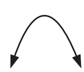
Now $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15$ $\therefore$ $v(0) = 15$

$\therefore$ the car was travelling at 15 m/s when the driver applied the brakes.

Note: In the reprint of the text, the function $v(t)$ in the question was altered to be more physically realistic.

b $v(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + 15$ has $a = -\frac{1}{2}$, $b = \frac{1}{2}$ and $c = 15$

Since $a < 0$, the graph is



The maximum velocity was when $t = -\dfrac{b}{2a} = -\dfrac{\frac{1}{2}}{2(-\frac{1}{2})} = \frac{1}{2}$ sec

i.e., the maximum velocity occurred half a second after the brake was applied.

Since the car was travelling downhill, it was accelerating. Therefore, when the brake was applied, it took a half a second before the deceleration, due to the brake, cancelled the acceleration due to the hill. Therefore the speed of the vehicle increased for a short time after the brake was applied.

c $v(\frac{1}{2}) = -\frac{1}{2}(\frac{1}{2})^2 + \frac{1}{2}(\frac{1}{2}) + 15$
$= -\frac{1}{8} + \frac{1}{4} + 15$
$= 15\frac{1}{8}$ m/s

$\therefore$ the maximum velocity was $15\frac{1}{8}$ m/s.

d The car is stopped when $v(t) = 0$.

$\therefore$ $-\frac{1}{2}t^2 + \frac{1}{2}t + 15 = 0$

$\therefore$ $t^2 - t - 30 = 0$

$\therefore$ $(t - 6)(t + 5) = 0$

$\therefore$ $t = 6$ or -5

but $t > 0$, so $t = 6$

$\therefore$ the car is stopped after 6 seconds.

4 **a** $P(n) = 84n - 45 - 2n^2$ has $a = -2$, $b = 84$ and $c = -45$

Since $a < 0$, the graph is 

$\therefore$ the maximum value of P is when $n = -\dfrac{b}{2a} = -\dfrac{84}{2(-2)} = 21$

$\therefore$ the maximum profit occurs for a fleet of 21 taxis.

b $P(21) = 84 \times 21 - 45 - 2 \times 21^2$
$= 837$

$\therefore$ the maximum profit is \$837 per hour.

c $P(0) = -45$, so if no taxis are on the road, \$45 is lost per hour.

5 **a** Dusk corresponds to $t = 0$.

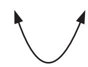
Now $T(t) = \frac{1}{4}t^2 - 5t + 30$

$\therefore$ $T(0) = 30^o\text{C}$

$\therefore$ the temperature at dusk was 30^oC.

b $T(t) = \frac{1}{4}t^2 - 5t + 30$

has $a = \frac{1}{4}$, $b = -5$ and $c = 30$

Since $a > 0$, the graph is 

$\therefore$ the temperature was a minimum

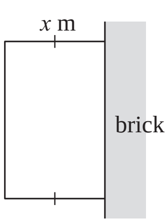
when $t = -\dfrac{b}{2a} = -\dfrac{(-5)}{2(\frac{1}{4})} = 10$

Now 10 hours after 7 pm is 5 am, so the temperature was a minimum at 5 am.

c $T(10) = \frac{1}{4} \times 10^2 - 5 \times 10 + 30$
$= 5^o\text{C}$

$\therefore$ the minimum temperature was 5^oC.

6 **a**



If the sides shown are x m long, and the total length of fencing is 40 m, then the other side must have length $(40 - 2x)$ m.

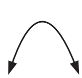
$\therefore$ the area is $A = x(40 - 2x)$ m^2

i.e., $A = -2x^2 + 40x$ m^2

b $A = -2x^2 + 40x$

has $a = -2$ and $b = 40$

Since $a < 0$,

the graph of A has shape 

$\therefore$ the area A is maximised

when $x = -\dfrac{b}{2a} = -\dfrac{40}{2(-2)} = 10$

$\therefore$ the area is maximised when $x = 10$.

c $A(10) = -2 \times 10^2 + 40 \times 10$
$= -200 + 400$
$= 200 \text{ m}^2$

$\therefore$ the maximum area is 200 m^2.

7 **a** With the axes as described, the parabola has vertex $(0, 70)$.

$\therefore$ its equation is $y = ax^2 + 70$ for some $a \neq 0$.

The end of the bridge is 80 m from A,
so the arch meets the vertical end supports at the point $(80, 6)$.

Since $(80, 6)$ must lie on the curve, $6 = a(80)^2 + 70$

$\therefore$ $6400a = -64$

$\therefore$ $a = -\frac{1}{100}$

$\therefore$ the arch has equation $y = -\frac{1}{100}x^2 + 70$

b The supports occur every 10 m.

When $x = 10$, $y = -\frac{1}{100} \times 10^2 + 70 = 69$ m

When $x = 20$, $y = -\frac{1}{100} \times 20^2 + 70 = 66$ m

When $x = 30$, $y = -\frac{1}{100} \times 30^2 + 70 = 61$ m

When $x = 40$, $y = -\frac{1}{100} \times 40^2 + 70 = 54$ m

When $x = 50$, $y = -\frac{1}{100} \times 50^2 + 70 = 45$ m

When $x = 60$, $y = -\frac{1}{100} \times 60^2 + 70 = 34$ m

When $x = 70$, $y = -\frac{1}{100} \times 70^2 + 70 = 21$ m

$\therefore$ the other supports have lengths 21 m, 34 m, 45 m, 54 m, 61 m, 66 m and 69 m.

8 a The maximum sag occurs in the middle of the cable.
This is halfway between the towers, which are at $x = 0$ and at $x = 60$ m

$\therefore$ the maximum sag is when $x = 30$ m.

The towers are 50 m and the sag is 20 m, $\therefore$ at the vertex $y = 50 - 20 = 30$ m

$\therefore$ the vertex of the parabola is at (30, 30).

b Using **a**, the parabola has equation

$y = a(x - 30)^2 + 30$ for some $a \neq 0$.

When $x = 0$, $y = 50$

$\therefore$ $50 = a(0 - 30)^2 + 30$

$\therefore$ $900a = 20$

$\therefore$ $a = \frac{20}{900} = \frac{1}{45}$

$\therefore$ the equation is $y = \frac{1}{45}(x - 30)^2 + 30$

c Point X lies at $x = 50$ m

$$\therefore\ y = \frac{1}{45}(50 - 30)^2 + 30 = \frac{1}{45} \times 400 + 30 = \frac{350}{9} \doteqdot 38.89$$

$\therefore$ the cable is about 38.9 m above X.

REVIEW SET 8A

1 a The x-intercepts are -2 and 1.

b The axis of symmetry lies midway between the x-intercepts, so its equation is $x = -\frac{1}{2}$.

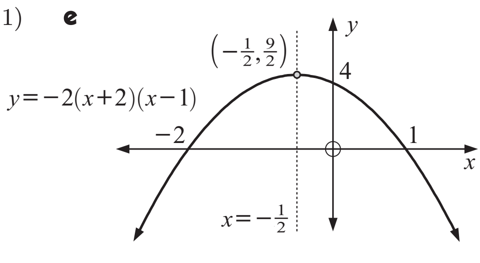
c When $x = -\frac{1}{2}$, $y = -2(-\frac{1}{2} + 2)(-\frac{1}{2} - 1)$
$= -2(\frac{3}{2})(-\frac{3}{2})$
$= \frac{9}{2}$

$\therefore$ the vertex is $(-\frac{1}{2}, \frac{9}{2})$

d When $x = 0$, $y = -2(2)(-1) = 4$

$\therefore$ the y-intercept is 4

e



2 a The axis of symmetry is $x = 2$.

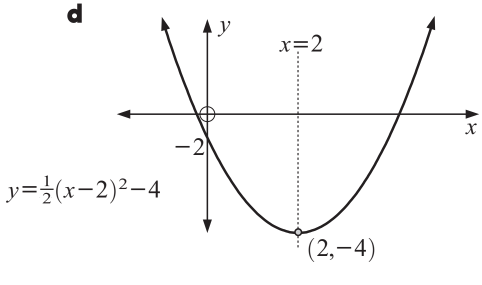
b When $x = 2$, $y = \frac{1}{2}(2 - 2)^2 - 4$
$= -4$

$\therefore$ the vertex is $(2, -4)$

c When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$
$= 2 - 4$
$= -2$

$\therefore$ the y-intercept is -2

d

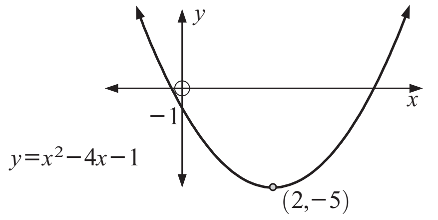


3 **a** $$\begin{aligned} y &= x^2-4x-1 \\ &= x^2-4x+4-4-1 \\ &= (x-2)^2-5 \end{aligned}$$

b The vertex is $(2, -5)$

c When $x=0$, $y=(-2)^2-5=-1$
$\therefore$ the y-intercept is -1

d

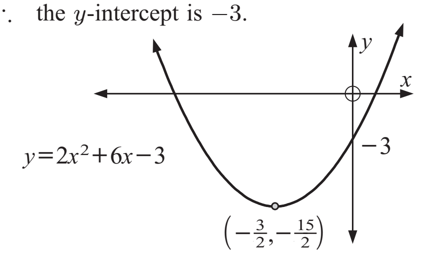


4 **a** $$\begin{aligned} y &= 2x^2+6x-3 \\ &= 2[x^2+3x-\tfrac{3}{2}] \\ &= 2\left[x^2+3x+(\tfrac{3}{2})^2-(\tfrac{3}{2})^2-\tfrac{3}{2}\right] \\ &= 2\left[x^2+3x+\tfrac{9}{4}-\tfrac{9}{4}-\tfrac{3}{2}\right] \\ &= 2\left(x^2+3x+\tfrac{9}{4}\right)-\tfrac{9}{2}-3 \\ &= 2(x+\tfrac{3}{2})^2-\tfrac{15}{2} \end{aligned}$$

b The vertex is $(-\frac{3}{2}, -\frac{15}{2})$.

c When $x=0$, $y=-3$
$\therefore$ the y-intercept is -3.

d



5 **a** $$\begin{aligned} x^2-11x &= 60 \\ \therefore\ x^2-11x-60 &= 0 \\ \therefore\ (x+4)(x-15) &= 0 \\ \therefore\ x &= -4 \text{ or } 15 \end{aligned}$$

b $$\begin{aligned} 3x^2-x-10 &= 0 \\ \therefore\ (3x+5)(x-2) &= 0 \\ \therefore\ x &= -\tfrac{5}{3} \text{ or } 2 \end{aligned}$$

c $$\begin{aligned} 3x^2-12x &= 0 \\ \therefore\ 3x(x-4) &= 0 \\ \therefore\ x &= 0 \text{ or } 4 \end{aligned}$$

6 **a** $$\begin{aligned} x^2+10 &= 7x \\ \therefore\ x^2-7x+10 &= 0 \\ \therefore\ (x-2)(x-5) &= 0 \\ \therefore\ x &= 2 \text{ or } 5 \end{aligned}$$

b $$\begin{aligned} x+\frac{12}{x} &= 7 \\ \therefore\ x^2+12 &= 7x \\ \therefore\ x^2-7x+12 &= 0 \\ \therefore\ (x-3)(x-4) &= 0 \\ \therefore\ x &= 3 \text{ or } 4 \end{aligned}$$

c $$\begin{aligned} 2x^2-7x+3 &= 0 \\ \therefore\ (2x-1)(x-3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } 3 \end{aligned}$$

7 $$\begin{aligned} x^2+7x-4 &= 0 \\ \therefore\ x^2+7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2-4 &= 0 \\ \therefore\ (x+\tfrac{7}{2})^2-\tfrac{49}{4}-4 &= 0 \\ \therefore\ (x+\tfrac{7}{2})^2 &= \tfrac{65}{4} \\ \therefore\ x+\tfrac{7}{2} &= \pm\tfrac{\sqrt{65}}{2} \\ \therefore\ x &= -\tfrac{7}{2}\pm\tfrac{\sqrt{65}}{2} \end{aligned}$$

8 $$\begin{aligned} x^2+4x+1 &= 0 \\ \therefore\ x^2+4x+4-4+1 &= 0 \\ \therefore\ (x+2)^2 &= 3 \\ \therefore\ x+2 &= \pm\sqrt{3} \\ \therefore\ x &= -2\pm\sqrt{3} \end{aligned}$$

9 **a** $x^2-7x+3=0$
has $a=1$, $b=-7$ and $c=3$

$$\begin{aligned} \therefore\ x &= \frac{-(-7)\pm\sqrt{(-7)^2-4(1)(3)}}{2(1)} \\ &= \frac{7\pm\sqrt{49-12}}{2} \\ &= \frac{7\pm\sqrt{37}}{2} \end{aligned}$$

i.e., $x = \frac{7}{2}\pm\frac{\sqrt{37}}{2}$

b $2x^2-5x+4=0$
has $a=2$, $b=-5$ and $c=4$

$$\begin{aligned} \therefore\ x &= \frac{-(-5)\pm\sqrt{(-5)^2-4(2)(4)}}{2(2)} \\ &= \frac{5\pm\sqrt{25-32}}{4} \end{aligned}$$

i.e., $x = \dfrac{5\pm\sqrt{-7}}{4}$

$\therefore$ x has no real solutions.

REVIEW SET 8B

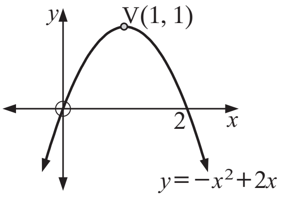
1 $y = -x^2 + 2x = x(2 - x)$

$\therefore$ the graph has x-intercepts 0 and 2, and y-intercept 0

Its axis of symmetry is midway between the x-intercepts,
i.e., at $x = 1$

and when $x = 1$, $y = -1^2 + 2 = 1$

$\therefore$ the vertex is $(1, 1)$



2 $y = -3x^2 + 8x + 7$ has $a = -3$, $b = 8$ and $c = 7$

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{8}{2(-3)}$

i.e., $x = \frac{4}{3}$

When $x = \frac{4}{3}$, $y = -3(\frac{4}{3})^2 + 8(\frac{4}{3}) + 7$
$= -\frac{16}{3} + \frac{32}{3} + 7$
$= \frac{37}{3}$

$\therefore$ the axis of symmetry is $x = \frac{4}{3}$ and the vertex is $(\frac{4}{3}, \frac{37}{3})$.

3 *Method 1*

$$\begin{aligned} y &= 2x^2 + 4x - 3 \\ &= 2[x^2 + 2x - \tfrac{3}{2}] \\ &= 2\left[x^2 + 2x + 1 - 1 - \tfrac{3}{2}\right] \\ &= 2\left[(x+1)^2 - 1 - \tfrac{3}{2}\right] \\ &= 2(x+1)^2 - 2 - 3 \\ &= 2(x+1)^2 - 5 \end{aligned}$$

$\therefore$ the axis of symmetry is $x = -1$
and the vertex is $(-1, -5)$

or

Method 2

$y = 2x^2 + 4x - 3$
has $a = 2$, $b = 4$, $c = -3$

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{4}{2(2)}$
i.e., $x = -1$

and when $x = -1$,

$$\begin{aligned} y &= 2(-1)^2 + 4(-1) - 3 \\ &= 2 - 4 - 3 \\ &= -5 \end{aligned}$$

$\therefore$ the axis of symmetry is $x = -1$
and the vertex is $(-1, -5)$.

4 **a** $3x^2 - 5x + 7 = 0$
has $a = 3$, $b = -5$ and $c = 7$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(7) \\ &= -59 \end{aligned}$$

Since $\Delta < 0$, there are no real solutions.

b $-2x^2 - 4x + 3 = 0$
has $a = -2$, $b = -4$ and $c = 3$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(-2)(3) \\ &= 40 \end{aligned}$$

Since $\Delta > 0$, there are two real solutions.

5 $5 + 7x + 3x^2$ has $a = 3$, $b = 7$ and $c = 5$

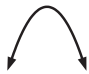
$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 7^2 - 4(3)(5) \\ &= -11 \end{aligned}$$

Since $\Delta < 0$, the graph of $y = 5 + 7x + 3x^2$ never cuts or touches the x-axis.

$\therefore$ since $a > 0$, the graph lies completely above the x-axis,

and $5 + 7x + 3x^2$ is positive definite.

6 $y = -2x^2 + 4x - 3$ has $a = -2$, $b = 4$ and $c = 3$

Since $a < 0$, the graph is  and will have a maximum.

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{4}{2(-2)}$

i.e., $x = 1$

When $x = 1$, $y = -2(1)^2 + 4(1) + 3$
$= 5$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

7 $y = x^2 - 3x$ meets $y = 3x^2 - 5x - 24$

when $x^2 - 3x = 3x^2 - 5x - 24$

$\therefore\ 2x^2 - 2x - 24 = 0$

$\therefore\ x^2 - x - 12 = 0$

$\therefore\ (x-4)(x+3) = 0$

$\therefore\ x = 4$ or -3

Substituting into $y = x^2 - 3x$,

when $x = 4$, $y = 4^2 - 3 \times 4 = 4$

and when $x = -3$, $y = (-3)^2 - 3(-3)$
$= 9 + 9 = 18$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

8 $y = -2x^2 + 5x + k$

has $a = -2$, $b = 5$ and $c = k$.

$\therefore\ \Delta = b^2 - 4ac$
$= 5^2 - 4(-2)k$
$= 25 + 8k$

The graph does not cut the x-axis

if $\Delta < 0$

$\therefore\ 25 + 8k < 0$

$\therefore\ 8k < -25$

$\therefore\ k < -\frac{25}{8}$

i.e., $k < -3\frac{1}{8}$

9 **a** The total length of wire for the fence is 60 m.

$\therefore$ AB + BC + CD = 60

Since the enclosure is rectangular,

CD = AB

$\therefore$ 2AB + x = 60

$\therefore$ 2AB = $60 - x$

$\therefore$ AB = $30 - \frac{1}{2}x$

$\therefore$ the area of the rectangle is

$A = x\left(30 - \frac{1}{2}x\right)$
$= \left(30x - \frac{1}{2}x^2\right)$ m^2

b $A = 30x - \frac{1}{2}x^2$

has $a = -\frac{1}{2}$ and $b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

When $x = 30$, AB $= 30 - \frac{1}{2} \times 30$
$= 15$ m

$\therefore$ the enclosure is 15 m by 30 m.

REVIEW SET 8C

1 **a** $x^2 + 5x + 3 = 0$

has $a = 1$, $b = 5$, $c = 3$

$$\therefore\ x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{25 - 12}}{2}$$

$$= -\tfrac{5}{2} \pm \tfrac{\sqrt{13}}{2}$$

b $3x^2 + 11x - 2 = 0$

has $a = 3$, $b = 11$, $c = -2$

$$\therefore\ x = \frac{-11 \pm \sqrt{11^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{-11 \pm \sqrt{121 + 24}}{6}$$

$$= -\tfrac{11}{6} \pm \tfrac{\sqrt{145}}{6}$$

2 **a** $x^2 - 5x - 3 = 0$
has $a = 1, \quad b = -5, \quad c = -3$

$$\therefore \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)} = \frac{5 \pm \sqrt{25+12}}{2} = \tfrac{5}{2} \pm \tfrac{\sqrt{37}}{2}$$

b $2x^2 - 7x - 3 = 0$
has $a = 2, \quad b = -7, \quad c = -3$

$$\therefore \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)} = \frac{7 \pm \sqrt{49+24}}{4} = \tfrac{7}{4} \pm \tfrac{\sqrt{73}}{4}$$

3 **a** $x = -5.828$ or -0.1716 **b** $x = -1.135$ or 1.468

4 **a** $x = 0.5858$ or 3.414 **b** $x = -0.1861$ or 2.686

5 **a** $2x^2 - 5x - 7 = 0$
has $a = 2, \quad b = -5, \quad c = -7$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(2)(-7) \\ &= 25 + 56 \\ &= 81\end{aligned}$$

$\therefore \quad \Delta > 0$ and $\sqrt{\Delta} = 9$

$\therefore$ there are two distinct real rational roots

b $3x^2 - 24x + 48 = 0$
has $a = 3, \quad b = -24, \quad c = 48$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= (-24)^2 - 4(3)(48) \\ &= 576 - 576 \\ &= 0\end{aligned}$$

$\therefore$ there is a repeated real root

6 $2x^2 - 3x + m = 0$
has $a = 2, \quad b = -3$ and $c = m$

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(2)m \\ &= 9 - 8m\end{aligned}$$

a There is a repeated root if $\Delta = 0$
i.e., $9 - 8m = 0$
$\therefore \quad m = \frac{9}{8}$

b There are two distinct real roots if $\Delta > 0$
i.e., $9 - 8m > 0$
$\therefore \quad 8m < 9$
$\therefore \quad m < \frac{9}{8}$

c There are no real roots if $\Delta < 0$
i.e., $9 - 8m < 0$
$\therefore \quad 8m > 9$
$\therefore \quad m > \frac{9}{8}$

7 $3x^2 + 4x + t = 0$
has $a = 3, \quad b = 4$ and $c = t$

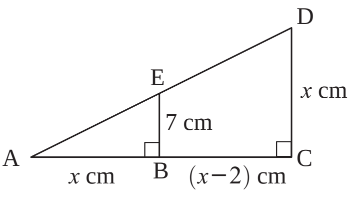
$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= 4^2 - 4(3)t \\ &= 16 - 12t\end{aligned}$$

a There is a repeated root if $\Delta = 0$
i.e., $16 - 12t = 0$
$\therefore \quad 12t = 16$
$\therefore \quad t = \frac{4}{3}$

b There are two distinct real roots if $\Delta > 0$
i.e., $16 - 12t > 0$
$\therefore \quad 12t < 16$
$\therefore \quad t < \frac{4}{3}$

c There are no real roots if $\Delta < 0$
i.e., $16 - 12t < 0$
$\therefore \quad 12t > 16$
$\therefore \quad t > \frac{4}{3}$

8 Suppose AB is x cm in length.
Then, using the information given, we can label the diagram:



Now by similar triangles, $\dfrac{BE}{AB} = \dfrac{CD}{AC}$

$$\therefore \quad \frac{7}{x} = \frac{x}{x+(x-2)}$$

$$\therefore \quad \frac{7}{x} = \frac{x}{2x-2}$$

$$\therefore \quad 7(2x-2) = x^2$$

$$\therefore \quad 14x - 14 = x^2$$

$$\therefore \quad x^2 - 14x + 14 = 0$$

which has $a = 1$, $b = -14$ and $c = 14$

$$\therefore \quad x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(14)}}{2(1)} = \frac{14 \pm \sqrt{140}}{2}$$

Now $x > 0$, so $x = \dfrac{14 + \sqrt{140}}{2} \doteqdot 12.92$ cm

$\therefore$ AB is approximately 12.9 cm long.

9 Let the hypotenuse have length x cm.

$\therefore$ the longer of the remaining sides has length $(x-2)$ cm,

and the third side has length $(x-9)$ cm.

By Pythagoras' theorem,

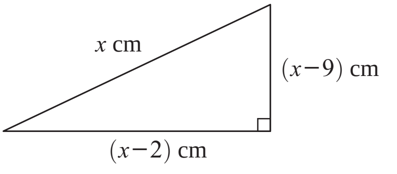
$$(x-2)^2 + (x-9)^2 = x^2$$

$$\therefore \quad x^2 - 4x + 4 + x^2 - 18x + 81 = x^2$$

$$\therefore \quad x^2 - 22x + 85 = 0$$

$$\therefore \quad (x-5)(x-17) = 0$$

$$\therefore \quad x = 5 \text{ or } 17$$

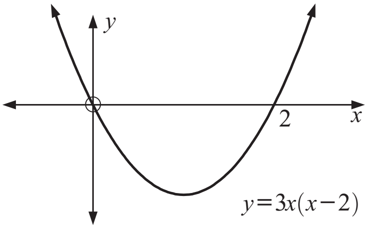


But if x was 5, the shortest side would have negative length.

$\therefore$ the only solution is that the hypotenuse has length 17 cm.

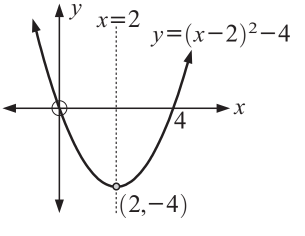
REVIEW SET 8D

1 $y = 3x(x-2)$ has x-intercepts 0 and 2 and y-intercept 0



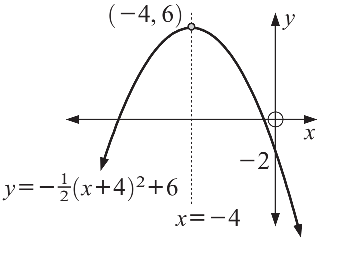
2 **a** $y = (x-2)^2 - 4$ has vertex $(2, -4)$ and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$ so the y-intercept is 0.



b $y = -\frac{1}{2}(x+4)^2 + 6$ has vertex $(-4, 6)$ and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$ so the y-intercept is -2.



3 **a** $y = 2x^2 + 4x - 1$

has $a = 2$, $b = 4$ and $c = -1$

The axis of symmetry is $x = -\dfrac{b}{2a}$

i.e., $x = -\dfrac{4}{2 \times 2}$

i.e., $x = -1$

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1$

$= 2 - 4 - 1$

$= -3$

$\therefore$ the vertex is $(-1, -3)$

c When $x = 0$, $y = -1$,

so the y-intercept is -1

When $y = 0$, $2x^2 + 4x - 1 = 0$

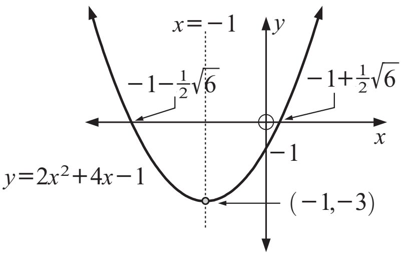
$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{24}}{4}$$

$$= \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \tfrac{1}{2}\sqrt{6}$$

$\therefore$ the x-intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$

d



4 **a** $y = 2x^2 + 3x - 7$

has $a = 2$, $b = 3$ and $c = -7$

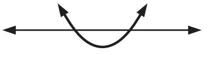
$\therefore \quad \Delta = b^2 - 4ac$

$= 3^2 - 4(2)(-7)$

$= 65$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a > 0$, the graph is



b $y = -3x^2 - 7x + 4$

has $a = -3$, $b = -7$ and $c = 4$

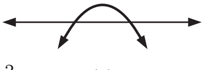
$\therefore \quad \Delta = b^2 - 4ac$

$= (-7)^2 - 4(-3)4$

$= 97$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a < 0$, the graph is



5 **a** $y = -2x^2 + 3x + 2$

has $a = -2$, $b = 3$ and $c = 2$

$\therefore \quad \Delta = b^2 - 4ac$

$= 3^2 - 4(-2)(2)$

$= 25$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

b $y = 3x^2 + x + 11$

has $a = 3$, $b = 1$ and $c = 11$

$\therefore \quad \Delta = b^2 - 4ac$

$= 1^2 - 4(3)(11)$

$= -131$

$\therefore \quad \Delta < 0$, and since $a > 0$, the function is positive definite.

6 **a** The graph has x-intercepts ± 3, so its equation is

$y = a(x + 3)(x - 3)$ for some $a \neq 0$.

Its y-intercept is -27, so

$a(3)(-3) = -27$

$\therefore \quad -9a = -27$

$\therefore \quad a = 3$

$\therefore$ the equation is $y = 3(x + 3)(x - 3)$

b The quadratic has vertex $(2, 25)$

$\therefore$ its equation is $y = a(x - 2)^2 + 25$

The y-intercept is 1, so

$a(-2)^2 + 25 = 1$

$\therefore \quad 4a = -24$

$\therefore \quad a = -6$

$\therefore$ the equation is $y = -6(x - 2)^2 + 25$

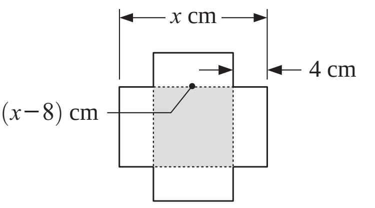
7 Let the number be x.

$\therefore$ its reciprocal is $\dfrac{1}{x}$.

$$\therefore \quad x + \frac{1}{x} = 2\tfrac{1}{30} = \tfrac{61}{30}$$
$$\therefore \quad x^2 + 1 = \tfrac{61}{30}x$$
$$\therefore \quad 30x^2 + 30 = 61x$$
$$\therefore \quad 30x^2 - 61x + 30 = 0$$
$$\therefore \quad (6x-5)(5x-6) = 0$$
$$\therefore \quad x = \tfrac{5}{6} \text{ or } \tfrac{6}{5}$$

$\therefore$ the number is $\frac{5}{6}$ or $\frac{6}{5}$

8



Since the container has a square base, the original tinplate must have been square.

Suppose its side was x cm long, so the base of the container is $(x-8)$ cm by $(x-8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x-8)(x-8)$ cm^3.

$$\therefore \quad 4(x-8)^2 = 120$$
$$\therefore \quad (x-8)^2 = 30$$
$$\therefore \quad x - 8 = \pm\sqrt{30}$$
$$\therefore \quad x = 8 \pm \sqrt{30}$$

Clearly, $x > 8$, so $x = 8 + \sqrt{30} \doteqdot 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm

9 $y = -x^2 - 5x + 3$ meets $y = x^2 + 3x + 11$

when $-x^2 - 5x + 3 = x^2 + 3x + 11$

$$\therefore \quad 2x^2 + 8x + 8 = 0$$
$$\therefore \quad x^2 + 4x + 4 = 0$$
$$\therefore \quad (x+2)^2 = 0$$
$$\therefore \quad x = -2$$

Substituting into $y = x^2 + 3x + 11$,

when $x = -2$,
$$\begin{aligned} y &= (-2)^2 + 3(-2) + 11 \\ &= 4 - 6 + 11 \\ &= 9 \end{aligned}$$

$\therefore$ the graphs touch at $(-2, 9)$.

REVIEW SET 8E

1 **a** The graph has x-intercepts -3 and 1, so its equation is

$y = a(x+3)(x-1)$ for some $a \neq 0$.

Its y-intercept is 18, so

$$a(3)(-1) = 18$$
$$\therefore \quad a = -6$$

So the equation is

$y = -6(x+3)(x-1)$.

b The graph has vertex $(2, -20)$, so its equation is

$y = a(x-2)^2 - 20$ for some $a \neq 0$.

Now an x-intercept is 5

$$\therefore \quad a(5-2)^2 - 20 = 0$$
$$\therefore \quad 9a = 20 \text{ and so } a = \tfrac{20}{9}$$

So the equation is $y = \frac{20}{9}(x-2)^2 - 20$.

2 **a** Since one x-intercept is 7 and the axis of symmetry is $x = 4$, the other x-intercept is $x = 1$.

$\therefore$ the graph has equation

$y = a(x-7)(x-1)$ for some $a \neq 0$.

The y-intercept is -2

$$\therefore \quad a(-7)(-1) = -2$$
$$\therefore \quad a = -\tfrac{2}{7}$$

$\therefore$ the equation is $y = -\frac{2}{7}(x-7)(x-1)$.

b The graph has vertex $(-3, 0)$, so its equation is

$y = a(x+3)^2$ for some $a \neq 0$.

The y-intercept is 2

$$\therefore \quad a(3)^2 = 2$$
$$\therefore \quad 9a = 2 \text{ and so } a = \tfrac{2}{9}$$

So the equation is $y = \frac{2}{9}(x+3)^2$.

3 The x-intercepts are 3 and -2, so the equation is $y = a(x-3)(x+2)$ for some $a \neq 0$.

But the y-intercept is 24 $\quad \therefore \; a(-3)(2) = 24$

$$\therefore \; -6a = 24$$
$$\therefore \; a = -4$$

$\therefore$ the equation is $y = -4(x-3)(x+2)$ i.e., $y = -4(x^2 - x - 6)$

i.e., $y = -4x^2 + 4x + 24$

4 The graph touches the x-axis at 4, so its vertex is $(4, 0)$.

$\therefore$ its equation is $y = a(x-4)^2$ for some $a \neq 0$.

The graph also passes through $(2, 12)$ $\quad \therefore \; a(2-4)^2 = 12$

$$\therefore \; 4a = 12$$
$$\therefore \; a = 3$$

$\therefore$ the equation is $y = 3(x-4)^2$ i.e., $y = 3(x^2 - 8x + 16)$

i.e., $y = 3x^2 - 24x + 48$

5 The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x+4)^2 + 1$ for some $a \neq 0$.

The graph also passes through $(1, 11)$

$\therefore \; 11 = a(1+4)^2 + 1$

$\therefore \; 25a = 10$

$\therefore \; a = \frac{2}{5}$

$\therefore$ the equation is $y = \frac{2}{5}(x+4)^2 + 1$ i.e., $y = \frac{2}{5}(x^2 + 8x + 16) + 1$

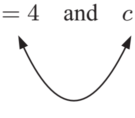
i.e., $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

6 **a** $y = 3x^2 + 4x + 7$

has $a = 3$, $b = 4$ and $c = 7$

Since $a > 0$,

the graph is



and so has a minimum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

i.e., $x = -\dfrac{4}{2(3)} = -\dfrac{2}{3}$

When $x = -\frac{2}{3}$,

$$\begin{aligned} y &= 3(-\tfrac{2}{3})^2 + 4(-\tfrac{2}{3}) + 7 \\ &= \tfrac{4}{3} - \tfrac{8}{3} + 7 \\ &= \tfrac{17}{3} \end{aligned}$$

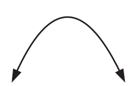
$\therefore$ the minimum is $\frac{17}{3}$ when $x = -\frac{2}{3}$

b $y = -2x^2 - 5x + 2$

has $a = -2$, $b = -5$ and $c = 2$

Since $a < 0$,

the graph is



and so has a maximum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

i.e., $x = -\dfrac{(-5)}{2(-2)} = -\dfrac{5}{4}$

When $x = -\frac{5}{4}$,

$$\begin{aligned} y &= -2(-\tfrac{5}{4})^2 - 5(-\tfrac{5}{4}) + 2 \\ &= -\tfrac{50}{16} + \tfrac{25}{4} + 2 \\ &= \frac{-25 + 50 + 16}{8} \\ &= \tfrac{41}{8} \end{aligned}$$

$\therefore$ the maximum is $\frac{41}{8}$ when $x = -\frac{5}{4}$

7 $y = x^2 - 2x + k$ has $a = 1$, $b = -2$ and $c = k$

$$\therefore \quad \Delta = b^2 - 4ac = (-2)^2 - 4(1)k = 4 - 4k$$

The graph cuts the x-axis twice if $\Delta > 0$

$$\therefore \quad 4 - 4k > 0$$
$$\therefore \quad 4k < 4$$
$$\therefore \quad k < 1$$

8 **a** The total length of fencing is $(8x + 9y)$ m

$$\therefore \quad 8x + 9y = 600$$
$$\therefore \quad 9y = 600 - 8x$$
$$\therefore \quad y = \frac{600 - 8x}{9}$$

b The area of each pen is

$$A = xy = x\left(\frac{600 - 8x}{9}\right) \text{ m}^2$$

c $A = x\left(\dfrac{600 - 8x}{9}\right) = \frac{600}{9}x - \frac{8}{9}x^2$

which has $a = -\frac{8}{9}$, $b = \frac{600}{9}$

Since $a < 0$, A is maximised at the axis of symmetry, which is $x = -\dfrac{b}{2a}$

i.e., $x = -\dfrac{\frac{600}{9}}{2(-\frac{8}{9})}$

i.e., $x = \frac{600}{16} = \frac{75}{2}$

When $x = \frac{75}{2}$, $y = \dfrac{600 - 8(\frac{75}{2})}{9} = \dfrac{600 - 300}{9} = \dfrac{300}{9} = \frac{100}{3}$

$\therefore$ for maximum area, each pen should be $37\frac{1}{2}$ m $\times$ $33\frac{1}{3}$ m

d The maximum area of each pen is

$$37\tfrac{1}{2} \times 33\tfrac{1}{3} = \tfrac{75}{2} \times \tfrac{100}{3} = 1250 \text{ m}^2$$

9 **a** AB = CD = x cm

$\therefore$ since AB + BC + CD = 24 cm,

BC = $24 - 2x$ cm

$\therefore$ the cross-sectional area is $A = x(24 - 2x)$ cm^2

b $A = x(24 - 2x)$

$= 24x - 2x^2$ which has $a = -2$ and $b = 24$

Since $a < 0$, A is maximised at the axis of symmetry, which is $x = -\dfrac{b}{2a}$

i.e., $x = -\dfrac{24}{2(-2)} = 6$

$\therefore$ the bends should be 6 cm from the ends of the sheet.

Chapter 9
THE BINOMIAL THEOREM

EXERCISE 9A

1 **a** $(x+1)^3$
$= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3$
$= x^3 + 3x^2 + 3x + 1$

b $(x+2)^3$
$= x^3 + 3x^2(2)^1 + 3x(2)^2 + (2)^3$
$= x^3 + 6x^2 + 12x + 8$

c $(x-4)^3$
$= x^3 + 3x^2(-4)^1 + 3x(-4)^2 + (-4)^3$
$= x^3 - 12x^2 + 48x - 64$

d $(2x+1)^3$
$= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$
$= 8x^3 + 12x^2 + 6x + 1$

e $(2x-1)^3$
$= (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3$
$= 8x^3 - 12x^2 + 6x - 1$

f $(3x-1)^3$
$= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3$
$= 27x^3 - 27x^2 + 9x - 1$

g $(2x+5)^3$
$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$
$= 8x^3 + 60x^2 + 150x + 125$

h $\left(2x + \frac{1}{x}\right)^3$
$= (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3$
$= 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

2 **a** $(x+2)^4 = x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4x(2)^3 + 2^4$
$= x^4 + 8x^3 + 24x^2 + 32x + 16$

b $(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
$= x^4 - 8x^3 + 24x^2 - 32x + 16$

c $(2x+3)^4 = (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4$
$= 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81$
$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

d $(3x-1)^4 = (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4$
$= 81x^4 - 4 \times 27x^3 + 6 \times 9x^2 - 4 \times 3x + 1$
$= 81x^4 - 108x^3 + 54x^2 - 12x + 1$

e $\left(x + \frac{1}{x}\right)^4 = x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$
$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

f $\left(2x - \frac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4$
$= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

3 **a** $(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5$
$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

b $(x-2)^5 = x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5$
$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

c $(2x+1)^5 = (2x)^5 + 5(2x)^4(1) + 10(2x)^3(1)^2 + 10(2x)^2(1)^3 + 5(2x)(1)^4 + (1)^5$
$= 32x^5 + 5 \times 16x^4 + 10 \times 8x^3 + 10 \times 4x^2 + 10x + 1$
$= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

d $\left(2x - \frac{1}{x}\right)^5$
$= (2x)^5 + 5(2x)^4\left(-\frac{1}{x}\right) + 10(2x)^3\left(-\frac{1}{x}\right)^2 + 10(2x)^2\left(-\frac{1}{x}\right)^3 + 5(2x)\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$
$= 32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5}$

4 **a** 1 5 10 10 5 1 ⟵ the 5th row
1 6 15 20 15 6 1 ⟵ the 6th row

b **i** $(x+2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6$
$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $(2x-1)^6 = (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4$
$+ 6(2x)(-1)^5 + (-1)^6$
$= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1$
$= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $\left(x + \frac{1}{x}\right)^6$
$= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5 **a** $\left(1+\sqrt{2}\right)^3 = (1)^3 + 3(1)^2\left(\sqrt{2}\right) + 3(1)\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^3$
$= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2}$
$= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$
$= 7 + 5\sqrt{2}$

b $\left(1+\sqrt{5}\right)^4 = (1)^4 + 4(1)^3\left(\sqrt{5}\right) + 6(1)^2\left(\sqrt{5}\right)^2 + 4(1)\left(\sqrt{5}\right)^3 + \left(\sqrt{5}\right)^4$
$= 1 + 4\sqrt{5} + 30 + 20\sqrt{5} + 25$
$= 56 + 24\sqrt{5}$

c $\left(2-\sqrt{2}\right)^5$
$= (2)^5 + 5(2)^4\left(-\sqrt{2}\right) + 10(2)^3\left(-\sqrt{2}\right)^2 + 10(2)^2\left(-\sqrt{2}\right)^3 + 5(2)^1\left(-\sqrt{2}\right)^4 + \left(-\sqrt{2}\right)^5$
$= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$
$= 232 - 164\sqrt{2}$

6 **a** $(2+x)^6 = (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6$
$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b $(2.01)^6$ is obtained by letting $x = 0.01$

$$\therefore \quad (2.01)^6 = 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3 + 60 \times (0.01)^4 + 12 \times (0.01)^5 + (0.01)^6$$
$$= 65.944\,160\,601\,201$$

$$\begin{array}{r} 64 \\ 1.92 \\ 0.024 \\ 0.000\,16 \\ 0.000\,000\,6 \\ 0.000\,000\,001\,2 \\ 0.000\,000\,000\,001 \\ \hline 65.944\,160\,601\,201 \end{array}$$

7 $(2x+3)(x+1)^4$
$= (2x+3)(x^4+4x^3+6x^2+4x+1)$
$= 2x^5+8x^4+12x^3+8x^2+2x$
$\quad + 3x^4+12x^3+18x^2+12x+3$
$= 2x^5+11x^4+24x^3+26x^2+14x+3$

8 **a** $(3a+b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + ...$

$\therefore$ the coefficient of a^3b^2 is $10 \times 3^3 = 270$

b $(2a+3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + ...$

$\therefore$ the coefficient of a^3b^3 is $20 \times 2^3 \times 3^3 = 4320$

EXERCISE 9B

1 **a** $(1+2x)^{11} = 1^{11} + \binom{11}{1}1^{10}(2x)^1 + \binom{11}{2}1^9(2x)^2 + ... + \binom{11}{10}1^1(2x)^{10} + \binom{11}{11}(2x)^{11}$
$= 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + ... + \binom{11}{10}(2x)^{10} + \binom{11}{11}(2x)^{11}$

b $\left(3x + \dfrac{2}{x}\right)^{15}$

$= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\dfrac{2}{x}\right) + \binom{15}{2}(3x)^{13}\left(\dfrac{2}{x}\right)^2 + ... + \binom{15}{14}(3x)\left(\dfrac{2}{x}\right)^{14} + \binom{15}{15}\left(\dfrac{2}{x}\right)^{15}$

c $\left(2x - \dfrac{3}{x}\right)^{20}$

$= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\dfrac{3}{x}\right) + \binom{20}{2}(2x)^{18}\left(-\dfrac{3}{x}\right)^2 + ... + \binom{20}{19}(2x)\left(-\dfrac{3}{x}\right)^{19} + \binom{20}{20}\left(-\dfrac{3}{x}\right)^{20}$

2 **a** For $(2x+5)^{15}$, $a = (2x)$, $b = 5$ and $n = 15$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 5$ gives $T_6 = \binom{15}{5}(2x)^{10}5^5$.

b For $\left(x^2 + \dfrac{5}{x}\right)^9$, $a = (x^2)$, $b = \left(\dfrac{5}{x}\right)$ and $n = 9$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 3$ gives $T_4 = \binom{9}{3}(x^2)^6\left(\dfrac{5}{x}\right)^3$.

c For $\left(x - \dfrac{2}{x}\right)^{17}$, $a = x$, $b = \left(-\dfrac{2}{x}\right)$ and $n = 17$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 9$ gives $T_{10} = \binom{17}{9}x^8\left(-\dfrac{2}{x}\right)^9$

d For $\left(2x^2 - \dfrac{1}{x}\right)^{21}$, $a = (2x^2)$, $b = \left(-\dfrac{1}{x}\right)$ and $n = 21$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ and letting $r = 8$ gives $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\dfrac{1}{x}\right)^8$.

3 **a** In $\left(3+2x^2\right)^{10}$, $a=3$, $b=(2x^2)$ and $n=10$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{10}{r}3^{10-r}(2x^2)^r$
$= \binom{10}{r}3^{10-r}2^r x^{2r}$

We now let $2r = 10$
$\therefore\ r = 5$

So, $T_6 = \underbrace{\binom{10}{5}3^5 2^5}x^{10}$

$\therefore$ the coefficient is $\binom{10}{5}3^5 2^5$.

b In $\left(2x^2-\dfrac{3}{x}\right)^6$, $a=(2x^2)$, $b=\left(-\dfrac{3}{x}\right)$ and $n=6$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{6}{r}\left(2x^2\right)^{6-r}\left(-\dfrac{3}{x}\right)^r$
$= \binom{6}{r}2^{6-r}x^{12-2r}\dfrac{(-3)^r}{x^r}$
$= \binom{6}{r}2^{6-r}(-3)^r x^{12-3r}$

We now let $12-3r = 3$
$\therefore\ 3r = 9$
$\therefore\ r = 3$

So, $T_4 = \underbrace{\binom{6}{3}2^3(-3)^3}x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3(-3)^3$.

c In $\left(2x^2-\dfrac{1}{x}\right)^{12}$, $a=(2x^2)$, $b=\left(-\dfrac{1}{x}\right)$ and $n=12$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{12}{r}\left(2x^2\right)^{12-r}\left(-\dfrac{1}{x}\right)^r$
$= \binom{12}{r}2^{12-r}x^{24-2r}\dfrac{(-1)^r}{x^r}$
$= \binom{12}{r}2^{12-r}(-1)^r x^{24-3r}$

We now let $24-3r = 12$
$\therefore\ 3r = 12$
$\therefore\ r = 4$

So, $T_5 = \underbrace{\binom{12}{4}2^8(-1)^4}x^{12}$

$\therefore$ the coefficient is $\binom{12}{4}2^8$.

4 **a** For $\left(x+\dfrac{2}{x^2}\right)^{15}$, $a=x$, $b=\dfrac{2}{x^2}$ and $n=15$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{15}{r}x^{15-r}\left(\dfrac{2}{x^2}\right)^r$
$= \binom{15}{r}x^{15-r}\dfrac{2^r}{x^{2r}}$
$= \binom{15}{r}2^r x^{15-3r}$

The constant term does not contain x.
$\therefore\ 15-3r = 0$
$\therefore\ r = 5$

and $T_6 = \binom{15}{5}2^5x^0$

$\therefore$ the constant term is $\binom{15}{5}2^5$.

b For $\left(x-\dfrac{3}{x^2}\right)^9$, $a=x$, $b=\left(-\dfrac{3}{x^2}\right)$ and $n=9$

Now $T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{9}{r}x^{9-r}\left(-\dfrac{3}{x^2}\right)^r$
$= \binom{9}{r}x^{9-r}\dfrac{(-3)^r}{x^{2r}}$
$= \binom{9}{r}(-3)^r x^{9-3r}$

The constant term does not contain x.
$\therefore\ 9-3r = 0$
$\therefore\ r = 3$

and $T_4 = \binom{9}{3}(-3)^3x^0$

$\therefore$ the constant term is $\binom{9}{3}(-3)^3$.

5 **a, b**

$$\begin{array}{lccl}
\text{Row 1} & 1\ \ 1 & \longleftarrow & \text{sum} = 1+1=2 & =2^1\\
\text{Row 2} & 1\ \ 2\ \ 1 & \longleftarrow & \text{sum} = 1+2+1=4 & =2^2\\
\text{Row 3} & 1\ \ 3\ \ 3\ \ 1 & \longleftarrow & \text{sum} = 1+3+3+1=8 & =2^3\\
\text{Row 4} & 1\ \ 4\ \ 6\ \ 4\ \ 1 & \longleftarrow & \text{sum} = 1+4+6+4+1=16 & =2^4\\
\text{Row 5} & 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 & \longleftarrow & \text{sum} = 1+5+10+10+5+1=32 & =2^5
\end{array}$$

c It seems that the sum of the numbers in row n of Pascal's triangle is 2^n.

d
$$\begin{aligned}(1+x)^n &= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}x + \binom{n}{2}1^{n-2}x^2 + \binom{n}{3}1^{n-3}x^3 + \dots\dots + \binom{n}{n-1}1^1x^{n-1} + \binom{n}{n}x^n\\
&= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots\dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\\
&\qquad \{\text{as all powers of 1 are 1}\}\end{aligned}$$

Now letting $x=1$ gives LHS $= (1+1)^n = 2^n$

and RHS $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots\dots + \binom{n}{n-1} + \binom{n}{n}$

$\therefore\ \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots\dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

6 **a**
$$\begin{aligned}&(x+2)(x^2+1)^8\\
&= (x+2)\left[(x^2)^8 + \binom{8}{1}(x^2)^7 1 + \binom{8}{2}(x^2)^6 1^2 + \dots\dots + \binom{8}{6}(x^2)^2 1^6 + \binom{8}{7}(x^2)^1 1^7 + \binom{8}{8}1^8\right]\end{aligned}$$

only terms which when multiplied give an x^5

$\therefore$ coefficient of x^5 is $1 \times \binom{8}{6} = \binom{8}{6} = 28.$

b
$$\begin{aligned}&(2-x)(3x+1)^9\\
&= (2-x)\left[(3x)^9 + \binom{9}{1}(3x)^8 + \binom{9}{2}(3x)^7 + \binom{9}{3}(3x)^6 + \binom{9}{4}(3x)^5 + \dots\dots\right]\end{aligned}$$

$\therefore$ coefficient of x^6 is $2 \times \binom{9}{3} \times 3^6 + (-1) \times \binom{9}{4} \times 3^5 = 2\binom{9}{3}3^6 - \binom{9}{4}3^5$

$= 91\,854$

7 **a** **Notice:**

$\binom{n}{1} = n$ {the 2nd member in each row of Pascal's triangle}

$$\begin{array}{c}
1\ \ 1\\
1\ \ 2\ \ 1\\
1\ \ 3\ \ 3\ \ 1\\
1\ \ 4\ \ 6\ \ 4\ \ 1\\
1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1
\end{array}$$

these are $\binom{n}{1}$ values

$\binom{3}{1} = 3$ and $\binom{3}{2} = \dfrac{3 \times 2}{2} = 3$ ✓

$\binom{8}{1} = 8$ and $\binom{8}{2} = \dfrac{8 \times 7}{2} = 28$ ✓

$\binom{20}{1} = 20$ and $\binom{20}{2} = \dfrac{20 \times 19}{2} = 190$ ✓

b $(1+x)^n$ has $T_3 = \binom{n}{2}1^{n-2}x^2$ and $n \geqslant 2$

$= \binom{n}{2}x^2$

But this term is $36x^2$ $\quad\therefore\ \binom{n}{2} = 36$

$$\begin{aligned}\therefore\ \frac{n(n-1)}{2} &= 36\\
\therefore\ n(n-1) &= 72\\
\therefore\ n^2 - n - 72 &= 0\\
\therefore\ (n-9)(n+8) &= 0\end{aligned}$$

$\therefore\ n = 9$ or -8

But $n \geqslant 2$ $\quad\therefore\ n = 9$

and $T_4 = \binom{n}{3}1^{n-3}x^3$

$= \binom{9}{3}x^3$

$= 84x^3$

c $(1+kx)^n = 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + \ldots..$

$= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \ldots..$

$\therefore \quad \binom{n}{1}k = -12 \qquad$ and $\quad \binom{n}{2}k^2 = 60$

$\therefore \quad nk = -12 \qquad$ and $\quad \dfrac{n(n-1)}{2}k^2 = 60$

$\therefore \quad n(n-1)k^2 = 120$

But $\quad k = -\dfrac{12}{n} \quad \therefore \quad n(n-1)\dfrac{144}{n^2} = 120$

$\therefore \quad 144(n-1) = 120n \quad \{n \geqslant 2\}$

$\therefore \quad 144n - 120n = 144$

$\therefore \quad 24n = 144$

$\therefore \quad n = 6$

and so, as $\quad k = -\dfrac{12}{n}, \quad k = -2$

REVIEW SET 9

1 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$\therefore \quad (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

a $(x-3)^6 = x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6$

$= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

b $\left(1+\dfrac{1}{x}\right)^6$

$= (1)^6 + 6(1)^5\left(\dfrac{1}{x}\right) + 15(1)^4\left(\dfrac{1}{x}\right)^2 + 20(1)^3\left(\dfrac{1}{x}\right)^3 + 15(1)^2\left(\dfrac{1}{x}\right)^4 + 6(1)\left(\dfrac{1}{x}\right)^5 + \left(\dfrac{1}{x}\right)^6$

$= 1 + \dfrac{6}{x} + \dfrac{15}{x^2} + \dfrac{20}{x^3} + \dfrac{15}{x^4} + \dfrac{6}{x^5} + \dfrac{1}{x^6}$

2 In the expansion of $(2x+5)^6$, $\quad a = (2x), \quad b = 5, \quad n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$

$= \binom{6}{r}(2x)^{6-r}5^r$

$= \binom{6}{r}2^{6-r}x^{6-r}5^r$

For the coefficient of x^3 we let $\quad 6 - r = 3$

$\therefore \quad r = 3$

and $\quad T_4 = \underbrace{\binom{6}{3}2^35^3}x^3$

$\therefore$ the coefficient is $\quad \binom{6}{3}2^35^3 = 20\,000.$

3 $\left(\sqrt{3}+2\right)^5 = \left(\sqrt{3}\right)^5 + 5\left(\sqrt{3}\right)^4(2) + 10\left(\sqrt{3}\right)^3(2)^2 + 10\left(\sqrt{3}\right)^2(2)^3 + 5\left(\sqrt{3}\right)^1(2)^4 + 2^5$

$= 9\sqrt{3} + 90 + 120\sqrt{3} + 240 + 80\sqrt{3} + 32$

$= 362 + 209\sqrt{3}$

4 $(4+x)^3 = 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3$

$= 64 + 48x + 12x^2 + x^3$

Letting $\ x = 0.02\ $ gives $\quad (4.02)^3 = 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3$

$= 64 + 0.96 + 0.0048 + 0.000\,008$

$= 64.964\,808$

5 For $\left(3x^2+\frac{1}{x}\right)^8$, $a=(3x^2)$, $b=\left(\frac{1}{x}\right)$, $n=8$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{8}{r}\left(3x^2\right)^{8-r}\left(\frac{1}{x}\right)^r \\ &= \binom{8}{r}3^{8-r}x^{16-2r-r} \\ &= \binom{8}{r}3^{8-r}x^{16-3r} \end{aligned}$$

Now a constant term does not contain x

$$\therefore\ 16-3r=0 \quad \therefore\ 3r=16 \quad \therefore\ r=5\tfrac{1}{3}$$

which is impossible as r is in $\mathbf{Z}$

$\therefore$ no constant term exists.

6 $\left(3a^2-\frac{2}{b}\right)^5$

$$= (3a^2)^5+5(3a^2)^4\left(-\frac{2}{b}\right)+10(3a^2)^3\left(-\frac{2}{b}\right)^2+10(3a^2)^2\left(-\frac{2}{b}\right)^3+5(3a^2)^1\left(-\frac{2}{b}\right)^4+\left(-\frac{2}{b}\right)^5$$

$$= 243a^{10}-\frac{810a^8}{b}+\frac{1080a^6}{b^2}-\frac{720a^4}{b^3}+\frac{240a^2}{b^4}-\frac{32}{b^5}$$

7 In $\left(2x-\frac{3}{x^2}\right)^{12}$, $a=(2x)$, $b=\left(-\frac{3}{x^2}\right)$, $n=12$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{12}{r}(2x)^{12-r}\left(-\frac{3}{x^2}\right)^r \\ &= \binom{12}{r}2^{12-r}x^{12-r}\,\frac{(-3)^r}{x^{2r}} \\ &= \binom{12}{r}2^{12-r}(-3)^r\,x^{12-3r} \end{aligned}$$

For the coefficient of x^{-6} we let $12-3r=-6$

$$\therefore\ 3r=18 \quad \therefore\ r=6$$

So, $T_7=\underbrace{\binom{12}{6}2^6(-3)^6}x^{-6}$

$\therefore$ the coefficient is $\binom{12}{6}2^6(-3)^6$.

8 For $\left(x^2+\frac{5}{x}\right)^{15}$, $a=(x^2)$, $b=\left(\frac{5}{x}\right)$, $n=15$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{15}{r}\left(x^2\right)^{15-r}\left(\frac{5}{x}\right)^r \\ &= \binom{15}{r}x^{30-2r}\,\frac{5^r}{x^r} \\ &= \binom{15}{r}5^r\,x^{30-3r} \end{aligned}$$

For the coefficient of x^{15} we let $30-3r=15$

$$\therefore\ 3r=15 \quad \therefore\ r=5$$

So, $T_6=\underbrace{\binom{15}{5}5^5}x^{15}$

$\therefore$ the coefficient is $\binom{15}{5}5^5$.

9 $(2x+3)(x-2)^6$

$$= (2x+3)\left[x^6+6x^5(-2)+15x^4(-2)^2+\ldots..\right]$$

$\therefore$ coefficient of x^5 is $2\times 15\times(-2)^2+3\times 6\times(-2)$

$$= 120-36$$
$$= 84$$

Chapter 10

PRACTICAL TRIGONOMETRY WITH RIGHT ANGLED TRIANGLES

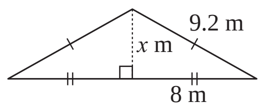
EXERCISE 10A

1 **a**
$$\begin{aligned} 1^2 + x^2 &= 1.2^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 1.2^2 - 1^2 \\ \therefore\ x &= \sqrt{1.2^2 - 1^2} \\ \therefore\ x &\doteqdot 0.663 \end{aligned}$$

b
$$\begin{aligned} x^2 &= 3.8^2 + 2.1^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{3.8^2 + 2.1^2} \\ \therefore\ x &\doteqdot 4.34 \end{aligned}$$

c
$$\begin{aligned} x^2 &= 1.8^2 + 1.32^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{1.8^2 + 1.32^2} \\ \therefore\ x &\doteqdot 2.23 \end{aligned}$$

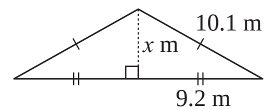
2 **a**



$$\begin{aligned} x^2 + 8^2 &= 9.2^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 9.2^2 - 8^2 \\ \therefore\ x &= \sqrt{9.2^2 - 8^2} \\ \therefore\ x &\doteqdot 4.54 \end{aligned}$$

$\therefore$ the height is 4.54 m

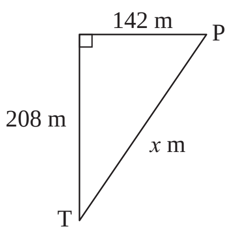
b



$$\begin{aligned} x^2 + 9.2^2 &= 10.1^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 10.1^2 - 9.2^2 \\ \therefore\ x &= \sqrt{10.1^2 - 9.2^2} \\ \therefore\ x &\doteqdot 4.17 \end{aligned}$$

$\therefore$ the height is 4.17 m

3



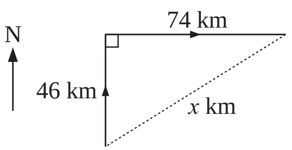
$$\begin{aligned} x^2 &= 208^2 + 142^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{208^2 + 142^2} \\ \therefore\ x &\doteqdot 252 \text{ m} \end{aligned}$$

$\therefore$ must hit the ball (in the air)

$(252 - 15)$ m

$= 237$ m

4 **a**

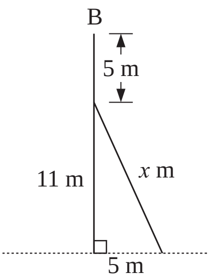


b
$$\begin{aligned} x^2 &= 46^2 + 74^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{46^2 + 74^2} \\ \therefore\ x &\doteqdot 87.1 \end{aligned}$$

$\therefore$ is 87.1 km from the start.

EXERCISE 10B

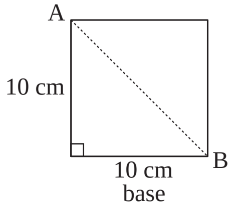
1



$$\begin{aligned} x^2 &= 5^2 + 11^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{5^2 + 11^2} \\ \therefore\ x &\doteqdot 12.083 \end{aligned}$$

But $4x + 2$ m is needed
i.e., $(4 \times 12.083 + 2)$ m
$\doteqdot 50.3$ m

2



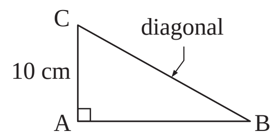
$$\begin{aligned} AB^2 &= 10^2 + 10^2 \quad \{\text{Pythagoras}\} \\ &= 100 + 100 \\ &= 200 \end{aligned}$$

But $BC^2 = AB^2 + AC^2$ {Pythagoras, again}

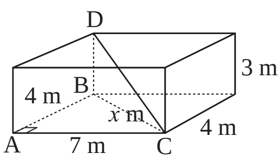
$\therefore\ BC^2 = 200 + 10^2$

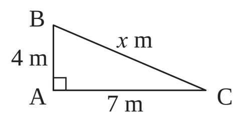
$\therefore\ BC = \sqrt{300} \doteqdot 17.3$

$\therefore$ the diagonal is 17.3 cm long.

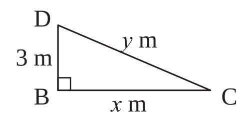


3





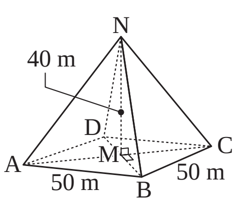
$x^2 = 4^2 + 7^2$ {Pythagoras}

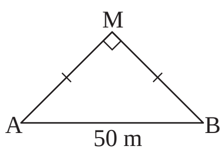


$$\begin{aligned} y^2 &= x^2 + 3^2 \quad \{\text{Pythagoras}\} \\ \therefore\ y^2 &= 4^2 + 7^2 + 3^2 \\ \therefore\ y &= \sqrt{4^2 + 7^2 + 3^2} \\ \therefore\ y &\doteqdot 8.60 \end{aligned}$$

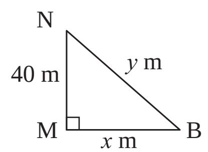
So, the distance is about 8.60 m

4





$$\begin{aligned} x^2 + x^2 &= 50^2 \quad \{\text{Pythagoras}\} \\ \therefore\ 2x^2 &= 50^2 \\ \therefore\ x^2 &= \tfrac{50^2}{2} \end{aligned}$$



$$\begin{aligned} y^2 &= 40^2 + x^2 \quad \{\text{Pythagoras}\} \\ \therefore\ y^2 &= 40^2 + \tfrac{50^2}{2} \\ \therefore\ y &= \sqrt{40^2 + \tfrac{50^2}{2}} \\ \therefore\ y &\doteqdot 53.4 \end{aligned}$$

$\therefore$ each slant edge is 53.4 m long.

5 **a** Both are right angles.

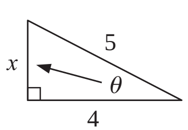
b In ΔPQR,
$PR^2 = a^2 + b^2$
{Pythagoras}

c In ΔKPR,

$$\begin{aligned} KR^2 &= KP^2 + PR^2 \quad \{\text{Pythagoras}\} \\ \therefore\ KR^2 &= c^2 + a^2 + b^2 \\ \therefore\ KR &= \sqrt{a^2 + b^2 + c^2} \end{aligned}$$

EXERCISE 10C

1 **a**



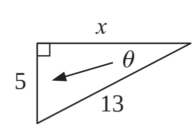
$$\begin{aligned} x^2 + 4^2 &= 5^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{5^2 - 4^2} \\ \therefore\ x &= 3 \end{aligned}$$

So, $\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$

b



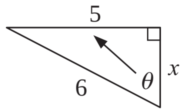
$$\begin{aligned} x^2 + 5^2 &= 13^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{13^2 - 5^2} \\ \therefore\ x &= 12 \end{aligned}$$

So, $\sin\theta = \frac{5}{13}$

$\cos\theta = \frac{12}{13}$

$\tan\theta = \frac{5}{12}$

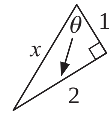
c



$$\begin{aligned} x^2 + 5^2 &= 6^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{6^2 - 5^2} \\ \therefore \quad x &= \sqrt{11} \end{aligned}$$

$$\text{So,} \quad \sin\theta = \tfrac{5}{6}, \quad \cos\theta = \tfrac{\sqrt{11}}{6}, \quad \tan\theta = \tfrac{5}{\sqrt{11}}$$

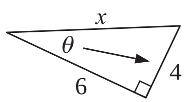
d



$$\begin{aligned} x^2 &= 1^2 + 2^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{1+4} \\ \therefore \quad x &= \sqrt{5} \end{aligned}$$

$$\text{So,} \quad \sin\theta = \tfrac{2}{\sqrt{5}}, \quad \cos\theta = \tfrac{1}{\sqrt{5}}, \quad \tan\theta = \tfrac{2}{1} = 2$$

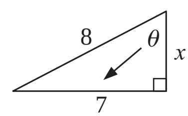
e



$$\begin{aligned} x^2 &= 4^2 + 6^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{52} \end{aligned}$$

$$\text{So,} \quad \sin\theta = \tfrac{4}{\sqrt{52}}, \quad \cos\theta = \tfrac{6}{\sqrt{52}}, \quad \tan\theta = \tfrac{4}{6} = \tfrac{2}{3}$$

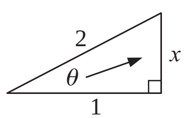
f



$$\begin{aligned} x^2 + 7^2 &= 8^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x^2 &= 8^2 - 7^2 \\ \therefore \quad x &= \sqrt{64 - 49} \\ \therefore \quad x &= \sqrt{15} \end{aligned}$$

$$\text{So,} \quad \sin\theta = \tfrac{7}{8}, \quad \cos\theta = \tfrac{\sqrt{15}}{8}, \quad \tan\theta = \tfrac{7}{\sqrt{15}}$$

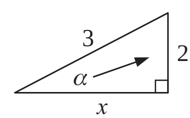
2 **a** as $\cos\theta = \tfrac{1}{2}$



$$\begin{aligned} x^2 + 1^2 &= 2^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x^2 &= 4 - 1 \\ \therefore \quad x &= \sqrt{3} \end{aligned}$$

$$\text{So,} \quad \sin\theta = \frac{x}{2} = \tfrac{\sqrt{3}}{2}, \quad \tan\theta = \frac{x}{1} = \sqrt{3}$$

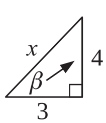
b as $\sin\alpha = \tfrac{2}{3}$



$$\begin{aligned} \therefore \quad x^2 + 2^2 &= 3^2 \\ \therefore \quad x^2 &= 9 - 4 \\ \therefore \quad x &= \sqrt{5} \end{aligned}$$

$$\text{So,} \quad \cos\alpha = \frac{x}{3} = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{x} = \frac{2}{\sqrt{5}}$$

c



as $\tan\beta = \tfrac{4}{3}$

$x = 5 \quad \{3\text{-}4\text{-}5\ \Delta\}$

$$\text{So,} \quad \sin\beta = \frac{4}{x} = \frac{4}{5}, \quad \cos\beta = \frac{3}{x} = \frac{3}{5}$$

3 **a**

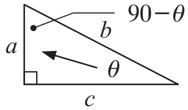
$$\sin\theta = \frac{b}{c}, \quad \cos\theta = \frac{a}{c}, \quad \tan\theta = \frac{b}{a}$$

b

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \tan\theta$$

4 **a**

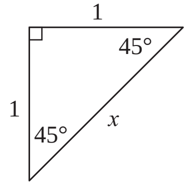


i $\sin\theta = \dfrac{a}{b}$ **ii** $\cos\theta = \dfrac{c}{b}$ **iii** $\sin(90-\theta) = \dfrac{c}{b}$ **iv** $\cos(90-\theta) = \dfrac{a}{b}$

b **i** The sine of an angle is the cosine of its complement.

ii The cosine of an angle is the sine of its complement.

5



a $x^2 = 1^2 + 1^2 = 2$

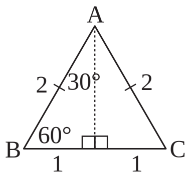
$\therefore \quad x = \sqrt{2}$ {Pythagoras}

b $\sin 45^o = \dfrac{1}{x} = \dfrac{1}{\sqrt{2}}$

$\cos 45^o = \dfrac{1}{x} = \dfrac{1}{\sqrt{2}}$

$\tan 45^o = \frac{1}{1} = 1$

6



a $\angle \text{ABN} = 60^o$

$\angle \text{BAN} = 30^o$

b ΔABC is equilateral and AN is perpendicular to BC.

$\therefore$ N is the midpoint of BC.

$\therefore$ BN $= 1$

$\therefore$ $\text{AN}^2 = 2^2 - 1^2$ {Pythagoras}

$\therefore$ AN $= \sqrt{3}$

c **i** $\sin 60^o = \dfrac{\text{AN}}{2} = \frac{\sqrt{3}}{2}$

$\cos 60^o = \frac{1}{2}$

$\tan 60^o = \dfrac{\text{AN}}{1} = \sqrt{3}$

ii $\sin 30^o = \frac{1}{2}$

$\cos 30^o = \dfrac{\text{AN}}{2} = \dfrac{\sqrt{3}}{2}$

$\tan 30^o = \dfrac{1}{\text{AN}} = \dfrac{1}{\sqrt{3}}$

EXERCISE 10D

1 **a** $\sin 35^o = \dfrac{x}{30}$

$\therefore \quad 30 \times \sin 35^o = x$

$\therefore \quad x \doteqdot 17.2$

b $\cos 50^o = \dfrac{x}{400}$

$\therefore \quad 400 \times \cos 50^o = x$

$\therefore \quad x \doteqdot 257$

c $\tan 60^o = \dfrac{x}{8.7}$

$\therefore \quad \tan 60^o \times 8.7 = x$

$\therefore \quad x \doteqdot 15.1$

d $\cos 65^o = \dfrac{3}{x}$

$\therefore \quad x = \dfrac{3}{\cos 65^o}$

$\therefore \quad x \doteqdot 7.10$

e $\tan 36.7^o = \dfrac{413}{x}$

$\therefore \quad x = \dfrac{413}{\tan 36.7}$

$\therefore \quad x \doteqdot 554$

f $\sin 53.9 = \dfrac{369}{x}$

$\therefore \quad x = \dfrac{369}{\sin 53.9}$

$\therefore \quad x \doteqdot 457$

2 **a** $\sin\theta = 0.9364$

$\therefore \quad \theta = \sin^{-1}(0.9364)$

$\therefore \quad \theta \doteqdot 69.5^o$

b $\cos\theta = 0.2381$

$\therefore \quad \theta = \cos^{-1}(0.2381)$

$\therefore \quad \theta \doteqdot 76.2^o$

c $\tan\theta = 1.7321$

$\therefore \quad \theta = \tan^{-1}(1.7321)$

$\therefore \quad \theta \doteqdot 60.0^o$

d $\cos\theta = \frac{2}{7}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{7}\right)$

$\therefore \quad \theta \doteqdot 73.4^o$

e $\sin\theta = \frac{1}{3}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\therefore \quad \theta \doteqdot 19.5^o$

f $\tan\theta = \frac{14}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{14}{3}\right)$

$\therefore \quad \theta \doteqdot 77.9^o$

g $\sin\theta = \frac{\sqrt{3}}{11}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{3}}{11}\right)$

$\therefore \quad \theta \doteqdot 9.06^o$

h $\cos\theta = \frac{5}{\sqrt{37}}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{\sqrt{37}}\right)$

$\therefore \quad \theta \doteqdot 34.7^o$

3 **a** $\sin\theta = \frac{5}{6}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{5}{6}\right)$

$\therefore \quad \theta \doteqdot 56.4$

b $\tan\alpha = \frac{1}{12}$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{1}{12}\right)$

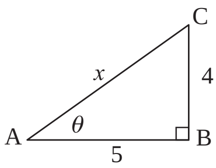
$\therefore \quad \alpha \doteqdot 4.76$

c $\cos\beta = \frac{4}{6}$

$\therefore \quad \beta = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore \quad \beta \doteqdot 48.2$

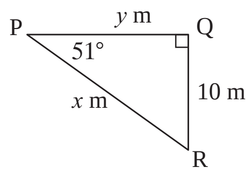
4 **a**



$$x^2 = 4^2 + 5^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad x = \sqrt{4^2+5^2}$$
$$\therefore \quad x = \sqrt{41}$$
$$\therefore \quad \text{AC} = \sqrt{41} \text{ m}$$
$$\text{or} \doteqdot 6.40 \text{ m}$$

$$\tan\theta = \tfrac{4}{5}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{4}{5}\right)$$
$$\therefore \quad \theta \doteqdot 38.7$$
$$\therefore \quad \measuredangle \text{A} \doteqdot 38.7^o, \quad \measuredangle \text{C} \doteqdot 51.3^o$$

b



$$\measuredangle \text{R} = 90^o - 51^o$$
$$= 39^o$$

$$\sin 51^o = \frac{10}{x}$$
$$\therefore \quad x = \frac{10}{\sin 51^o}$$
$$\therefore \quad x \doteqdot 12.9$$
So, $\text{PR} \doteqdot 12.9$ m

$$\tan 51^o = \frac{10}{y}$$
$$\therefore \quad y = \frac{10}{\tan 51^o}$$
$$\therefore \quad y \doteqdot 8.10$$
So, $\text{PQ} \doteqdot 8.10$ m

5 **a**

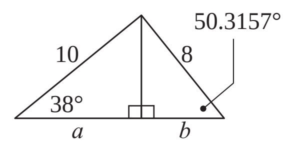
$$x^2 + 3^2 = 4^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad x = \sqrt{4^2 - 3^2}$$
$$\therefore \quad x = \sqrt{7} \quad (\text{or } 2.65)$$

$$\tan\theta = \frac{2}{x} = \frac{2}{\sqrt{7}}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{2}{\sqrt{7}}\right) \quad \text{and so} \quad \theta \doteqdot 37.1$$

b

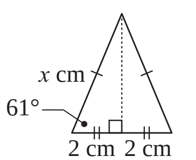
$$\sin 38^o = \frac{x}{10}$$
$$\therefore \quad 10 \times \sin 38^o = x$$
$$\therefore \quad x \doteqdot 6.16$$

$$\sin\theta = \frac{x}{8}$$
$$\therefore \quad \sin\theta \doteqdot \frac{6.1566}{8}$$
$$\therefore \quad \theta \doteqdot \sin^{-1}\left(\tfrac{6.1566}{8}\right)$$
$$\therefore \quad \theta \doteqdot 50.3$$



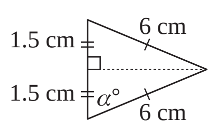
$$\cos 38^o = \frac{a}{10} \quad \text{and} \quad \cos 50.3157^o = \frac{b}{8}$$
$$\text{Now} \quad y = a + b$$
$$= 10\cos 38^o + 8\cos 50.3157^o$$
$$\doteqdot 13.0$$

6 **a**



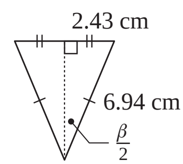
$$\cos 61^o = \frac{2}{x}$$
$$\therefore \quad x = \frac{2}{\cos 61^o}$$
$$\therefore \quad x \doteqdot 4.13$$

b



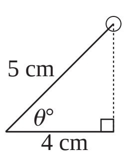
$$\cos\alpha = \frac{1.5}{6}$$
$$\therefore \quad \alpha = \cos^{-1}\left(\frac{1.5}{6}\right)$$
$$\therefore \quad \alpha \doteqdot 75.5$$

c



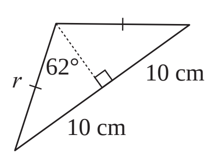
$$\sin\left(\frac{\beta}{2}\right) = \tfrac{2.43}{6.94}$$
$$\therefore \quad \beta = 2 \times \sin^{-1}\left(\tfrac{2.43}{6.94}\right)$$
$$\therefore \quad \beta \doteqdot 41.0$$

7 **a**



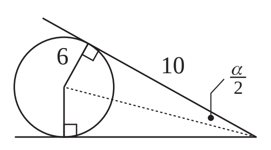
$$\cos\theta = \tfrac{4}{5}$$
$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{4}{5}\right)$$
$$\therefore \quad \theta \doteqdot 36.9$$

b



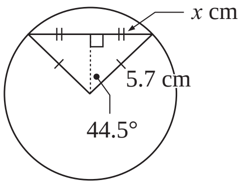
$$\sin 62^o = \frac{10}{r}$$
$$\therefore \quad r = \frac{10}{\sin 62^o}$$
$$\therefore \quad r \doteqdot 11.3$$

c



$$\tan\left(\frac{\alpha}{2}\right) = \tfrac{6}{10} = 0.6$$
$$\therefore \quad \alpha = 2 \times \tan^{-1}(0.6)$$
$$\therefore \quad \alpha \doteqdot 61.9$$

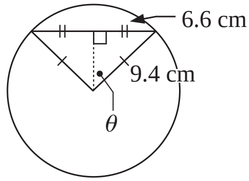
8



$$\begin{aligned} \sin 44.5^o &= \frac{x}{5.7} \\ \therefore\quad 5.7 \times \sin 44.5^o &= x \\ \therefore\quad x &\doteqdot 3.995 \\ \therefore\quad 2x &\doteqdot 7.99 \end{aligned}$$

$\therefore$ chord is 7.99 cm long.

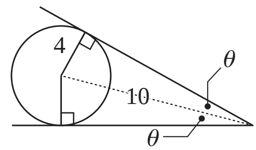
9



$$\begin{aligned} \sin\theta &= \tfrac{6.6}{9.4} \\ \therefore\quad \theta &= \sin^{-1}\left(\tfrac{6.6}{9.4}\right) \\ \therefore\quad \theta &\doteqdot 44.6 \\ \therefore\quad 2\theta &\doteqdot 89.2 \end{aligned}$$

$\therefore$ angle is 89.2^o

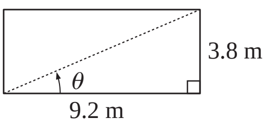
10



$$\begin{aligned} \sin\theta &= \tfrac{4}{10} = 0.4 \\ \therefore\quad \theta &= \sin^{-1}(0.4) \\ \therefore\quad 2\theta &= 2 \times \sin^{-1}(0.4) \\ &\doteqdot 47.2 \end{aligned}$$

$\therefore$ angle is 47.2^o

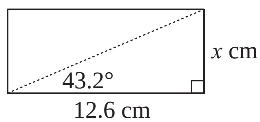
11



$$\begin{aligned} \tan\theta &= \frac{3.8}{9.2} \\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{3.8}{9.2}\right) \\ \therefore\quad \theta &\doteqdot 22.4 \end{aligned}$$

$\therefore$ the angle is 22.4^o

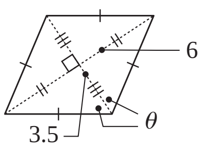
12



$$\begin{aligned} \tan 43.2^o &= \frac{x}{12.6} \\ \therefore\quad 12.6 \times \tan 43.2^o &= x \\ \therefore\quad x &\doteqdot 11.8 \end{aligned}$$

$\therefore$ the shorter side is 11.8 cm long

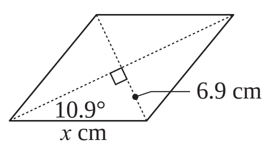
13



$$\begin{aligned} \tan\theta &= \frac{6}{3.5} \\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{6}{3.5}\right) \\ \therefore\quad 2\theta &= 2 \times \tan^{-1}\left(\tfrac{6}{3.5}\right) \doteqdot 119.49 \end{aligned}$$

So, the larger angle is $\doteqdot 119^o$

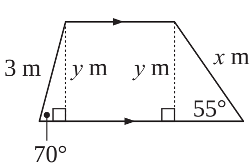
14



$$\begin{aligned} \sin 10.9^o &= \frac{6.9}{x} \\ \therefore\quad x &= \frac{6.9}{\sin 10.9^o} \doteqdot 36.5 \end{aligned}$$

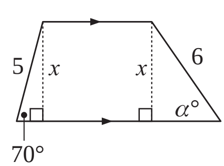
$\therefore$ sides are 36.5 cm long

15 **a**



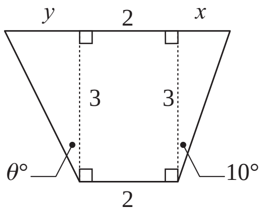
$$\begin{aligned} \sin 70^o &= \frac{y}{3} \\ \therefore\quad y &= 3 \times \sin 70^o \\ \text{But}\quad \sin 55^o &= \frac{y}{x} \\ \therefore\quad \sin 55^o &= \frac{3 \times \sin 70^o}{x} \\ \therefore\quad x &= \frac{3 \times \sin 70^o}{\sin 55^o} \doteqdot 3.44 \end{aligned}$$

b



$$\begin{aligned} \sin 70^o &= \frac{x}{5} \\ \therefore\quad 5 \times \sin 70^o &= x \\ \text{Now}\quad \sin\alpha &= \frac{x}{6} \\ \therefore\quad \sin\alpha &= \frac{5 \times \sin 70^o}{6} \\ \therefore\quad \alpha &= \sin^{-1}\left(\frac{5 \times \sin 70^o}{6}\right) \doteqdot 51.5 \end{aligned}$$

16

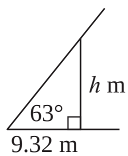


$$\begin{aligned} \tan 10^o &= \frac{x}{3} \\ \therefore \quad 3 \times \tan 10^o &= x \\ \therefore \quad x &\doteqdot 0.5290 \\ \text{But} \quad y + 2 + x &= 5 \\ \therefore \quad y &\doteqdot 3 - 0.5290 \\ \therefore \quad y &\doteqdot 2.471 \end{aligned}$$

$$\begin{aligned} \text{Now} \quad \tan\theta &= \frac{y}{3} \\ \therefore \quad \tan\theta &= 0.8237 \\ \therefore \quad \theta &\doteqdot 39.48^o \\ \text{So,} \quad \beta &\doteqdot 90 + 39.48^o \\ \therefore \quad \beta &\doteqdot 129^o \end{aligned}$$

EXERCISE 10E

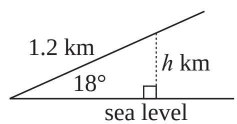
1



$$\begin{aligned} \tan 63^o &= \frac{h}{9.32} \\ \therefore \quad \tan 63^o \times 9.32 &= h \\ \therefore \quad h &\doteqdot 18.3 \end{aligned}$$

$\therefore$ the height is 18.3 m

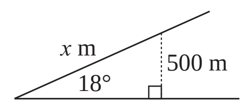
2 **a**



$$\begin{aligned} \sin 18^o &= \frac{h}{1.2} \\ \therefore \quad 1.2 \times \sin 18^o &= h \\ \therefore \quad h &\doteqdot 0.371 \end{aligned}$$

$\therefore$ height is 371 m above sea level.

b



$$\begin{aligned} \sin 18^o &= \frac{500}{x} \\ \therefore \quad x &= \frac{500}{\sin 18^o} \\ \therefore \quad x &\doteqdot 1618 \end{aligned}$$

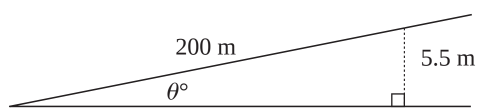
$\therefore$ have walked 1.62 km up the hill.

3

$$\begin{aligned} \text{Let} \quad \text{AB} &= x \text{ m} \\ \text{Now} \quad \tan 37^o &= \frac{120}{x} \\ \therefore \quad x &= \frac{120}{\tan 37^o} \\ \therefore \quad x &\doteqdot 159 \end{aligned}$$

$\therefore$ the canal is 159 m wide

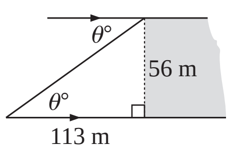
4



$$\begin{aligned} \sin\theta &= \tfrac{5.5}{200} \\ \therefore \quad \theta &= \sin^{-1}\left(\tfrac{5.5}{200}\right) \\ \therefore \quad \theta &\doteqdot 1.58 \end{aligned}$$

i.e., an incline of 1.58^o

5



$$\begin{aligned} \tan\theta &= \tfrac{56}{113} \\ \therefore \quad \theta &\doteqdot \tan^{-1}\left(\tfrac{56}{113}\right) \\ \therefore \quad \theta &\doteqdot 26.4 \end{aligned}$$

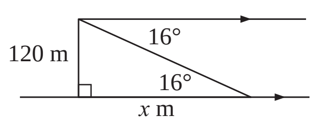
$\therefore$ angle of elevation is about 26.4^o

The angle of depression is also θ^o,

i.e., about 26.4^o

{equal alternate angles}

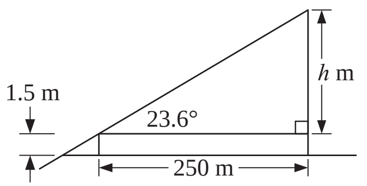
6



$$\begin{aligned} \tan 16^o &= \frac{120}{x} \\ \therefore \quad x &= \frac{120}{\tan 16^o} \\ \therefore \quad x &\doteqdot 418 \end{aligned}$$

$\therefore$ boat is 418 m out from the base of the cliff.

7



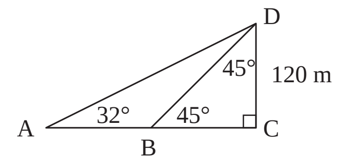
$$\tan 23.6^o = \frac{h}{250}$$

$$\therefore \quad 250 \times \tan 23.6^o = h$$

$$\therefore \quad h = 109.2$$

$$\begin{aligned} \therefore \quad \text{tree height} &\doteqdot 109.2 + 1.5 \\ &\doteqdot 110.7 \text{ m} \\ &\doteqdot 111 \text{ m} \end{aligned}$$

8



ΔBCD is right-angled isosceles

$$\therefore \quad \text{BC} = 120 \text{ m also}$$

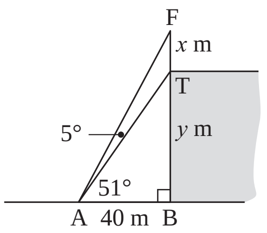
In ΔACD, $\tan 32^o = \dfrac{120}{\text{AC}}$

$$\therefore \quad \text{AC} = \frac{120}{\tan 32^o}$$

$$\therefore \quad \text{AC} \doteqdot 192$$

$$\begin{aligned} \therefore \quad \text{she walks} \quad &(192 - 120) \text{ m} \\ &\doteqdot 72 \text{ m} \end{aligned}$$

9



$$\tan 51^o = \frac{y}{40}$$

$$\therefore \quad 40 \times \tan 51^o = y$$

$$\therefore \quad y \doteqdot 49.396$$

$$\tan 56^o = \frac{x+y}{40}$$

$$\therefore \quad 40 \times \tan 56^o = x + y$$

$$\therefore \quad x + y \doteqdot 59.302$$

$$\therefore \quad x \doteqdot 59.302 - 49.396$$

$$\therefore \quad x \doteqdot 9.91$$

i.e., it is 9.91 m high

10 **a**

$$\begin{aligned} v &= \sqrt{gr \tan\theta} \\ &= \sqrt{9.8 \times 100 \times \tan 15^o} \\ &\doteqdot 16.2 \text{ m/s} \end{aligned}$$

b

$$v = \sqrt{gr \tan\theta}$$

$$\therefore \quad 20 = \sqrt{9.8 \times 200 \times \tan\theta}$$

$$\therefore \quad 400 = 9.8 \times 200 \times \tan\theta$$

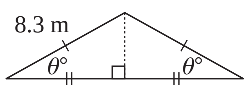
$$\therefore \quad \tfrac{2}{9.8} = \tan\theta$$

$$\therefore \quad \theta = \tan^{-1}(\tfrac{2}{9.8})$$

$$\therefore \quad \theta \doteqdot 11.5$$

i.e., a banked angle of 11.5^o

11



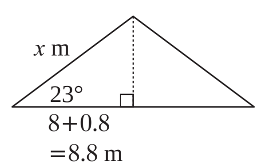
$$7.5 + 0.6 = 8.1 \text{ m}$$

$$\cos\theta = \frac{8.1}{8.3}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8.1}{8.3}\right)$$

$$\therefore \quad \theta \doteqdot 12.6$$

12



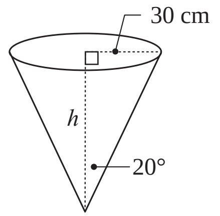
$$\cos 23^o = \frac{8.8}{x}$$

$$\therefore \quad x = \frac{8.8}{\cos 23^o}$$

$$\therefore \quad x \doteqdot 9.56$$

$\therefore$ a beam is 9.56 m long

13



$$\tan 20^o = \frac{30}{h}$$

$$\therefore \quad h = \frac{30}{\tan 20^o}$$

$$\therefore \quad h \doteqdot 82.42 \text{ cm}$$

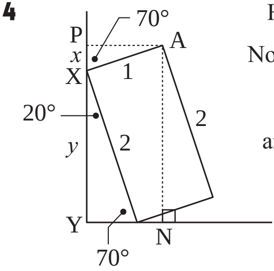
$$V = \tfrac{1}{3}\pi r^2 h$$

$$= \tfrac{1}{3}\pi \times 30^2 \times 82.42$$

$$\doteqdot 77\,683 \text{ cm}^3$$

$\therefore$ capacity is 77 683 mL
i.e., $\doteqdot$ 77.7 L

14



Height AN = PY also

$$\text{Now} \quad \cos 70^o = \frac{x}{1}$$

$$\therefore \quad x = \cos 70^o$$

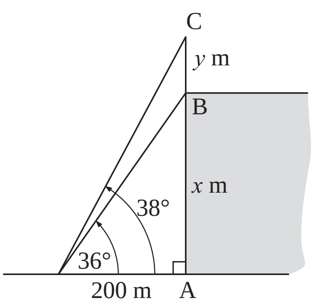
$$\text{and} \quad \sin 70^o = \frac{y}{2}$$

$$\therefore \quad y = 2\sin 70^o$$

Now PY $= x + y$
$= \cos 70^o + 2\sin 70^o$
$\doteqdot$ 2.22 m

$\therefore$ A is 2.22 m above the floor

15



$$\tan 36^o = \frac{x}{200}$$

$$\therefore \quad 200 \times \tan 36^o = x$$

$$\therefore \quad x \doteqdot 145.31 \quad \ldots\ldots (1)$$

$$\tan 38^o = \frac{x+y}{200}$$

$$\therefore \quad 200 \times \tan 38^o = x + y$$

$$\therefore \quad x + y \doteqdot 156.26 \quad \ldots\ldots (2)$$

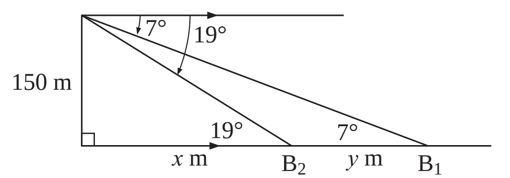
Using (1) and (2)

$$y \doteqdot 156.26 - 145.31$$

$$\therefore \quad y \doteqdot 10.9$$

$\therefore$ the pole is about 10.9 m long.

16



$$\tan 19^o = \frac{150}{x}$$

$$\therefore \quad x = \frac{150}{\tan 19^o} \doteqdot 435.63 \quad \ldots\ldots (1)$$

$$\tan 7^o = \frac{150}{x+y}$$

$$\therefore \quad x + y = \frac{150}{\tan 7^o} \doteqdot 1221.65 \quad \ldots\ldots (2)$$

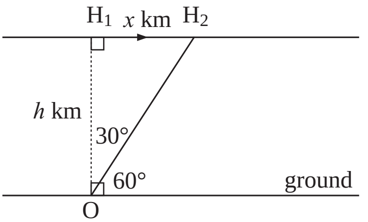
From (1) and (2)

$$y \doteqdot 1221.65 - 435.63$$

$$\therefore \quad y \doteqdot 786$$

i.e., needs to be 786 m closer in.

17



$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$\therefore \quad 100 = \frac{x}{\frac{20}{3600}\text{ hours}}$$

$$\therefore \quad x = 100 \times \tfrac{20}{3600}$$

$$\therefore \quad x = \tfrac{20}{36} = \tfrac{5}{9}$$

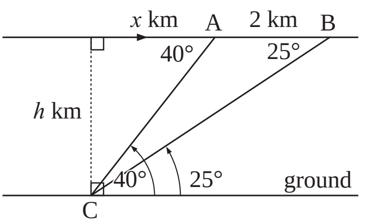
$$\text{Now} \quad \tan 30^o = \frac{\frac{5}{9}}{h}$$

$$\therefore \quad h = \frac{\frac{5}{9}}{\tan 30^o}$$

$$\therefore \quad h \doteqdot 0.962$$

$\therefore$ is about 962 m above ground level.

18



Notice that

$$\tan 40^o = \frac{h}{x}$$

$$\text{and} \quad \tan 25^o = \frac{h}{x+2}$$

$$\therefore \quad h = x\tan 40^o$$

$$\text{and} \quad h = (x+2)\tan 25^o$$

$$\therefore \quad x\tan 40^o = (x+2)\tan 25^o$$

$$\therefore \quad \frac{x+2}{x} = \frac{\tan 40^o}{\tan 25^o}$$

$$\therefore \quad \frac{x+2}{x} \doteqdot 1.799\,45$$

$$\therefore \quad x+2 \doteqdot 1.79945x$$

$$\therefore \quad 2 \doteqdot 0.79945x$$

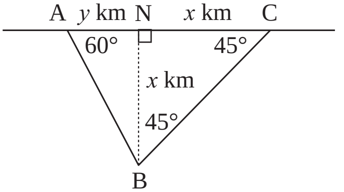
$$\therefore \quad x \doteqdot 2.502$$

$$\text{So} \quad h = 2.502 \times \tan 40^o$$

$$\therefore \quad h \doteqdot 2.10$$

i.e., 2.10 km about ground level

19



ΔNBC is right angled isosceles

$$\therefore \quad \text{BN} = x \text{ km}$$

$$\text{Now} \quad \tan 60^o = \frac{x}{y}$$

$$\therefore \quad \sqrt{3} = \frac{x}{5-x} \qquad \{\text{as} \quad x+y=5\}$$

$$\therefore \quad 5\sqrt{3} - x\sqrt{3} = x$$

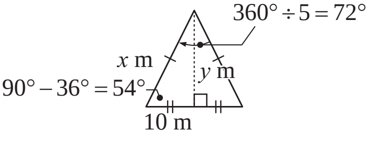
$$\therefore \quad 5\sqrt{3} = x(1+\sqrt{3})$$

$$\therefore \quad \frac{5\sqrt{3}}{1+\sqrt{3}} = x$$

$$\therefore \quad x \doteqdot 3.17$$

$\therefore$ is 3.17 km from the shore

20 Each triangle is:



$$\cos 54^o = \frac{10}{x} \qquad \therefore \quad x = \frac{10}{\cos 54^o}$$

$$\therefore \quad x \doteqdot 17.01$$

$$\tan 54^o = \frac{y}{10} \qquad \therefore \quad y = 10\tan 54^o$$

$$\therefore \quad y \doteqdot 13.76$$

$$\text{But} \quad d = x+y$$

$$\doteqdot 17.01 + 13.76$$

$$\doteqdot 30.8$$

i.e., width of land is 30.8 m

EXERCISE 10F

1 **a**
$$\tan\theta = \frac{4-5}{-1-2}$$
$$\therefore \tan\theta = \tfrac{-1}{-3} = \tfrac{1}{3}$$
$$\therefore \theta = \tan^{-1}\left(\tfrac{1}{3}\right)$$
$$\therefore \theta \doteqdot 18.4^o$$

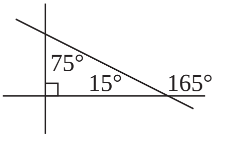
b
$$\tan\theta = \frac{-4--2}{-1-3} = \tfrac{-2}{-4} = \tfrac{1}{2}$$
$$\therefore \theta = \tan^{-1}\left(\tfrac{1}{2}\right)$$
$$\therefore \theta \doteqdot 26.6^o$$

c
$$\tan\theta = \frac{-5-1}{1--2} = -\tfrac{6}{3} = -2$$
$$\therefore \theta = 180^o - \tan^{-1}(2)$$
$$\therefore \theta \doteqdot 116.6^o$$

d
$$\tan\theta = \frac{-1-4}{-2--7} = -\tfrac{5}{5} = -1$$
$$\therefore \theta = 135^o$$

2 **a**
$$m = \tan 60^o \doteqdot 1.73 \quad \text{or} \quad \sqrt{3}$$

b



$$m = \tan 165^o \doteqdot -0.268$$

3 **a**
$$\theta = 60^o$$
$$\therefore \tan\theta = \sqrt{3} = m$$
$$\therefore y = \sqrt{3}x + 2$$

b
$$\theta = 120^o$$
$$\therefore m = \tan 120^o$$
$$\therefore m = -\sqrt{3}$$
$$\therefore y = -\sqrt{3}x$$

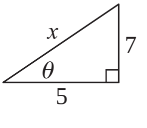
c
$$\theta = 30^o$$
$$\therefore \tan\theta = \tfrac{1}{\sqrt{3}}$$
So, $y = \frac{1}{\sqrt{3}}x + c$

But $(2\sqrt{3}, 0)$ lies on the line
$$\therefore 0 = \tfrac{1}{\sqrt{3}}(2\sqrt{3}) + c$$
$$\therefore 0 = 2 + c$$
$$\therefore c = -2$$
i.e., line is $y = \frac{1}{\sqrt{3}}x - 2$

REVIEW SET 10A

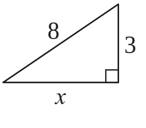
1 $\sin\theta = \frac{7}{11}$

2



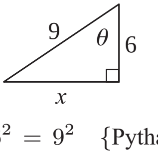
$$x^2 = 5^2 + 7^2 \quad \{\text{Pythagoras}\}$$
$$\therefore x = \sqrt{5^2+7^2}$$
$$\therefore x = \sqrt{74}$$
So, $\sin\theta = \frac{7}{\sqrt{74}}$, $\cos\theta = \frac{5}{\sqrt{74}}$

3



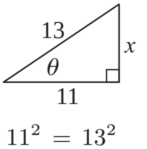
$$x^2 + 3^2 = 8^2 \quad \{\text{Pythagoras}\}$$
$$\therefore x = \sqrt{8^2-3^2}$$
$$\therefore x = \sqrt{55}$$
So, $\tan\theta = \frac{3}{\sqrt{55}}$

4



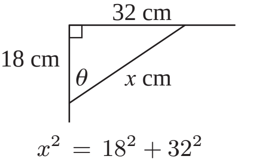
$$x^2 + 6^2 = 9^2 \quad \{\text{Pythagoras}\}$$
$$\therefore x = \sqrt{9^2-6^2}$$
$$\therefore x = \sqrt{45}$$
So, $\sin\theta = \frac{\sqrt{45}}{9}$
$\cos\theta = \frac{6}{9} = \frac{2}{3}$
$\tan\theta = \frac{\sqrt{45}}{6}$

5



$$x^2 + 11^2 = 13^2$$
$$\therefore x = \sqrt{13^2-11^2}$$
$$\therefore x = \sqrt{48}$$
So, $\sin\theta = \frac{\sqrt{48}}{13}$
$\cos\theta = \frac{11}{13}$
$\tan\theta = \frac{\sqrt{48}}{11}$

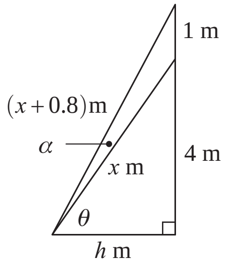
6



a
$$x^2 = 18^2 + 32^2$$
$$\therefore x = \sqrt{18^2+32^2}$$
$$\therefore x \doteqdot 36.715$$
So, the support is 36.7 cm long

b
$$\tan\theta = \tfrac{32}{18}$$
$$\therefore \theta = \tan^{-1}\left(\tfrac{32}{18}\right) \doteqdot 60.6$$
$\therefore$ makes 60.6^o with the wall.

7



a
$$\begin{aligned} x^2 &= h^2 + 4^2 \text{ .... (1)} \\ \text{and} \quad (x+0.8)^2 &= h^2 + 5^2 \\ \therefore \quad (x+0.8)^2 - 25 &= x^2 - 16 \\ \therefore \quad x^2 + 1.6x + 0.64 - 25 &= x^2 - 16 \\ \therefore \quad 1.6x &= 8.36 \\ \therefore \quad x &\doteqdot 5.225 \\ \text{and} \quad x + 0.8 &\doteqdot 6.025 \end{aligned}$$

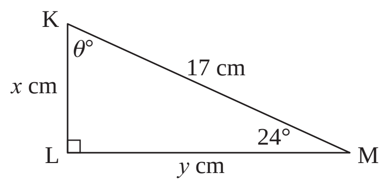
$\therefore$ the extended ladder is 6.025 m long

b $\sin\theta = \dfrac{4}{5.225}$ $\quad \sin(\theta+\alpha) = \dfrac{5}{6.025}$

$\therefore \ \theta \doteqdot 49.956 \qquad \therefore \ \theta + \alpha \doteqdot 56.086$

$\therefore \ \alpha \doteqdot 56.086 - 49.956 \doteqdot 6.13 \qquad \therefore$ angle increases by 6.13^o

8



$$\theta^o = 90^o - 24^o = 66^o$$

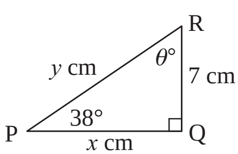
$$\cos 24^o = \frac{y}{17}, \quad \sin 24^o = \frac{x}{17}$$

$\therefore \ y = 17\cos 24^o \qquad \therefore \ x = 17\sin 24^o$

$\therefore \ y \doteqdot 15.5 \qquad \therefore \ x \doteqdot 6.91$

So, KL $\doteqdot$ 6.91 cm
LM $\doteqdot$ 15.5 cm

9



$$\theta^o = 90^o - 38^o = 52^o$$

$\therefore \ \measuredangle\text{PRQ} = 52$

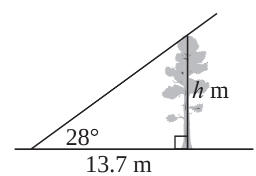
$\tan 38^o = \dfrac{7}{x}$ and $\sin 38^o = \dfrac{7}{y}$

$\therefore \ x = \dfrac{7}{\tan 38^o} \qquad \therefore \ y = \dfrac{7}{\sin 38^o}$

$\therefore \ x \doteqdot 8.96 \qquad \therefore \ y \doteqdot 11.4$

So PQ $\doteqdot$ 8.96 cm $\quad$ PR $\doteqdot$ 11.4 cm

10

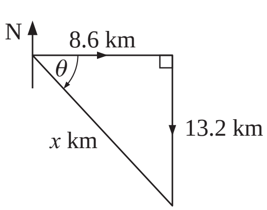


$$\begin{aligned} \tan 28^o &= \frac{h}{13.7} \\ \therefore \quad 13.7 \times \tan 28 &= h \\ \therefore \quad h &\doteqdot 7.28 \end{aligned}$$

$\therefore$ the height is 7.28 m

REVIEW SET 10B

1



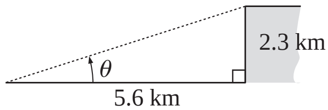
$$\begin{aligned} x^2 &= 8.6^2 + 13.2^2 \\ \therefore \quad x &= \sqrt{8.6^2 + 13.2^2} \\ \therefore \quad x &\doteqdot 15.75 \end{aligned}$$

So, distance is 15.75 km

$$\begin{aligned} \tan\theta &= \frac{13.2}{8.6} \\ \therefore \quad \theta &= \tan^{-1}\left(\frac{13.2}{8.6}\right) \\ \therefore \quad \theta &\doteqdot 56.9^o \\ \text{and} \quad 90^o + 56.9^o &\doteqdot 147^o \end{aligned}$$

$\therefore$ bearing is 147^o

2



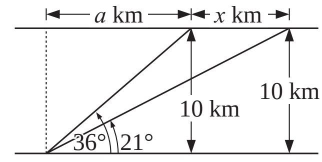
$\tan\theta = \dfrac{2.3}{5.6}$ {assuming a vertical mountain}

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{2.3}{5.6}\right)$

$\therefore \quad \theta \doteqdot 22.3^o$

So, the angle of elevation is 22.3^o

3



$\tan 36^o = \dfrac{10}{a}$ and $\tan 21^o = \dfrac{10}{a+x}$

$\therefore \quad a = \dfrac{10}{\tan 36^o}$ and $a + x = \dfrac{10}{\tan 21^o}$

$\therefore \quad a \doteqdot 13.764$ and $a + x \doteqdot 26.051$

$\therefore \quad x \doteqdot 26.051 - 13.764$

$\therefore \quad x \doteqdot 12.287$

$\text{speed} = \dfrac{12.287 \text{ km}}{\frac{2}{60} \text{ hours}} \doteqdot 369 \text{ km/h}$

4 **a** $\sin\theta = 0.8147$
$\therefore \quad \theta = \sin^{-1}(0.8147)$
$\therefore \quad \theta \doteqdot 54.6^o$

b $\cos\theta = 0.0917$
$\therefore \quad \theta = \cos^{-1}(0.0917)$
$\therefore \quad \theta \doteqdot 84.7^o$

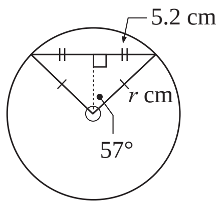
c $\tan\theta = 5.23$
$\therefore \quad \theta = \tan^{-1}(5.23)$
$\therefore \quad \theta \doteqdot 79.2^o$

5 **a** $\sin\theta = \frac{\sqrt{11}}{5}$
$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{11}}{5}\right)$
$\therefore \quad \theta \doteqdot 41.6^o$

b $\cos\theta = \frac{5}{7}$
$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{7}\right)$
$\therefore \quad \theta \doteqdot 44.4^o$

c $\tan\theta = 0.7452$
$\therefore \quad \theta = \tan^{-1}(0.7452)$
$\therefore \quad \theta \doteqdot 36.7^o$

6



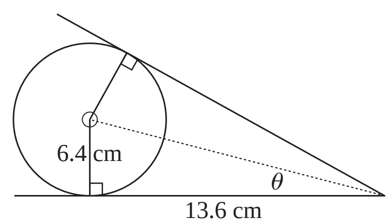
$\sin 57^o = \dfrac{5.2}{r}$

$\therefore \quad r = \dfrac{5.2}{\sin 57^o}$

$\therefore \quad r \doteqdot 6.20$

$\therefore$ the radius is 6.20 cm

7



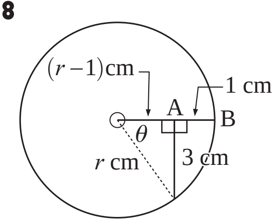
$\tan\theta = \dfrac{6.4}{13.6}$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{6.4}{13.6}\right)$

$\therefore \quad \theta \doteqdot 25.2$

So, the angle measures 25.2^o

8



a If the radius OC is r cm

$$\begin{aligned} \text{then OA} &= (r-1) \text{ cm} \\ \therefore \quad (r-1)^2 + 3^2 &= r^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad r^2 - 2r + 1 + 9 &= r^2 \\ \therefore \quad -2r + 10 &= 0 \\ \therefore \quad r &= 5 \end{aligned}$$

So, the radius is 5 cm

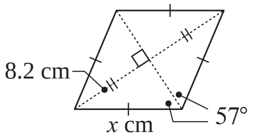
b $\sin\theta = \dfrac{3}{r} = \dfrac{3}{5}$

$\therefore \quad \theta = \sin^{-1}(0.6)$

$\therefore \quad \theta \doteqdot 36.9$

So, BC subtends 36.9^o at O.

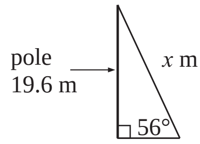
9



$$\sin 57^o = \frac{8.2}{x}$$

$$\therefore \quad x = \frac{8.2}{\sin 57^o} \doteqdot 9.78 \qquad \text{i.e., sides are 9.78 cm long}$$

10



$$\sin 56^o = \frac{19.6}{x}$$

$$\therefore \quad x = \frac{19.6}{\sin 56^o}$$

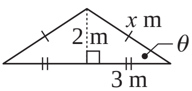
$$\therefore \quad x \doteqdot 23.64$$

and $3x \doteqdot 70.9$

So, the total length is 70.9 m

REVIEW SET 10C

1



a $x^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore \quad x = \sqrt{2^2+3^2}$

$\therefore \quad x \doteqdot 3.61$

$\therefore$ beam AB is 3.61 m long.

b $\tan\theta = \frac{2}{3}$

$\therefore \quad \theta = \tan^{-1}(\frac{2}{3})$

$\therefore \quad \theta \doteqdot 33.7$

$\therefore$ beam is inclined at 33.7^o to the horizontal.

2 $\tan\theta = \dfrac{6-(-4)}{1-7} = \frac{10}{-6} = -\frac{5}{3}$

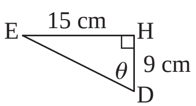
$\therefore \quad \theta = 180^o - \tan^{-1}\left(\frac{5}{3}\right)$

$\therefore \quad \theta \doteqdot 121^o$

3 $m = \tan 30^o = \frac{1}{\sqrt{3}}$

$\therefore$ equation is $y = \frac{1}{\sqrt{3}}x - 3$

4 **a**



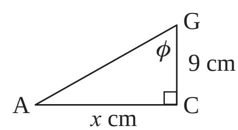
b $\tan\theta = \frac{15}{9} = \frac{5}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{5}{3}\right)$

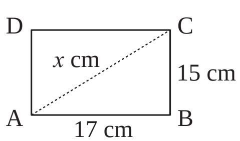
$\therefore \quad \theta \doteqdot 59.0$

So, $\measuredangle$HDE is 59.0^o

c



d



By Pythagoras $x^2 = 15^2 + 17^2$

$\therefore \quad x = \sqrt{15^2+17^2}$

$\therefore \quad x \doteqdot 22.67$

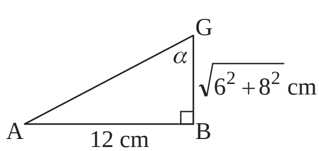
and $\tan\phi = \dfrac{x}{9}$

$\therefore \quad \phi = \tan^{-1}\left(\frac{22.67}{9}\right)$

$\therefore \quad \phi \doteqdot 68.3$

i.e., $\measuredangle$AGC is 68.3^o

5 **a**



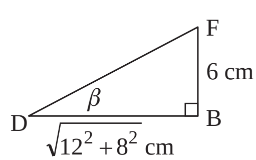
$$\tan\alpha = \frac{12}{\sqrt{6^2+8^2}}$$

$\therefore \quad \tan\alpha = \frac{12}{10}$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{6}{5}\right) \doteqdot 50.2$

So, the angle measures 50.2^o

b

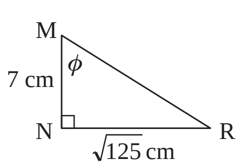


$$\tan\beta = \frac{6}{\sqrt{12^2+8^2}} = \frac{6}{\sqrt{208}}$$

$$\therefore \quad \beta = \tan^{-1}\left(\frac{6}{\sqrt{208}}\right) \doteqdot 22.6$$

So, the angle measures 22.6^o

6 **a**



b $\text{RN} = \sqrt{5^2+10^2}$

$= \sqrt{125}$

$\doteqdot 11.180......$

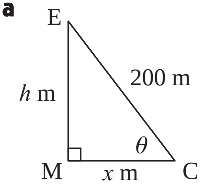
$\doteqdot 11.2$ cm

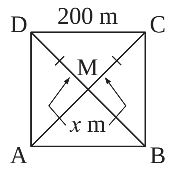
c $\tan\phi = \frac{\sqrt{125}}{7}$

$\therefore \quad \phi = \tan^{-1}\left(\frac{\sqrt{125}}{7}\right) \doteqdot 57.9$

i.e., $\measuredangle$RMN is 57.9^o

7 **a**

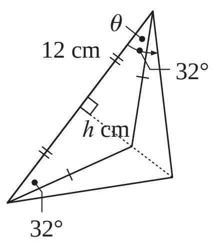




$$x^2 + x^2 = 200^2 \quad \{\text{Pythgoras}\}$$
$$\therefore \quad 2x^2 = 40\,000$$
$$\therefore \quad x^2 = 20\,000$$
Also
$$h^2 + x^2 = 200^2$$
$$\therefore \quad h^2 = 200^2 - 20\,000$$
$$\therefore \quad h = \sqrt{200^2 - 20\,000} \doteqdot 141$$
i.e., 141 m high

b
$$\sin\theta = \frac{h}{200} \doteqdot \frac{141.42}{200}$$
$$\therefore \quad \theta = \sin^{-1}\left(\frac{141.42}{200}\right) \doteqdot 45^o$$

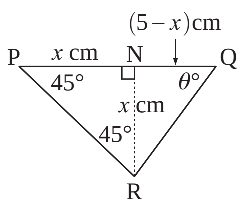
8



$$\tan 32^o = \frac{h}{12}$$
$$\therefore \quad 12 \times \tan 32^o = h$$
$$\therefore \quad h \doteqdot 7.4984$$
$$\text{Area} = \tfrac{1}{2}\text{ base} \times \text{height}$$
$$= \tfrac{1}{2} \times 24 \times 7.4984$$
$$\doteqdot 89.981$$
$$\therefore \quad \text{new area} = 2 \times \text{old area}$$
$$\doteqdot 179.96\text{ cm}^2$$

Now if the new height is H cm
$$\tfrac{1}{2} \times 24 \times H \doteqdot 179.96$$
$$\therefore \quad 12H \doteqdot 179.96$$
$$\therefore \quad H \doteqdot 14.997$$
$$\therefore \quad \tan\theta = \frac{14.997}{12}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{14.997}{12}\right)$$
$$\doteqdot 51.3$$
So, the new base angles are 51.3^o

9



Let NR $= x$ cm

Now ΔPNR is right angled isosceles

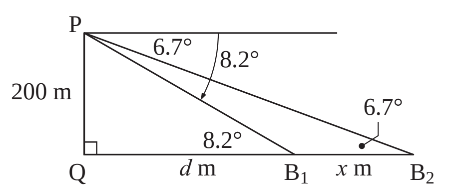
$\therefore$ PN $= x$ cm also

and so QN $= (5 - x)$ cm

Now $\tan\theta = \dfrac{x}{5 - x}$ {in ΔRQN}

$$\therefore \quad (5 - x)\tan\theta = x$$
$$\therefore \quad 5\tan\theta - x\tan\theta = x$$
$$\therefore \quad 5\tan\theta = x + x\tan\theta$$
$$\therefore \quad 5\tan\theta = x(1 + \tan\theta)$$
$$\therefore \quad \frac{5\tan\theta}{1 + \tan\theta} = x$$
i.e., $\text{RN} = \dfrac{5\tan\theta}{1 + \tan\theta}$ cm

10



Let $\text{QB}_1 = d$ m and $\text{B}_1\text{B}_2 = x$ m

$$\therefore \quad \tan 8.2^o = \frac{200}{d} \quad \text{and} \quad \tan 6.7^o = \frac{200}{d + x}$$
$$\therefore \quad d = \frac{200}{\tan 8.2} \doteqdot 1387.9$$
$$\text{and} \quad d + x = \frac{200}{\tan 6.7} \doteqdot 1702.5$$
So $x = 1702.5 - 1387.9 \doteqdot 315$

i.e., the boats are 315 m apart.

Chapter 11

THE UNIT CIRCLE

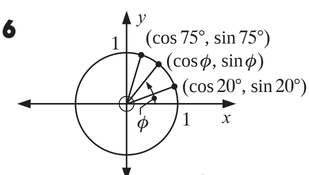
EXERCISE 11A

1 **a** The y-coordinate at 0^o is 0, $\therefore$ $\sin 0^o = 0$
b The y-coordinate at 15^o is $\doteqdot 0.26$, $\therefore$ $\sin 15^o = 0.26$
c The y-coordinate at 25^o is $\doteqdot 0.42$, $\therefore$ $\sin 25^o = 0.42$
d The y-coordinate at 30^o is $\doteqdot 0.5$, $\therefore$ $\sin 30^o = 0.50$
e The y-coordinate at 45^o is $\doteqdot 0.71$, $\therefore$ $\sin 45^o = 0.71$
f The y-coordinate at 60^o is $\doteqdot 0.87$, $\therefore$ $\sin 60^o = 0.87$
g The y-coordinate at 75^o is $\doteqdot 0.97$, $\therefore$ $\sin 75^o = 0.97$
h The y-coordinate at 90^o is 1, $\therefore$ $\sin 90^o = 1$

3 **a** The x-coordinate at 0^o is 1, $\therefore$ $\cos 0^o = 1$
b The x-coordinate at 15^o is $\doteqdot 0.97$, $\therefore$ $\cos 15^o \doteqdot 0.97$
c The x-coordinate at 25^o is $\doteqdot 0.91$, $\therefore$ $\cos 25^o \doteqdot 0.91$
d The x-coordinate at 30^o is $\doteqdot 0.87$, $\therefore$ $\cos 30^o \doteqdot 0.87$
e The x-coordinate at 45^o is $\doteqdot 0.71$, $\therefore$ $\cos 45^o \doteqdot 0.71$
f The x-coordinate at 60^o is $\doteqdot 0.5$, $\therefore$ $\cos 60^o \doteqdot 0.5$
g The x-coordinate at 75^o is $\doteqdot 0.26$, $\therefore$ $\cos 75^o \doteqdot 0.26$
h The x-coordinate at 90^o is 0, $\therefore$ $\cos 90^o = 0$

5 The coordinates
$(\cos 55^o, \sin 55^o)$
i.e., $(0.57, 0.82)$

6



7 **a** As $\cos^2\theta + \sin^2\theta = 1$,

then $\sin\theta = \sqrt{1-\cos^2\theta}$ as $\sin\theta > 0$
$= \sqrt{1-(0.8)^2}$
$= 0.6$

b As $\cos^2\theta + \sin^2\theta = 1$,

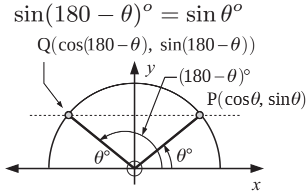
then $\cos\theta = \sqrt{1-\sin^2\theta}$ as $\cos\theta > 0$
$= \sqrt{1-(0.7)^2}$
$\doteqdot 0.714$

EXERCISE 11B

1 **a** 0.98 **b** 0.98 **c** 0.87 **d** 0.87 **e** 0.5 **f** 0.5 **g** 0 **h** 0

3 **a** It appears that
$\sin(180-\theta)^o = \sin\theta^o$

b

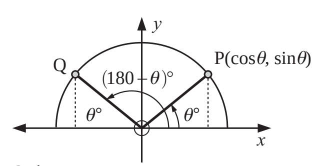


P and Q have the same y-coordinate.
$\therefore$ $\sin(180-\theta)^o = \sin\theta^o$

4 **a** -0.34
b 0.34
c -0.64
d 0.64
e -0.77
f 0.77
g -1
h 1

6 **a** $\cos(180-\theta)^o = -\cos\theta^o$

b



Q is
$(\cos(180-\theta))^o, \sin(180-\theta)^o)$
The x-coordinate of Q is the negative of the x-coordinate of P.
$\therefore$ $\cos(180-\theta)^o = -\cos\theta^o$

7 **a** $\sin 45^o$
$= \sin(180-45)^o$
$= \sin 135^o$
i.e., $\theta = 135^o$

b $\sin 51^o$
$= \sin(180-51)^o$
$= \sin 129^o$
i.e., $\theta = 129^o$

c $\sin 74^o$
$= \sin(180-74)^o$
$= \sin 106^o$
i.e., $\theta = 106^o$

d $\sin 82^o$
$= \sin(180-82)^o$
$= \sin 98^o$
i.e., $\theta = 98^o$

8 **a** $\sin 130^o$
$= \sin(180 - 130)^o$
$= \sin 50^o$
i.e., $\theta = 50^o$

b $\sin 146^o$
$= \sin(180 - 146)^o$
$= \sin 34^o$
i.e., $\theta = 34^o$

c $\sin 162^o$
$= \sin(180 - 162)^o$
$= \sin 18^o$
i.e., $\theta = 18^o$

d $\sin 171^o$
$= \sin(180 - 171)^o$
$= \sin 9^o$
i.e., $\theta = 9^o$

9 **a** $\sin 137^o$
$= \sin(180 - 137)^o$
$= \sin 43^o$
$\doteqdot 0.6820$

b $\sin 59^o$
$= \sin(180 - 59)^o$
$= \sin 121^o$
$\doteqdot 0.8572$

c $\cos 143^o$
$= -\cos(180 - 143)^o$
$= -\cos 37^o$
$\doteqdot -0.7986$

d $\cos 24^o$
$= -\cos(180 - 24)^o$
$= -\cos 156^o$
$= 0.9135$

e $\sin 115^o$
$= \sin(180 - 115)^o$
$= \sin 65^o$
$\doteqdot 0.9063$

f $\cos 132^o$
$= -\cos(180 - 132)^o$
$= -\cos 48^o$
$= -0.6691$

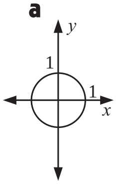
10 **a** $\measuredangle AOQ + \measuredangle BOQ = 180^o$
$\therefore \quad \theta^o + \measuredangle BOQ = 180^o$
$\measuredangle BOQ = (180 - \theta)^o$

b OQ is a reflection of OP in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$

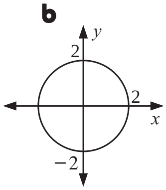
c $\cos(180 - \theta)^o = -\cos\theta^o, \quad \sin(180 - \theta)^o = \sin\theta^o$

EXERCISE 11C

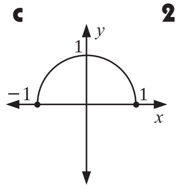
1 **a**



b

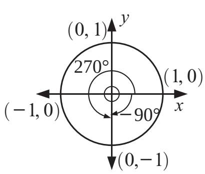


c



2 **a** **i** A$(\cos 26^o, \sin 26^o)$ B$(\cos 146^o, \sin 146^o)$ C$(\cos 199^o, \sin 199^o)$
ii A(0.899, 0.438) B(−0.829, 0.559) C(−0.946, −0.326)
b **i** A$(\cos 123^o, \sin 123^o)$ B$(\cos 251^o, \sin 251^o)$ C$(\cos(-35^o), \sin(-35^o))$
ii A(−0.545, 0.839) B(−0.326, −0.946) C(0.819, −0.574)

3



a $\cos 0^o = 1, \quad \sin 0^o = 0$
b $\cos 90^o = 0, \quad \sin 90^o = 1$
c $\cos 180^o = -1, \quad \sin 180^o = 0$
d $\cos 270^o = 0, \quad \sin 270^o = -1$
e $\cos(-90^o) = 0, \quad \sin(-90^o) = -1$
f $\cos 450^o = 0, \quad \sin 450^o = 1$

REVIEW SET 11

1 **a** $\sin 70^o \doteqdot 0.94$ **b** $\cos 35^o \doteqdot 0.82$

2 M$(\cos 73^o, \sin 73^o) \doteqdot (0.292, 0.956)$ N$(\cos 190^o, \sin 190^o) \doteqdot (-0.985, -0.174)$
P$(\cos 307^o, \sin 307^o) \doteqdot (0.602, -0.799)$

3 The x-coordinate of A $= -0.222$
$\therefore \quad \cos\theta = -0.222$
$\therefore \quad \theta = \cos^{-1}(-0.222)$
$\therefore \quad \theta \doteqdot 102.8^o$

4 **a** $\sin 120^o = \sin(180 - 120)^o = \sin 60^o$
$\therefore \quad \theta = 60^o$
b $\sin 165^o = \sin(180 - 165)^o = \sin 15^o$
$\therefore \quad \theta = 15^o$
c $\sin 95^o = \sin(180 - 95)^o = \sin 85^o$
$\therefore \quad \theta = 85^o$

5 **a** $\sin 47^o$
$= \sin(180 - 47)^o$
$= \sin 133^o$
$\therefore \quad \theta = 133^o$

b $\sin 8^o$
$= \sin(180 - 8)^o$
$= \sin 172^o$
$\therefore \quad \theta = 172^o$

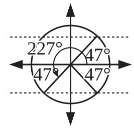
c $\sin 86^o$
$= \sin(180 - 86)^o$
$= \sin 94^o$
$\therefore \quad \theta = 94^o$

6 **a** $\sin 159^o$
$= \sin(180 - 159)^o$
$= \sin 21^o$
$\doteqdot 0.358$

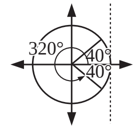
b $\cos 92^o$
$= -\cos(180 - 92)^o$
$= -\cos 88^o$
$\doteqdot -0.035$

c $\cos 75^o$
$= -\cos(180 - 75)^o$
$= -\cos 105^o$
$\doteqdot 0.259$

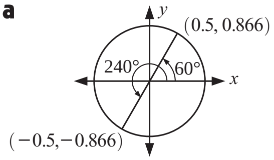
d $\sin 227^o = \sin(-47)^o$
$= -\sin 47^o$
$\doteqdot -0.731$



e $\cos 320^o$
$= \cos 40^o$
$\doteqdot 0.766$

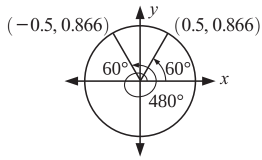


7 **a**



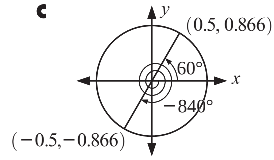
$\therefore \quad \cos 240^o = -0.5$
$\sin 240^o = -0.866$

b



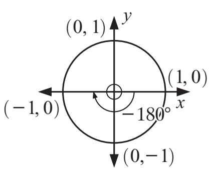
$\therefore \quad \cos 480^o = -0.5$
$\sin 480^o = 0.866$

c



$\therefore \quad \cos(-840^o) = -0.5$
$\sin(-840^o) = -0.866$

8



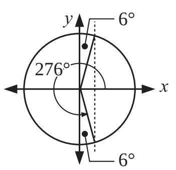
a $\cos 360^o = 1, \quad \sin 360^o = 0$
b $\cos(-180^o) = -1, \quad \sin(-180^o) = 0$
c $\cos(630^o) = 0, \quad \sin(630^o) = -1$

9 **a** $\sin 101^o$
$= \sin(180 - 101)^o$
$= \sin 79^o$
$\therefore \quad \theta = 79^o$

b $\sin 127^o$
$= \sin(180 - 127)^o$
$= \sin 53^o$
$\therefore \quad \theta = 53^o$

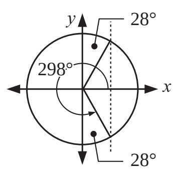
c $\sin 168^o$
$= \sin(180 - 168)^o$
$= \sin 12^o$
$\therefore \quad \theta = 12^o$

10 **a**



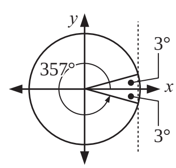
$\therefore \quad \theta = (90 - 6)^o = 84^o$

b



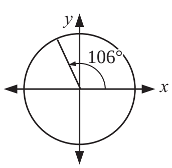
$\therefore \quad \theta = (90 - 28)^o = 62^o$

c



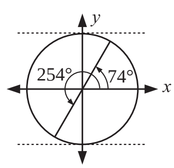
$\therefore \quad \theta = 3^o$

11 **a**



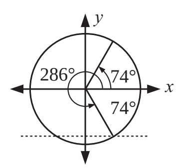
$\sin 106^o$
$= \sin(180 - 106)^o$
$= \sin 74^o$
$\doteqdot 0.961$

b



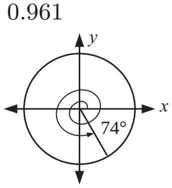
$254^o = 74^o + 180^o$
$\therefore \quad \sin 254^o = -\sin 74^o$
$\doteqdot -0.961$

c



$\sin 286^o = -\sin 74^o$
$= -0.961$

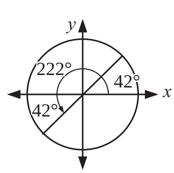
d



$646^o = 360^o + 286^o$
$\therefore \quad \sin 646^o = \sin 286^o$
$= -0.961 \quad$ {from **c**}

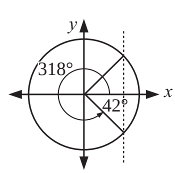
12 **a** $\cos 138^o$
$= -\cos(180 - 138)^o$
$= -\cos 42^o$
$\doteqdot -0.743$

b



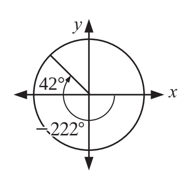
$\cos 222^o$
$= -\cos 42^o$
$= -0.743$

c



$\cos 318^o$
$= \cos 42^o$
$= 0.743$

d



$\cos(-222)^o$
$= -\cos 42^o$
$= -0.743$

Chapter 12

NON RIGHT ANGLED TRIANGLE TRIGONOMETRY

EXERCISE 12A

1 **a** Area
$= \frac{1}{2} \times 9 \times 10 \times \sin 40^o$
$\doteqdot 28.9 \text{ cm}^2$

b Area
$= \frac{1}{2} \times 25 \times 31 \times \sin 82^o$
$\doteqdot 384 \text{ km}^2$

c Area
$= \frac{1}{2} \times 10.2 \times 6.4 \times \sin 125^o$
$\doteqdot 26.7 \text{ cm}^2$

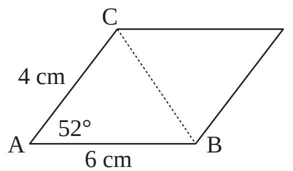
2 Area $= 150 \text{ cm}^2$

$\therefore \quad \frac{1}{2} \times 17 \times x \times \sin 68^o = 150$

$\therefore \quad x = \dfrac{2 \times 150}{17 \times \sin 68^o}$

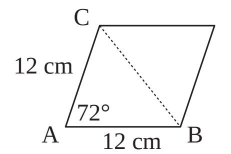
$\therefore \quad x \doteqdot 19.0$

3



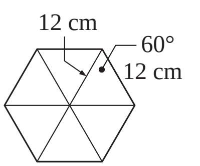
Area
$= 2\times$ area ΔABC
$= 2 \times \frac{1}{2} \times 4 \times 6 \times \sin 52^o$
$\doteqdot 18.9 \text{ cm}^2$

4



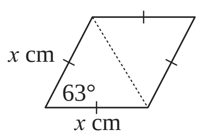
Area $= 2 \times$ area ΔABC
$= 2 \times \frac{1}{2} \times 12^2 \times \sin 72^o$
$\doteqdot 137 \text{ cm}^2$

5



Area
$= 6 \times$ area of Δ
$= 6 \times \frac{1}{2} \times 12^2 \times \sin 60^o$
$\doteqdot 374 \text{ cm}^2$

6



Area $= 2 \times \frac{1}{2} x^2 \sin 63^o$

$\therefore \quad x^2 \sin 63^o = 50$

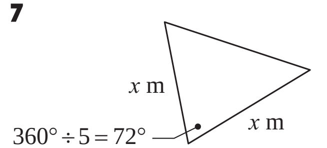
$\therefore \quad x^2 = \dfrac{50}{\sin 63^o}$

$\therefore \quad x = \sqrt{\dfrac{50}{\sin 63^o}}$

$\therefore \quad x \doteqdot 7.49$

So, sides are 7.49 cm long.

7



Area of $\Delta = \frac{338}{5}$

$\therefore \quad \frac{1}{2} x^2 \sin 72^o = \frac{338}{5}$

$\therefore \quad x^2 = \dfrac{2 \times 338}{5 \times \sin 72^o}$

$\therefore \quad x = \sqrt{\dfrac{2 \times 338}{5 \times \sin 72^o}}$

$\therefore \quad x \doteqdot 11.9$

So, OA $\doteqdot$ 11.9 m long.

8 **a** If the included angle is θ

then $\frac{1}{2} \times 5 \times 8 \times \sin\theta = 15$

$\therefore \quad 20 \sin\theta = 15$

$\therefore \quad \sin\theta = \frac{3}{4}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{3}{4}\right)$

$\therefore \quad \theta \doteqdot 48.6^o$ or $(180 - 48.6)^o$

i.e., $\theta \doteqdot 48.6^o$ or 131.4^o

b Likewise,

$\frac{1}{2} \times 45 \times 53 \times \sin\theta = 800$

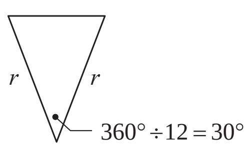
$\therefore \quad \sin\theta = \frac{800 \times 2}{45 \times 53}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1600}{45 \times 53}\right)$

$\therefore \quad \theta \doteqdot 42.1^o$ or $(180 - 42.1)^o$

$\therefore \quad \theta \doteqdot 42.1^o$ or 137.9^o

9

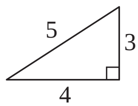


Total area of 8 coins
$= 8 \times 12 \times \frac{1}{2} r^2 \sin 30^o$
$= 48r^2 \left(\frac{1}{2}\right)$
$= 24r^2$

Area of \$10 note
$= 8r \times 4r$
$= 32r^2$

Fraction covered
$= \dfrac{24r^2}{32r^2}$
$= \frac{3}{4} \quad \therefore \quad \frac{1}{4}$ is uncovered

10 **a** **i**



$$\text{Area} = \tfrac{1}{2}\text{ base} \times \text{alt}$$
$$= \tfrac{1}{2} \times 4 \times 3$$
$$= 6 \text{ cm}^2$$

ii $$s = \frac{3+4+5}{2} = 6$$
$$\therefore \text{ area} = \sqrt{6(6-3)(6-4)(6-5)}$$
$$= \sqrt{6 \times 3 \times 2 \times 1}$$
$$= 6 \text{ cm}^2$$

b **i** $$s = \frac{6+8+12}{2} = 13$$
$$\therefore A = \sqrt{13(13-6)(13-8)(13-12)}$$
$$= \sqrt{13 \times 7 \times 5 \times 1}$$
$$\doteqdot 21.3 \text{ cm}^2$$

ii $$s = \frac{7.2+8.9+9.7}{2} = 12.9$$
$$\therefore A = \sqrt{12.9(12.9-7.2)(12.9-8.9)(12.9-9.7)}$$
$$= \sqrt{12.9 \times 5.7 \times 4 \times 3.2}$$
$$\doteqdot 30.7 \text{ cm}^2$$

EXERCISE 12B

1 **a** **i** arc length
$$= \left(\tfrac{41.6}{360}\right) \times 2\pi \times 9$$
$$\doteqdot 6.53 \text{ cm}$$

ii area
$$= \left(\tfrac{41.6}{360}\right) \times \pi \times 9^2$$
$$\doteqdot 29.4 \text{ cm}^2$$

b **i** arc length
$$= \left(\tfrac{122}{360}\right) \times 2\pi \times 4.93$$
$$\doteqdot 10.5 \text{ cm}$$

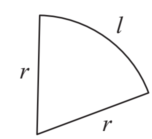
ii area
$$= \left(\tfrac{122}{360}\right) \times \pi \times 4.93^2$$
$$\doteqdot 25.9 \text{ cm}^2$$

2 **a** $\theta = 107.9^o, \quad l = 5.92$
$$\therefore \left(\tfrac{107.9}{360}\right) \times 2\pi \times r = 5.92$$
$$\therefore r = \frac{5.92 \times 360}{107.9 \times 2 \times \pi}$$
$$\therefore r \doteqdot 3.14 \text{ m}$$

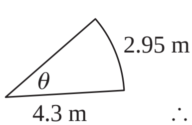
b $$\text{Area} = \left(\tfrac{107.9}{360}\right) \times \pi \times (3.1436)^2$$
$$\doteqdot 9.30 \text{ m}^2$$

3 **a** $$\text{Area} = \left(\tfrac{\theta}{360}\right) \times \pi r^2$$
$$\therefore 20.8 = \left(\tfrac{68.2}{360}\right) \times \pi r^2$$
$$\therefore \frac{20.8 \times 360}{68.2 \times \pi} = r^2$$
$$\therefore r = \sqrt{\frac{20.8 \times 360}{68.2 \times \pi}}$$
$$\therefore r \doteqdot 5.91 \text{ cm}$$

b Perimeter
$$= l + 2r$$
$$= \left(\tfrac{68.2}{360}\right) \times 2\pi \times 5.912 +$$
$$+ 2 \times 5.912$$
$$\doteqdot 18.9 \text{ cm}$$



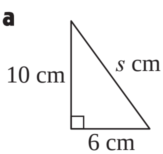
4 **a**



$$l = \left(\tfrac{\theta}{360}\right) \times 2\pi \times r$$
$$\therefore 2.95 = \left(\tfrac{\theta}{360}\right) \times 2\pi \times 4.3$$
$$\therefore \frac{2.95 \times 360}{2 \times \pi \times 4.3} = \theta$$
$$\therefore \theta \doteqdot 39.3^o$$

b $$\text{Area} = \left(\tfrac{\theta}{360}\right) \times \pi r^2$$
$$\therefore 30 = \left(\tfrac{\theta}{360}\right) \times \pi \times 10^2$$
$$\therefore \frac{30 \times 360}{\pi \times 100} = \theta$$
$$\therefore \theta \doteqdot 34.4^o$$

5 **a**



$$s^2 = 6^2 + 10^2 \quad \{\text{Pythagoras}\}$$
$$\therefore s = \sqrt{6^2 + 10^2}$$
$$\therefore s \doteqdot 11.6619$$
$$\therefore s \doteqdot 11.7 \text{ cm}$$

b $r = s \doteqdot 11.7$ cm

c $$\text{arc length} = 2\pi \times 6 \doteqdot 37.6991 \ldots.$$
$$\doteqdot 37.7 \text{ cm}$$

d $$\text{arc length} = \left(\tfrac{\theta}{360}\right) \times 2\pi r$$
$$\therefore \tfrac{\theta}{360} \times 2 \times \pi \times 11.6619 \doteqdot 37.6991$$
$$\therefore \frac{37.6991 \times 360}{2 \times \pi \times 11.6619} \doteqdot \theta$$
$$\therefore \theta \doteqdot 185^o$$

6 **a** $l = \left(\frac{\theta}{360}\right) \times 2\pi r$

$= \frac{\frac{1}{60}}{360} \times 2 \times \pi \times 6370$ km

$\doteqdot 1.852\,957$ ...... km

$\doteqdot 1.853$ km

b speed $= \dfrac{\text{distance}}{\text{time}}$ $\quad\therefore$ time $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{2130 \text{ km}}{480 \text{ n miles/h}}$

$= \dfrac{2130 \text{ km}}{480 \times 1.853 \text{ km/h}}$

$\doteqdot 2.3947$ ..... hours

$\doteqdot 2$ hours 23.68 min

$\doteqdot 2$ hours 24 min

7 **a** Area

$= \frac{1}{2} \times 12 \times 12 \times \sin 110^o$

$\doteqdot 67.657$.....

$\doteqdot 67.7$ cm^2

b Area of sector

$= \left(\frac{110}{360}\right) \times \pi \times 12^2$

$\doteqdot 138$ cm^2

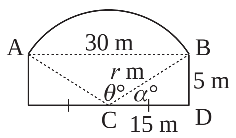
c Area of segment

= area of sector $-$ area of Δ

$\doteqdot 138.2 - 67.7$

$\doteqdot 70.5$ cm^2

8



a $\tan\alpha = \frac{5}{15}$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{1}{3}\right)$

$\therefore \quad \alpha \doteqdot 18.43$

b $\theta + 2\alpha = 180$ {angles on a line}

$\therefore \quad \theta = 180 - 2 \times 18.43$

$\therefore \quad \theta \doteqdot 143.1$

Note: $r^2 = 5^2 + 15^2$

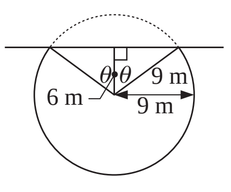
$\therefore \quad r^2 = 250$

c Area $= 2 \times$ area ΔCDB + area sector

$= 2 \times \frac{1}{2} \times 15 \times 5 + \left(\frac{143.1}{360}\right) \times \pi \times 250$

$\doteqdot 387.3$ m^2

9



$\cos\theta = \frac{6}{9} = \frac{2}{3}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore \quad \theta \doteqdot 48.19^o$

So, $360 - 2\theta \doteqdot 263.62^o$

Area

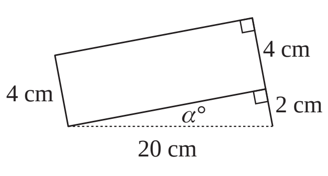
= area of Δ + area of sector

$= \frac{1}{2} \times 9^2 \times \sin 96.38^o$

$+ \left(\frac{263.62}{360}\right) \times \pi \times 9^2$

$\doteqdot 227$ m^2

10



a $\sin\alpha = \frac{2}{20} = 0.1$

$\therefore \quad \alpha = \sin^{-1}(0.1)$

$\therefore \quad \alpha \doteqdot 5.7392$.....

$\therefore \quad \alpha \doteqdot 5.739$

b $\theta + 90 + 90 + 2\alpha = 360$

$\therefore \quad \theta = 180 - 2\alpha$

$\doteqdot 180 - 2 \times 5.739$

$\doteqdot 168.5$

c $\phi + \theta = 360$

$\therefore \quad \phi \doteqdot 360 - 168.5$

$\therefore \quad \phi \doteqdot 191.5$

d length of belt

$= 2 \times 20 + \frac{\theta}{360} \times 2\pi \times 4$

$+ \frac{\phi}{360} \times 2\pi \times 6$

$\doteqdot 71.82$ cm

EXERCISE 12C

1 **a** $\text{BC}^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^o$

$\therefore \quad \text{BC} = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^o} \doteqdot 28.8$ cm

b $\text{PQ}^2 = 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^o$

$\therefore \quad \text{PQ} = \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^o} \doteqdot 3.38$ km

c $\text{KM}^2 = 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^o$

$\therefore \quad \text{KM} = \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^o} \doteqdot 14.2$ m

2 $\cos A = \dfrac{12^2+13^2-11^2}{2\times12\times13}$

$\therefore \quad A = \cos^{-1}\left(\frac{192}{312}\right)$

$\therefore \quad A \doteqdot 52.0^o$

$\cos B = \dfrac{13^2+11^2-12^2}{2\times13\times11}$

$\therefore \quad B = \cos^{-1}\left(\frac{146}{286}\right)$

$\therefore \quad B \doteqdot 59.3^o$

$C = (180-A-B)^o$

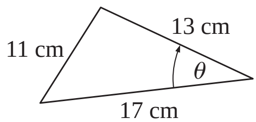
$\doteqdot 68.7^o$

3 $\cos Q = \dfrac{5^2+7^2-10^2}{2\times5\times7}$

$\therefore \quad Q = \cos^{-1}\left(\frac{-26}{70}\right)$

$\therefore \quad Q \doteqdot 112^o$

4 **a**



The smallest angle is opposite the shortest side.

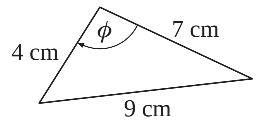
$\cos\theta = \dfrac{13^2+17^2-11^2}{2\times13\times17}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{337}{442}\right)$

$\therefore \quad \theta \doteqdot 40.3$

So, the smallest angle measures 40.3^o.

b



The largest angle is opposite the longest side.

$\cos\phi = \dfrac{4^2+7^2-9^2}{2\times4\times7}$

$\therefore \quad \phi = \cos^{-1}\left(-\frac{16}{56}\right)$

$\therefore \quad \phi \doteqdot 106.60$

So, the largest angle measures 107^o, approx.

5 **a** $\cos\theta = \dfrac{2^2+5^2-4^2}{2\times2\times5}$

$= \frac{13}{20}$

$= 0.65$

b $x^2 = 5^2+3^2-2\times5\times3\times\cos\theta$

$\therefore \quad x = \sqrt{5^2+3^2-2\times5\times3\times0.65}$

$\therefore \quad x \doteqdot 3.81$

EXERCISE 12D.1

1 **a** By the sine rule,

$\dfrac{x}{\sin 48^o} = \dfrac{23}{\sin 37^o}$

$\therefore \quad x = \dfrac{23\times\sin 48^o}{\sin 37^o}$

$\therefore \quad x \doteqdot 28.4$

b By the sine rule,

$\dfrac{x}{\sin 115^o} = \dfrac{11}{\sin 48^o}$

$\therefore \quad x = \dfrac{11\times\sin 115^o}{\sin 48^o}$

$\therefore \quad x \doteqdot 13.4$

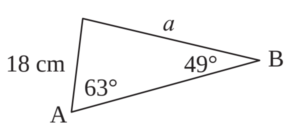
c By the sine rule,

$\dfrac{x}{\sin 51^o} = \dfrac{4.8}{\sin 80^o}$

$\therefore \quad x = \dfrac{4.8\times\sin 51^o}{\sin 80^o}$

$\therefore \quad x \doteqdot 3.79$

2 **a**

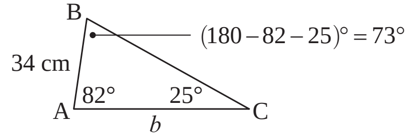


$\dfrac{a}{\sin 63^o} = \dfrac{18}{\sin 49^o}$ {sine rule}

$\therefore \quad a = \dfrac{18\times\sin 63^o}{\sin 49^o}$

$\therefore \quad a \doteqdot 21.25$ cm

b

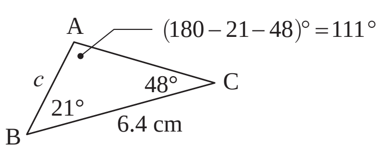


By the sine rule, $\dfrac{b}{\sin 73^o} = \dfrac{34}{\sin 25^o}$

$\therefore \quad b = \dfrac{34\times\sin 73^o}{\sin 25^o}$

$\therefore \quad b \doteqdot 76.9$ cm

c



By the sine rule, $\dfrac{c}{\sin 48^o} = \dfrac{6.4}{\sin 111^o}$

$\therefore \quad c = \dfrac{6.4\times\sin 48^o}{\sin 111^o}$

$\therefore \quad c \doteqdot 5.09$ cm

EXERCISE 12D.2

1 By the sine rule

$$\frac{\sin C}{11} = \frac{\sin 40^o}{8}$$

$$\therefore \quad \sin C = \frac{11 \times \sin 40^o}{8}$$

$$\therefore \quad C = \sin^{-1}\left(\frac{11 \times \sin 40^o}{8}\right)$$

$$\therefore \quad C \doteqdot 62.1^o \quad \text{or} \quad (180 - 62.1)^o$$

$$\therefore \quad C \doteqdot 62.1^o \quad \text{or} \quad 117.9^o$$

2 **a**

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \quad \sin A = \frac{14.6 \times \sin 65^o}{17.4}$$

$$\therefore \quad A = \sin^{-1}\left(\frac{14.6 \times \sin 65^o}{17.4}\right)$$

$$\therefore \quad A \doteqdot 49.5^o \quad \text{or} \quad 180^o - 49.5^o$$

$$\therefore \quad A \doteqdot 49.5^o \quad \text{or} \quad 130.5^o$$

Check: $A = 130.5^o$ is impossible as $A + B = 130.5^o + 65^o$ is already over 180^o. $\therefore$ $A \doteqdot 49.5^o$

b

$$\frac{\sin B}{43.8} = \frac{\sin 43^o}{31.4}$$

$$\therefore \quad \sin B = \frac{43.8 \times \sin 43^o}{31.4}$$

$$\therefore \quad B = \sin^{-1}\left(\frac{43.8 \times \sin 43^o}{31.4}\right)$$

$$\therefore \quad B \doteqdot 72.0^o \quad \text{or} \quad 108^o$$

both of which are possible as $108 + 43 = 151$ which is < 180.

c

$$\frac{\sin C}{4.8} = \frac{\sin 71^o}{6.5}$$

$$\therefore \quad \sin C = \frac{4.8 \times \sin 71^o}{6.5}$$

$$\therefore \quad C = \sin^{-1}\left(\frac{4.8 \times \sin 71^o}{6.5}\right)$$

$$\therefore \quad C \doteqdot 44.3^o \quad \text{or} \quad 135.7^o$$

But $135.7 + 71$ is already > 180

$\therefore$ this case is impossible $\therefore$ $C \doteqdot 44.3^o$

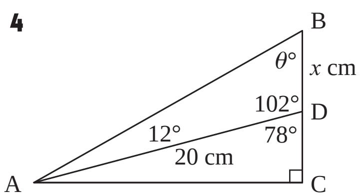
3 The third angle is $180^o - 85^o - 68^o = 27^o$

$$\frac{\sin 85^o}{11.4} = 0.087\,38.... \quad \text{and} \quad \frac{\sin 27^o}{9.8} = 0.046\,32....$$

$\therefore$ it is not possible as $\dfrac{\sin 85^o}{11.4} \neq \dfrac{\sin 27^o}{9.8}$

i.e., the sine rule is violated.

4



In ΔABD,

$$\theta = (180 - 12 - 102)^o$$

$$\therefore \quad \theta = 66^o$$

Now

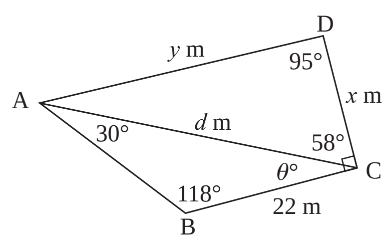
$$\frac{x}{\sin 12^o} = \frac{20}{\sin 66^o}$$

$$\therefore \quad x = \frac{20 \times \sin 12^o}{\sin 66^o}$$

$$\therefore \quad x \doteqdot 4.55$$

$\therefore$ BD is 4.55 cm long

5 First we find the length of the diagonal, d m.



$$\frac{d}{\sin 118^o} = \frac{22}{\sin 30^o}$$

$$\therefore \quad d = \frac{22 \times \sin 118^o}{\sin 30^o}$$

$$\therefore \quad d \doteqdot 38.85$$

$$\theta = 180^o - 30^o - 118^o = 32^o$$

$$\therefore \quad \angle\text{ACD} = 58^o$$

Using the sine rule

$$\frac{y}{\sin 58^o} = \frac{38.85}{\sin 95^o}$$

$$\therefore \quad y = \frac{38.85 \times \sin 58^o}{\sin 95^o}$$

$$\therefore \quad y \doteqdot 33.1$$

and

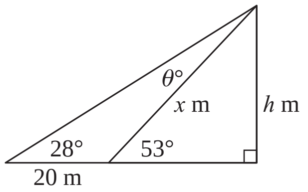
$$\frac{x}{\sin(180 - 95 - 58)} = \frac{38.85}{\sin 95^o}$$

$$\therefore \quad x = \frac{38.85 \times \sin 27^o}{\sin 95^o}$$

$$\therefore \quad x \doteqdot 17.7$$

EXERCISE 12E

1



$\theta^o + 28^o = 53^o$

{exterior angle of a Δ theorem}

$\therefore \quad \theta = 25^o$

By the Sine Rule,

$$\frac{x}{\sin 28^o} = \frac{20}{\sin 25^o}$$

$$\therefore \quad x = \frac{20 \times \sin 28^o}{\sin 25^o}$$

$$\therefore \quad x \doteqdot 22.22$$

and $\sin 53^o = \dfrac{h}{x}$

$$\therefore \quad h = x \sin 53^o$$
$$= 22.22 \times \sin 53^o$$
$$\doteqdot 17.7 \text{ m}$$

$\therefore$ the pole is 17.7 m high

2

$$\text{PR}^2 = 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^o$$

$$\therefore \quad \text{PR} = \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^o}$$

$$\therefore \quad \text{PR} \doteqdot 207 \text{ m}$$

3

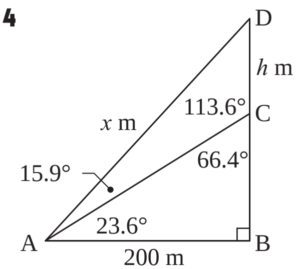
$$\cos T = \frac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340}$$

$$\therefore \quad T = \cos^{-1}\left(\frac{136\,775}{149\,600}\right)$$

$$\therefore \quad T \doteqdot 23.9$$

$\therefore$ was 23.9^o off line.

4



In ΔABD,

$$\cos(23.6 + 15.9)^o = \frac{200}{x}$$

$$\therefore \quad x = \frac{200}{\cos 39.5^o}$$

$$\therefore \quad x \doteqdot 259.2$$

In ΔACD,

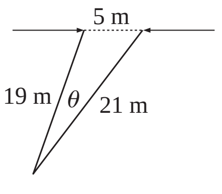
$$\frac{h}{\sin 15.9^o} = \frac{259.2}{\sin 113.6^o}$$

$$\therefore \quad h = \frac{259.2 \times \sin 15.9^o}{\sin 113.6^o}$$

$$\therefore \quad h \doteqdot 77.5$$

$\therefore$ is 77.5 m high.

5



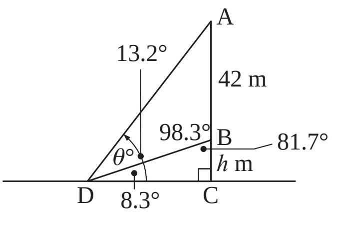
$$\cos\theta = \frac{19^2 + 21^2 - 5^2}{2 \times 19 \times 21}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{777}{798}\right)$$

$$\therefore \quad \theta = 13.2$$

$\therefore$ angle of view is 13.2^o

6



$$\theta = 13.2^o - 8.3^o = 4.9^o$$

In ΔABD,

$$\frac{\text{AD}}{\sin 98.3^o} = \frac{42}{\sin 4.9^o}$$

$$\therefore \quad \text{AD} = \frac{42 \times \sin 98.3^o}{\sin 4.9^o}$$

$$\therefore \quad \text{AD} \doteqdot 486.56 \text{ m}$$

In ΔADC,

$$\sin 13.2^o = \frac{h + 42}{486.56}$$

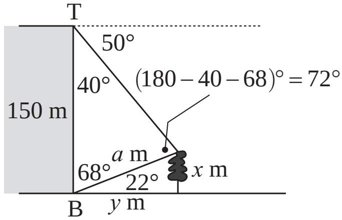
$$\therefore \quad h + 42 = 486.56 \times \sin 13.2^o$$

$$\therefore \quad h + 42 \doteqdot 111.1$$

$$\therefore \quad h \doteqdot 69.1$$

$\therefore$ the hill is 69.1 m high

7



By the sine rule

$$\frac{a}{\sin 40^o} = \frac{150}{\sin 72^o}$$

$$\therefore \quad a = \frac{150 \times \sin 40^o}{\sin 72^o}$$

$$\therefore \quad a \doteqdot 101.38$$

Now $\sin 22^o = \dfrac{x}{101.38}$

$$\therefore \quad x = 101.38 \times \sin 22^o$$

$$\therefore \quad x \doteqdot 38.0$$

and $\cos 22^o = \dfrac{y}{101.38}$

$$\therefore \quad y = 101.38 \times \cos 22^o$$

$$\therefore \quad y \doteqdot 94.0$$

$\therefore$ the tree is 38.0 m high and 94.0 m from the building.

8 Using Pythagoras' theorem

$$\text{RQ} = \sqrt{4^2 + 7^2} = \sqrt{65} \text{ m}$$

$$\text{PQ} = \sqrt{8^2 + 7^2} = \sqrt{113} \text{ cm}$$

$$\text{PR} = \sqrt{8^2 + 4^2} = \sqrt{80} \text{ cm}$$

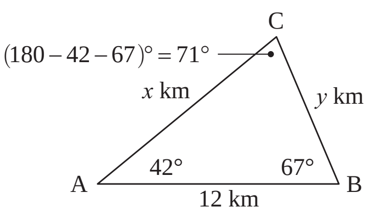
Now $\cos Q = \dfrac{(\sqrt{113})^2 + (\sqrt{65})^2 - (\sqrt{80})^2}{2 \times \sqrt{113} \times \sqrt{65}}$

$$\therefore \quad \cos\theta \doteqdot \left(\tfrac{98}{171.4}\right)$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{98}{171.4}\right)$$

$\therefore \quad \theta \doteqdot 55.1$ So $\measuredangle$PQR measures 55.1^o

9



$$\underbrace{\frac{x}{\sin 67^o} = \frac{12}{\sin 71^o}} = \frac{y}{\sin 42^o}$$

$$\therefore \quad x = \frac{12 \times \sin 67^o}{\sin 71^o} \quad \text{and} \quad y = \frac{12 \times \sin 42^o}{\sin 71^o}$$

$$\therefore \quad x \doteqdot 11.7 \qquad \therefore \quad y \doteqdot 8.49$$

So, C is 11.7 km from A and 8.49 km from B.

10 **a**

$$\text{QS} = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70^o}$$

$$\doteqdot 11.93$$

$$\therefore \quad \text{area} = \tfrac{1}{2} \times 8 \times 12 \times \sin 70^o + \tfrac{1}{2} \times 10 \times 11.93 \times \sin 30^o$$

$$\doteqdot 74.9 \text{ km}^2$$

b

$$1 \text{ ha is } 100 \text{ m} \times 100 \text{ m}$$

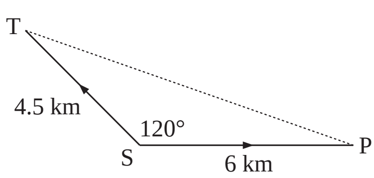
$$= 0.1 \text{ km} \times 0.1 \text{ km}$$

$$= 0.01 \text{ km}^2$$

$$\therefore \quad 1 \text{ km}^2 = 100 \text{ ha}$$

$$\therefore \quad \text{area} \doteqdot 7490 \text{ ha}$$

11



Distance = speed × time

So, after 45 min = 0.75 h

$$\text{ST} = 6 \times 0.75 = 4.5 \text{ km}$$

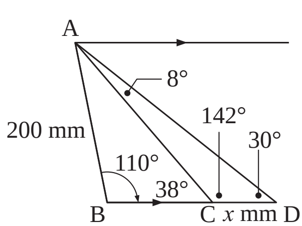
$$\text{SP} = 8 \times 0.75 = 6 \text{ km}$$

Now $\text{PT} = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120^o}$

$$\therefore \quad \text{PT} \doteqdot 9.12$$

So, they are 9.12 km apart.

12



In ΔABC $\dfrac{\text{AC}}{\sin 110^o} = \dfrac{200}{\sin 38^o}$

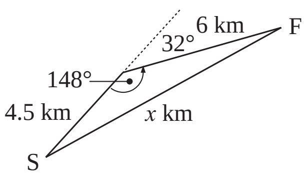
$$\therefore \quad \text{AC} = \frac{200 \times \sin 110^o}{\sin 38^o} \doteqdot 305.26$$

and in ΔACD $\dfrac{x}{\sin 8^o} = \dfrac{305.26}{\sin 30^o}$

$$\therefore \quad x = \frac{305.26 \times \sin 8^o}{\sin 30^o} \doteqdot 84.968$$

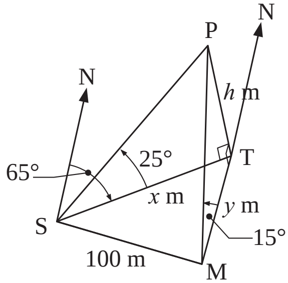
$\therefore$ the metal strip is 85.0 mm wide.

13



$$x = \sqrt{6^2 + (4.5)^2 - 2 \times 6 \times 4.5 \times \cos 148^o}$$
$$\therefore \quad x \doteqdot 10.1$$
$\therefore$ is 10.1 km from the start.

14



In ΔPST, $\tan 25^o = \dfrac{h}{x}$

$$\therefore \quad x = \frac{h}{\tan 25^o} \doteqdot 2.145h$$

In ΔPMT, $\tan 15^o = \dfrac{h}{y}$

$$\therefore \quad y = \frac{h}{\tan 15^o} \doteqdot 3.732h$$

But $\measuredangle$STM $= 65^o$ {equal alternate angles}

and $100^2 = x^2 + y^2 - 2xy\cos 65^o$

$$\therefore \quad 10\,000 = (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2 \cos 65^o$$
$$\therefore \quad 10\,000 \doteqdot 11.763h^2$$
$$\therefore \quad h^2 \doteqdot 850.15$$
$$\therefore \quad h \doteqdot 29.2$$

So, the tree is 29.2 m high.

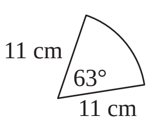
REVIEW SET 12A

1 Area $= \frac{1}{2} \times 7.3 \times 9.4 \times \sin 38^o \doteqdot 21.1$ km^2

2 **a** Area $= \left(\frac{80}{360}\right) \times T \times 13^2 \doteqdot 118$ cm^2

b Area $= \frac{1}{2} \times 11 \times 9 \times \sin 65^o \doteqdot 44.9$ cm^2

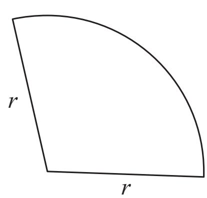
3



a Perimeter $= 2 \times 11 + \left(\frac{63}{360}\right) \times 2\pi \times 11 \doteqdot 34.1$ cm

b Area $= \left(\frac{63}{360}\right) \times \pi \times 11^2 \doteqdot 66.5$ cm^2

4



Perimeter $= 2r + \left(\frac{120}{360}\right) \times 2\pi r$

$$\therefore \quad 36 = 2r + \tfrac{1}{3} \times 2\pi r$$
$$\therefore \quad 36 = r\left(2 + \tfrac{2\pi}{3}\right)$$
$$\therefore \quad r = \frac{36}{2 + \frac{2\pi}{3}} \text{ cm}$$
$$\therefore \quad r \doteqdot 8.7925.....$$
$$\therefore \quad r \doteqdot 8.79 \text{ cm}$$

Area $= \left(\frac{120}{360}\right) \times \pi r^2 = \frac{1}{3} \times \pi \times (8.7925)^2 \doteqdot 81.0$ cm^2

5 Area $= 42$ cm^2 $\quad \therefore \quad \frac{1}{2} \times 7 \times 13 \times \sin x = 42$

$$\therefore \quad \sin x = \frac{42 \times 2}{7 \times 13}$$
$$\therefore \quad x = \sin^{-1}\left(\tfrac{84}{91}\right)$$
$$\therefore \quad x \doteqdot 67.4 \quad \text{or} \quad 180 - 67.4$$
$$\therefore \quad x = 67.4 \quad \text{or} \quad 112.6$$

{assuming the figure is not drawn accurately}

6 Area $= 80$ cm^2

$$\therefore \quad \tfrac{1}{2} \times 11.3 \times 19.2 \sin x^o = 80$$
$$\therefore \quad \sin x^o = \frac{160}{11.3 \times 19.2}$$
$$\therefore \quad x = \sin^{-1}\left(\frac{160}{11.3 \times 19.2}\right)$$
$$\therefore \quad x \doteqdot 47.5 \quad \text{or} \quad 180 - 47.5$$
$$\therefore \quad x \doteqdot 47.5 \quad \text{or} \quad 132.5$$

When $x \doteq 47.5$, $\text{AC} \doteq \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 47.5^o} \doteq 14.3$ cm

When $x \doteq 132.5$, $\text{AC} \doteq \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 132.5^o} \doteq 28.1$ cm

7 Non shaded area

$= \text{area of } \Delta + \text{area of sector}$

$= \frac{1}{2} \times 7 \times 7 \times \sin 130^o + \left(\frac{230}{360}\right) \times \pi \times 7^2$

$\doteq 117.12 \text{ cm}^2$

$\therefore$ shaded area

$\doteq \pi \times 7^2 - 117.12 \text{ cm}^2$

$\doteq 36.8 \text{ cm}^2$

8 **a** $\text{BD} = \sqrt{125^2 + 120^2 - 2 \times 125 \times 120 \times \cos 75^o} \doteq 149.2$

$\therefore$ area $= \frac{1}{2} \times 120 \times 125 \sin 75^o + \frac{1}{2} \times 149.2 \times 90 \times \sin 30^o$

$\doteq 10\,600 \text{ m}^2$

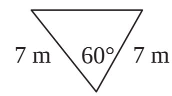
b $\doteq 1.06$ ha $(10\,000 \text{ m}^2 = 1 \text{ ha})$

9 Shaded area $=$ area of circle $-$ area of hexagon

$= \pi \times 7^2 - 6 \times$ area of one $\triangledown$

$= 49\pi - 6 \times \frac{1}{2} \times 7 \times 7 \times \sin 60^o$

$\doteq 26.6 \text{ m}^2$



REVIEW SET 12B

1 **a** $\cos x = \dfrac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19}$

$\therefore \cos x = \frac{409}{494}$

$\therefore x = \cos^{-1}\left(\frac{409}{494}\right)$

$\therefore x \doteq 34.1$

b $x = \sqrt{15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72^o}$

$\therefore x \doteq 18.9$

2 **a** $\cos x = \dfrac{11^2 + 19^2 - 13^2}{2 \times 11 \times 19}$

$\therefore \cos x = \frac{313}{418}$

$\therefore x = \cos^{-1}\left(\frac{313}{418}\right)$

$\therefore x \doteq 41.5$

b $x = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^o}$

$\therefore x \doteq 15.4$

3 $\text{AC} = \sqrt{11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74^o}$

$\therefore \text{AC} \doteq 12.554.....$

$\therefore \text{AC} \doteq 12.6$ cm

Now $\dfrac{\sin C}{11} = \dfrac{\sin 74^o}{12.554}$

$\therefore \sin C = \dfrac{11 \times \sin 74^o}{12.554}$

$\therefore C = \sin^{-1}\left(\dfrac{11 \times \sin 74^o}{12.554}\right)$

$\therefore C = 57.4^o$ or 122.6^o

$\uparrow$

impossible as $122.6 + 74 > 180$

$\therefore \angle$C measures 57.4^o

$\therefore \angle$A measures 48.6^o

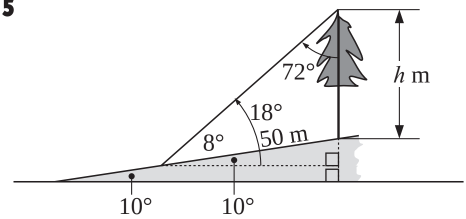
4 $\text{DB} = \sqrt{7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110^o}$

$\doteq 14.922$

$\therefore$ total area $= \frac{1}{2} \times 7 \times 11 \times \sin 110^o + \frac{1}{2} \times 16 \times 14.922 \times \sin 40^o$

$\doteq 113 \text{ cm}^2$

5



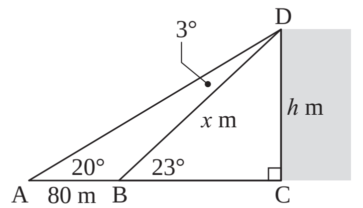
$$\frac{h}{\sin 8^o} = \frac{50}{\sin 72^o}$$

$$\therefore \quad h = \frac{50 \times \sin 8^o}{\sin 72^o}$$

$$\therefore \quad h \doteqdot 7.32$$

i.e., the tree is 7.32 m high

6



In ΔABD, $\dfrac{x}{\sin 20^o} = \dfrac{80}{\sin 3^o}$

$$\therefore \quad x = \frac{80 \times \sin 20^o}{\sin 3^o}$$

$$\doteqdot 522.8$$

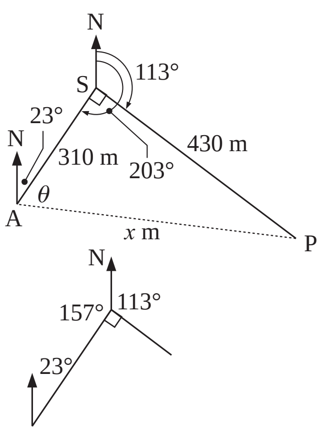
Now $\sin 23^o = \dfrac{h}{x}$

$$\therefore \quad h = 522.8 \times \sin 23^o$$

$$\therefore \quad h \doteqdot 204$$

So the building is 204 m tall.

7



$$\measuredangle ASP = 203^o - 113^o = 90^o$$

Now $x^2 = 310^2 + 430^2$ {Pythagoras}

$$\therefore \quad x = \sqrt{310^2 + 430^2}$$

$$\therefore \quad x \doteqdot 530$$

$\therefore$ they are 530 m apart.

and $\tan\theta = \frac{430}{310}$

$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{430}{310}\right) \doteqdot 54.2$$

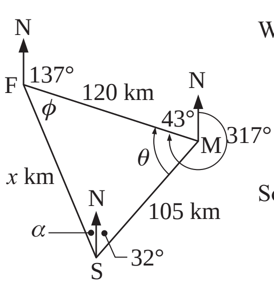
and $23 + \theta = 77.2$

$\therefore$ bearing of P from A is 077.2^o

8

In 45 minutes, $140 \times \frac{3}{4} = 105$ km is travelled.

In 40 minutes, $180 \times \frac{2}{3} = 120$ km is travelled.



We notice that $\theta + 43 + 32 = 180$ {allied angles add to 180^o}

$$\therefore \quad \theta = 105$$

and so, $x = \sqrt{120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^o}$

$$\therefore \quad x \doteqdot 178.74$$

So, is 179 km from the start.

Now $\dfrac{\sin\phi}{105} = \dfrac{\sin 105^o}{178.74}$

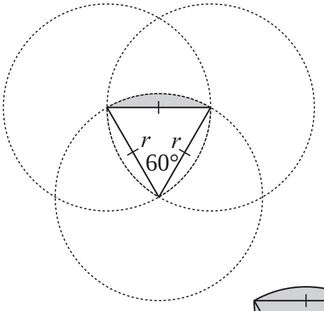
$$\therefore \quad \sin\phi = \frac{105 \times \sin 105^o}{178.74}$$

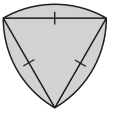
$$\therefore \quad \phi \doteqdot 34.6$$

$$\therefore \quad \theta = 180 - 105 - 34.6 - 32 = 8.4 \doteqdot 8$$

So, the bearing is 352^o.

9





shaded area of sector

$= \text{area of sector} - \text{area of } \Delta$

$= \frac{1}{6}\pi r^2 - \frac{1}{2} \times r \times r \times \sin 60^o$

$= \frac{\pi}{6} r^2 - \frac{1}{2} r^2 \left(\frac{\sqrt{3}}{2}\right)$

$\therefore$ shaded area of figure

$= 3\left[\frac{\pi}{6} r^2 - \frac{\sqrt{3}}{4} r^2\right] + \frac{1}{2} r^2 \left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{2} r^2 - \frac{3\sqrt{3}}{4} r^2 + \frac{\sqrt{3}}{4} r^2$

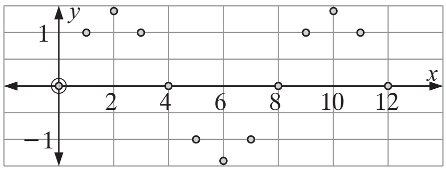
$= \frac{\pi}{2} r^2 - \frac{1}{2}\sqrt{3} r^2$

$= \dfrac{r^2}{2}(\pi - \sqrt{3})$

Chapter 13
PERIODIC PHENOMENA

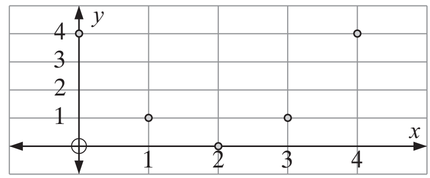
EXERCISE 13A

1 **a**



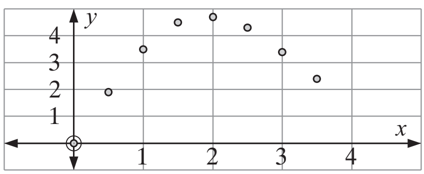
Data exhibits periodic behaviour.

b



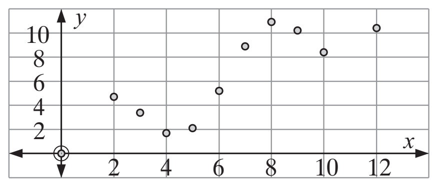
Not enough information to say data is periodic. It may in fact be quadratic.

c



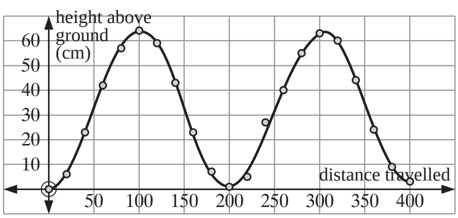
Not enough information to say data is periodic. It may in fact be quadratic.

d



Not enough information to say data is periodic.

2 **a**



b The data is periodic.

i The minimum value from the table is 0 and the maximum value is 64.

So, the principle axis is $y \doteqdot \frac{0+64}{2}$, i.e., $y \doteqdot 32$.

ii The maximum value is 64 cm (approx).

iii The period is $\doteqdot$ 200 cm.

iv The amplitude is $\doteqdot$ 32 cm.

c A curve can be fitted to the data as the distance travelled is continuous.

3 **a** periodic **b** periodic **c** periodic **d** not periodic **e** periodic **f** periodic

EXERCISE 13B.1

1 **a** $\pi \equiv 180^o$ $\therefore\ \frac{\pi}{4} \equiv 45^o$

b $-\frac{\pi}{4} \equiv -45^o$

c $\pi \equiv 180^o$ $\therefore\ \frac{3\pi}{2} \equiv \frac{3}{2} \times 180^o \equiv 270^o$

d $-\frac{3\pi}{2} \equiv -270^o$

e $\pi \equiv 180^o$ $\therefore\ \frac{\pi}{3} \equiv 60^o$

f $\pi \equiv 180^o$ $\therefore\ \frac{\pi}{6} \equiv 30^o$

2 **a** 30^o is $\frac{1}{12}$ of 360^o $\therefore\ d = \frac{1}{12} \times 2\pi$ $\therefore\ d = \frac{\pi}{6}$

b 60^o is $\frac{1}{6}$ of 360^o $\therefore\ d = \frac{1}{6} \times 2\pi$ $\therefore\ d = \frac{\pi}{3}$

c 150^o is $\frac{15}{36}$ of 360^o $\therefore\ d = \frac{15}{36} \times 2\pi$ $\therefore\ d = \frac{5\pi}{6}$

d 135^o is $\frac{135}{360}$ of 360^o $\therefore\ d = \frac{135}{360} \times 2\pi$ $\therefore\ d = \frac{3\pi}{4}$

e 240^o is $\frac{2}{3}$ of 360^o $\therefore\ d = \frac{2}{3} \times 2\pi$ $\therefore\ d = \frac{4\pi}{3}$

f 45^o is $\frac{1}{8}$ of 360^o $\therefore\ d = -\frac{1}{8} \times 2\pi$ $\therefore\ d = -\frac{\pi}{4}$

g From **d**, $d = -\frac{3\pi}{4}$

h 270^o is $\frac{27}{36}$ of 360^o
$\therefore \; d = -\frac{3}{4} \times 2\pi$
$\therefore \; d = -\frac{3\pi}{2}$

3 **a** ΔOAP is isosceles, so $\measuredangle AOP = 60^o$ $\therefore$ $\theta = 60$.

b 60^o is $\frac{1}{6}$ of 360^o, $\therefore$ $d = \frac{1}{6} \times 2\pi$ i.e., $d = \frac{\pi}{3}$.

c As AP $<$ arc AP, then AP < 1 $\therefore$ $\theta < 60^o$, i.e., θ decreases.

d When $d = \pi$, $\theta = 180^o$
So, when $d = 1$, $\theta = \dfrac{180^o}{\pi} \doteqdot 57.3$

EXERCISE 13B.2

1 **a** $180^o = \pi$ radians
$\therefore \; 90^o = \frac{\pi}{2}$ radians

b $180^o = \pi$ radians
$\therefore \; 60^o = \frac{\pi}{3}$ radians

c $180^o = \pi$ radians
$\therefore \; 30^o = \frac{\pi}{6}$ radians

d $180^o = \pi$ radians
$\therefore \; 18^o = \frac{\pi}{10}$ radians

e $180^o = \pi$ radians
$\therefore \; 9^o = \frac{\pi}{20}$ radians

f $180^o = \pi$ radians
$\therefore \; 45^o = \frac{\pi}{4}$ radians
$\therefore \; 135^o = \frac{3\pi}{4}$ radians

g $180^o = \pi$ radians
$\therefore \; 45^o = \frac{\pi}{4}$ radians
$\therefore \; 225^o = \frac{5\pi}{4}$ radians

h $180^o = \pi$ radians
$\therefore \; 90^o = \frac{\pi}{2}$ radians
$\therefore \; 270^o = \frac{3\pi}{2}$ radians

i $360^o = 2 \times 180^o$
$= 2\pi$ radians

j $720^o = 4 \times 180^o$
$= 4\pi$ radians

k $180^o = \pi$ radians
$\therefore \; 45^o = \frac{\pi}{4}$ radians
$\therefore \; 315^o = \frac{7\pi}{4}$ radians

l $180^o = \pi$ radians
$\therefore \; 540^o = 3\pi$ radians

m $180^o = \pi$ radians
$\therefore \; 36^o = \frac{\pi}{5}$ radians

n $180^o = \pi$ radians
$\therefore \; 10^o = \frac{\pi}{18}$ radians
$\therefore \; 80^o = \frac{8\pi}{18}$ radians
$= \frac{4\pi}{9}$ radians

o $180^o = \pi$ radians
$\therefore \; 10^o = \frac{\pi}{18}$ radians
$\therefore \; 230^o = \frac{23\pi}{18}$ radians

2 **a** 36.7^o
$= 36.7 \times \frac{\pi}{180}$ radians
$\doteqdot 0.641$ radians

b 137.2^o
$= 137.2 \times \frac{\pi}{180}$ radians
$\doteqdot 2.39$ radians

c 317.9^o
$= 317.9 \times \frac{\pi}{180}$ radians
$\doteqdot 5.55$ radians

d 219.6^o
$= 219.6 \times \frac{\pi}{180}$ radians
$\doteqdot 3.83$ radians

e 396.7^o
$= 396.7 \times \frac{\pi}{180}$ radians
$\doteqdot 6.92$ radians

3 **a** $\frac{\pi}{5}$
$= \frac{180^o}{5}$
$= 36^o$

b $\frac{3\pi}{5}$
$= \frac{3 \times 180^o}{5}$
$= 108^o$

c $\frac{3\pi}{4}$
$= \frac{3 \times 180^o}{4}$
$= 135^o$

d $\frac{\pi}{18}$
$= \frac{180^o}{18}$
$= 10^o$

e $\frac{\pi}{9}$
$= \frac{180^o}{9}$
$= 20^o$

f $\frac{7\pi}{9}$
$= \frac{7 \times 180^o}{9}$
$= 140^o$

g $\frac{\pi}{10}$
$= \frac{180^o}{10}$
$= 18^o$

h $\frac{3\pi}{20}$
$= \frac{3 \times 180^o}{20}$
$= 27^o$

i $\frac{5\pi}{6}$
$= \frac{5 \times 180^o}{6}$
$= 150^o$

j $\frac{\pi}{8}$
$= \frac{180^o}{8}$
$= 22\frac{1}{2}^o$

4 **a** 2^c
$= 2 \times \frac{180}{\pi}$ degrees
$\doteqdot 114.59^o$

b 1.53^c
$= 1.53 \times \frac{180}{\pi}$ degrees
$\doteqdot 87.66^o$

c 0.867^c
$= 0.867 \times \frac{180}{\pi}$ degrees
$\doteqdot 49.68^o$

d 3.179^c
$= 3.179 \times \frac{180}{\pi}$ degrees
$\doteqdot 182.14^o$

e 5.267^c
$= 5.267 \times \frac{180}{\pi}$ degrees
$\doteqdot 301.78^o$

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

6 **a** **i** arc length, AB

$$l = \frac{\theta^c}{2\pi^c} \times 2\pi r$$
$$= r\theta$$

i.e., $l = r\theta$

$\{\frac{\theta^c}{2\pi^c}$ is the fraction of the whole circle$\}$

ii sector area

$$A = \frac{\theta^c}{2\pi^c} \times \pi r^2$$
$$= \tfrac{1}{2}r^2\theta$$

i.e., $A = \frac{1}{2}r^2\theta$

b **i** $l = r\theta$
$= 12 \text{ cm} \times 0.8$
$= 9.6 \text{ cm}$

ii $l = r\theta$
$= 8 \text{ cm} \times 1.75$
$= 14 \text{ cm}$

iii $l = r\theta$
$= 11 \text{ cm} \times (2\pi - 2)$
$\doteqdot 47.1 \text{ cm}$

c **i** $l = r\theta$
$\therefore 6 = 8\theta$
$\therefore \theta = 0.75^c$

ii $l = r\theta$
$\therefore 8.4 = 5\theta$
$\therefore \theta = \frac{8.4}{5}$
$\therefore \theta = 1.68^c$

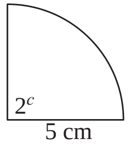
iii $l = r\theta$
$\therefore 31.7 = 8(2\pi - \theta)$
$\therefore 2\pi - \theta = \frac{31.7}{8}$
$\therefore \theta = 2\pi - \frac{31.7}{8}$
$\therefore \theta \doteqdot 2.32^c$

d **i** $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 5^2 \times 0.7$
$= 8.75 \text{ cm}^2$

ii $A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$
$= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \times 12^2(1.5 - \sin 1.5)$
$\doteqdot 36.2 \text{ cm}^2$

iii $A = \frac{1}{2}(\text{OA})(\text{OP})\sin(0.66) - \frac{1}{2}(\text{OA})^2 \times (0.66)$
$= \frac{1}{2}(12)(30)\sin(0.66) - \frac{1}{2}(12)^2 \times (0.66)$
$\doteqdot 62.8 \text{ cm}^2$

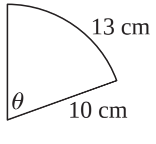
e



arc length
$= r\theta$
$= 5 \times 2^c$
$= 10 \text{ cm}$

area
$= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 5^2 \times 2$
$= 25 \text{ cm}^2$

f

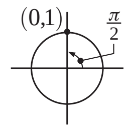


$l = r\theta$
$\therefore 13 = 10\theta$
$\therefore \theta = 1.3$

and $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 10^2 \times 1.3$
$= 65 \text{ cm}^2$

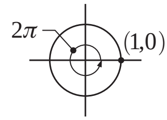
EXERCISE 13C.1

1 **a**



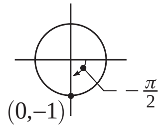
$\cos(\frac{\pi}{2}) = 0$
$\sin(\frac{\pi}{2}) = 1$

b



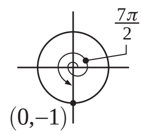
$\cos 2\pi = 1$
$\sin 2\pi = 0$

c



$\cos(-\frac{\pi}{2}) = 0$
$\sin(-\frac{\pi}{2}) = -1$

d



$\cos(\frac{7\pi}{2}) = 0$
$\sin(\frac{7\pi}{2}) = -1$

2 **a**
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{1}{4} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{3}{4} \\ \therefore\ \cos\theta &= \pm\tfrac{\sqrt{3}}{2} \end{aligned}$$

b
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{1}{9} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{8}{9} \\ \therefore\ \cos\theta &= \pm\tfrac{\sqrt{8}}{3} \\ &= \pm\tfrac{2\sqrt{2}}{3} \end{aligned}$$

c
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + 0 &= 1 \\ \therefore\ \cos\theta &= \pm 1 \end{aligned}$$

d
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + 1 &= 1 \\ \therefore\ \cos\theta &= 0 \end{aligned}$$

3 **a**
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \tfrac{16}{25} + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= \tfrac{9}{25} \\ \therefore\ \sin\theta &= \pm\tfrac{3}{5} \end{aligned}$$

b
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \tfrac{9}{16} + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= \tfrac{7}{16} \\ \therefore\ \sin\theta &= \pm\tfrac{\sqrt{7}}{4} \end{aligned}$$

c
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ 1 + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= 0 \\ \therefore\ \sin\theta &= 0 \end{aligned}$$

d
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ 0 + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= 1 \\ \therefore\ \sin\theta &= \pm 1 \end{aligned}$$

4 **a**

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve
2	$90 < \theta < 180$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve
3	$180 < \theta < 270$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve
4	$270 < \theta < 360$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve

b **i** 1 and 4
ii 2 and 3
iii 3
iv 2

5 **a**
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \tfrac{4}{9} + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= \tfrac{5}{9} \\ \therefore\ \sin\theta &= \pm\tfrac{\sqrt{5}}{3} \end{aligned}$$
But θ is in quadrant 1 where $\sin\theta > 0$
$\therefore\ \sin\theta = \frac{\sqrt{5}}{3}$

b
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{4}{25} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{21}{25} \\ \therefore\ \cos\theta &= \pm\tfrac{\sqrt{21}}{5} \end{aligned}$$
But θ is in quadrant 2 where $\cos\theta < 1$
$\therefore\ \cos\theta = -\frac{\sqrt{21}}{5}$

c
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{9}{25} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{16}{25} \\ \therefore\ \cos\theta &= \pm\tfrac{4}{5} \end{aligned}$$
But θ is in quadrant 4 where $\cos\theta > 0$
$\therefore\ \cos\theta = \frac{4}{5}$

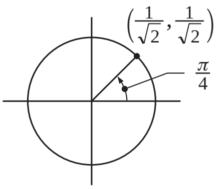
d
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \tfrac{25}{169} + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= \tfrac{144}{169} \\ \therefore\ \sin\theta &= \pm\tfrac{12}{13} \end{aligned}$$
But θ is in quadrant 3 where $\sin\theta < 0$
$\therefore\ \sin\theta = -\frac{12}{13}$

e
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \tfrac{1}{4} + \sin^2\theta &= 1 \\ \therefore\ \sin^2\theta &= \tfrac{3}{4} \\ \therefore\ \sin\theta &= \pm\tfrac{\sqrt{3}}{2} \end{aligned}$$
But θ is in quadrant 4 where $\sin\theta < 0$
$\therefore\ \sin\theta = -\frac{\sqrt{3}}{2}$

f
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{1}{2} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{1}{2} \\ \therefore\ \cos\theta &= \pm\tfrac{1}{\sqrt{2}} \end{aligned}$$
But θ is in quadrant 3 where $\cos\theta < 0$
$\therefore\ \cos\theta = -\frac{1}{\sqrt{2}}$

EXERCISE 13C.2

1



So $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

Draw separate unit circle diagrams for each case.

	a	b	c	d	e
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$

2 **Likewise, draw separate unit circle diagrams for each angle.**

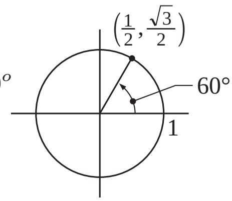
	a	b	c	d	e
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

3 **a** $\sin^2 60^o$

$= \sin 60^o \times \sin 60^o$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$

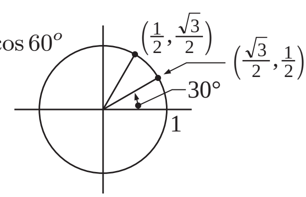
$= \frac{3}{4}$



b $\sin 30^o \cos 60^o$

$= \frac{1}{2} \times \frac{1}{2}$

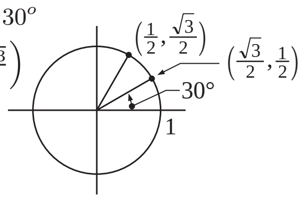
$= \frac{1}{4}$



c $4\sin 60^o \cos 30^o$

$= 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= 3$

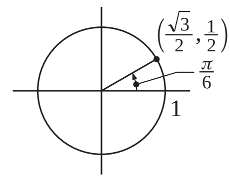


d $1 - \cos^2(\frac{\pi}{6})$

$= 1 - \left(\frac{\sqrt{3}}{2}\right)^2$

$= 1 - \frac{3}{4}$

$= \frac{1}{4}$

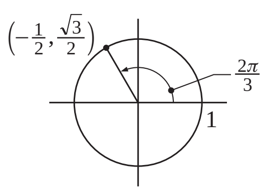


e $\sin^2(\frac{2\pi}{3}) - 1$

$= \left(\frac{\sqrt{3}}{2}\right)^2 - 1$

$= \frac{3}{4} - 1$

$= -\frac{1}{4}$

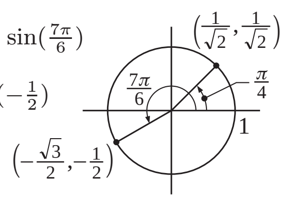


f $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$

$= \left(\frac{1}{\sqrt{2}}\right)^2 - (-\frac{1}{2})$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$



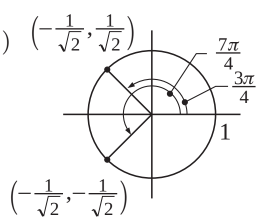
g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$

$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$



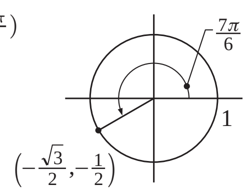
h $1 - 2\sin^2(\frac{7\pi}{6})$

$= 1 - 2(-\frac{1}{2})^2$

$= 1 - 2 \times \frac{1}{4}$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

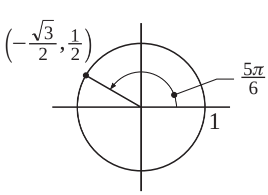


i $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

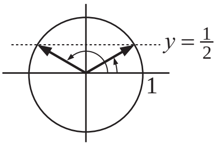
$= \left(-\frac{\sqrt{3}}{2}\right)^2 - (\frac{1}{2})^2$

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{1}{2}$

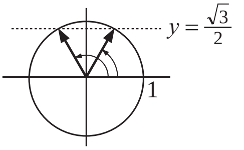


4 **a**



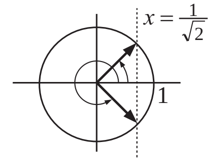
$\theta = 30^o$ or 150^o

b



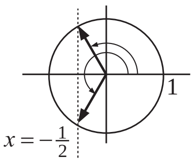
$\theta = 60^o$ or 120^o

c



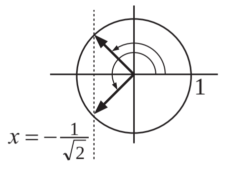
$\theta = 45^o$ or 315^o

d



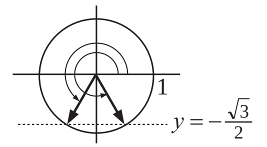
$\theta = 120^o$ or 240^o

e



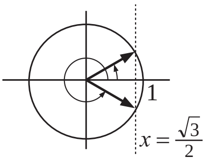
$\theta = 135^o$ or 225^o

f



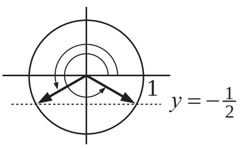
$\theta = 240^o$ or 300^o

5 **a**



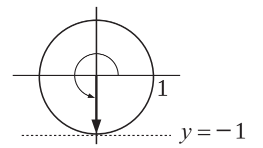
$\theta = 30^o, 330^o, 390^o, 690^o$

b



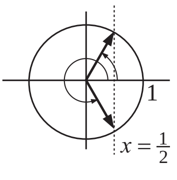
$\theta = 210^o, 330^o, 570^o, 690^o$

c



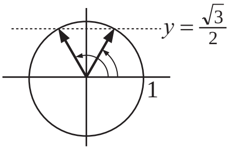
$\theta = 270^o, 630^o$

6 **a** $\cos\theta = \frac{1}{2}$



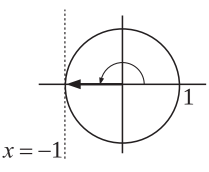
$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

b $\sin\theta = \frac{\sqrt{3}}{2}$



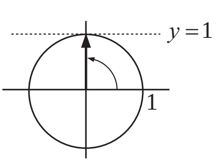
$\therefore \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

c $\cos\theta = -1$



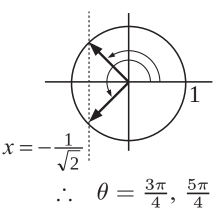
$\therefore \quad \theta = \pi$

d $\sin\theta = 1$



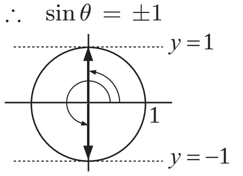
$\therefore \quad \theta = \frac{\pi}{2}$

e $\cos\theta = -\frac{1}{\sqrt{2}}$



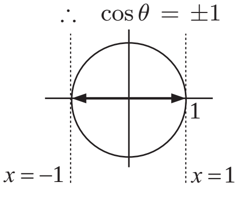
$\therefore \quad \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$

f
$$\sin^2\theta = 1$$
$$\therefore \quad \sin\theta = \pm 1$$



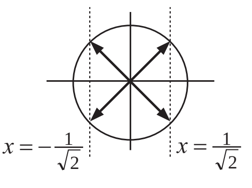
$\therefore \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

g
$$\cos^2\theta = 1$$
$$\therefore \quad \cos\theta = \pm 1$$



$\therefore \quad \theta = 0, \pi, 2\pi$

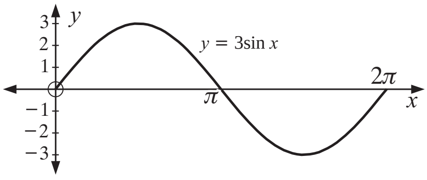
h $\cos^2\theta = \frac{1}{2} \quad \therefore \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$



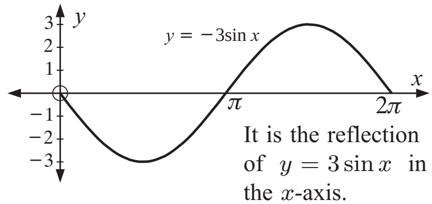
$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

EXERCISE 13D.1

1 **a** $y = 3\sin x$
has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$
When $x = 0$, $y = 0$.

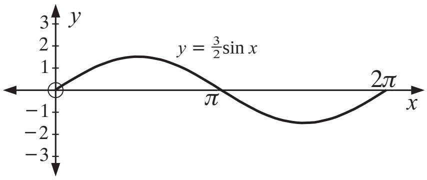


b $y = -3\sin x$
has amplitude $|-3| = 3$ and period $\frac{2\pi}{1}$
$= 2\pi$.
When $x = 0$, $y = 0$.

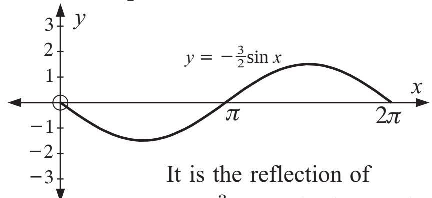


It is the reflection of $y = 3\sin x$ in the x-axis.

c $y = \frac{3}{2}\sin x$
has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.
When $x = 0$, $y = 0$.

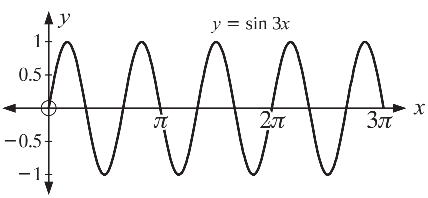


d $y = -\frac{3}{2}\sin x$
has amplitude $\left|-\frac{3}{2}\right| = \frac{3}{2}$ and
period $\frac{2\pi}{1} = 2\pi$.

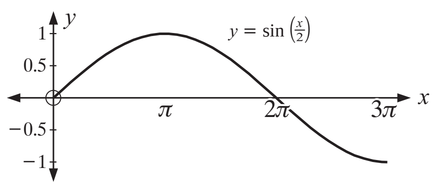


It is the reflection of $y = \frac{3}{2}\sin x$ in the x-axis.

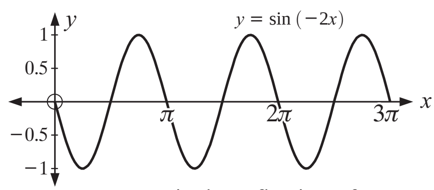
2 **a** $y = \sin 3x$
has amplitude 1 and period $\frac{2\pi}{3}$.
When $x = 0$, $y = 0$.



b $y = \sin\left(\frac{x}{2}\right)$
has amplitude 1 and period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.
When $x = 0$, $y = 0$.



c $y = \sin(-2x)$
has amplitude 1 and period $\dfrac{2\pi}{|2|} = \pi$.
When $x = 0$, $y = 0$.



It is the reflection of $y = \sin 2x$ in the x-axis.

3 **a** period $= \dfrac{2\pi}{|4|}$
$= \frac{\pi}{2}$

b period $= \dfrac{2\pi}{|-4|}$
$= \frac{\pi}{2}$

c period $= \dfrac{2\pi}{\left|\frac{1}{3}\right|}$
$= 6\pi$

d period $= \dfrac{2\pi}{0.6}$
$= \dfrac{20\pi}{6}$
$= \dfrac{10\pi}{3}$

4 **a** $\dfrac{2\pi}{B} = 5\pi$
$\therefore\ B = \frac{2}{5}$

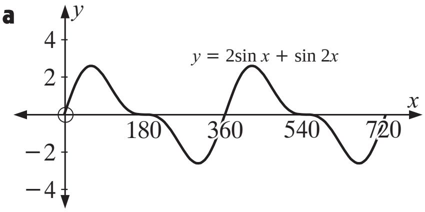
b $\dfrac{2\pi}{B} = \dfrac{2\pi}{3}$
$\therefore\ B = 3$

c $\dfrac{2\pi}{B} = 12\pi$
$\therefore\ B = \frac{1}{6}$

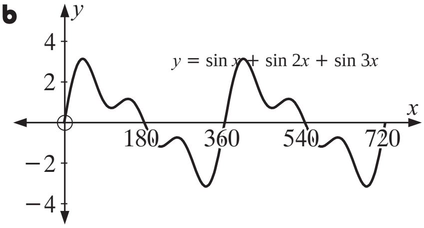
d $\dfrac{2\pi}{B} = 4$
$\therefore\ B = \frac{\pi}{2}$

e $\dfrac{2\pi}{B} = 100$
$\therefore\ B = \dfrac{2\pi}{100} = \dfrac{\pi}{50}$

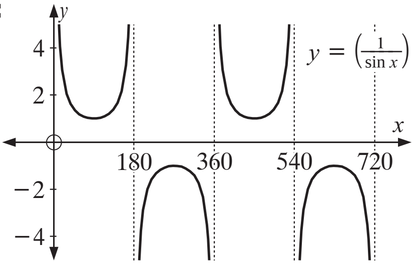
5 **a**



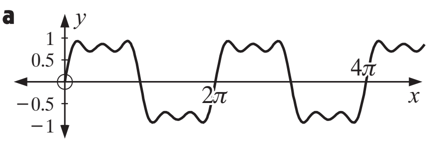
b



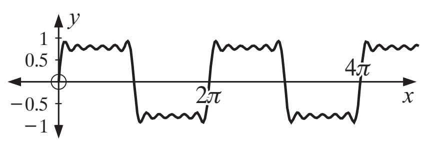
c



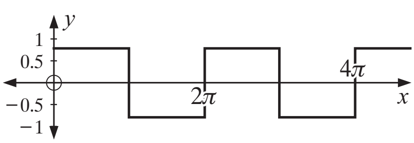
6 **a**



b

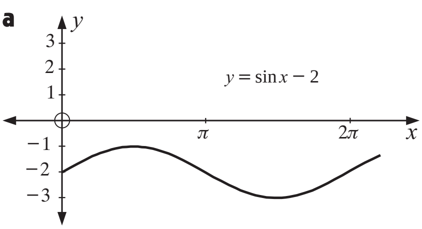


Prediction:



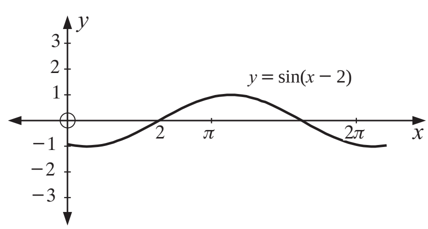
EXERCISE 13D.2

1 **a**



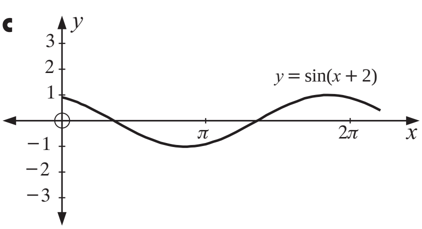
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

b



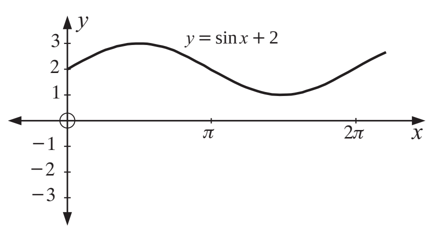
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

c



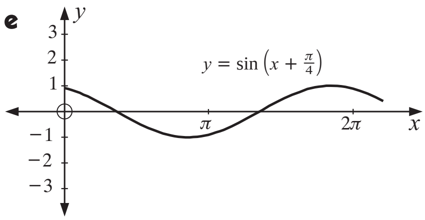
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

d



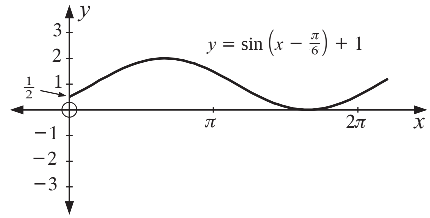
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

e



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -\frac{\pi}{4} \\ 0 \end{bmatrix}$.

f



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

3 **a** period $= \dfrac{2\pi}{|5|} = \frac{2\pi}{5}$ **b** period $= \dfrac{2\pi}{\left|\frac{1}{4}\right|} = 8\pi$ **c** period $= \dfrac{2\pi}{|-2|} = \pi$

4 **a** $\dfrac{2\pi}{B} = 3\pi$ $\therefore\ B = \frac{2}{3}$ **b** $\dfrac{2\pi}{B} = \dfrac{\pi}{10}$ $\therefore\ B = 20$ **c** $\dfrac{2\pi}{B} = 100\pi$ $\therefore\ B = \frac{2}{100} = \frac{1}{50}$ **d** $\dfrac{2\pi}{B} = 50$ $\therefore\ B = \frac{2\pi}{50} = \frac{\pi}{25}$

5 **a** A translation of $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, i.e., vertically down 1 unit.

b A translation of $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$, i.e., horizontally $\frac{\pi}{4}$ units right.

c A vertical dilation of factor 2. **d** A horizontal dilation of factor $\frac{1}{4}$.

e A vertical dilation of factor $\frac{1}{2}$. **f** A horizontal dilation of factor $\dfrac{1}{\frac{1}{4}} = 4$.

g A reflection in the x-axis. **h** A translation of $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

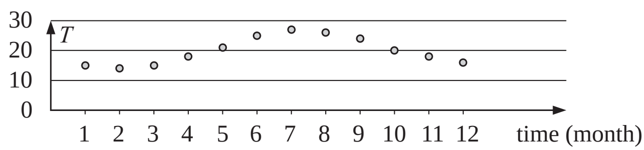
i A vertical dilation of factor 2 followed by a horizontal dilation of factor $\frac{1}{3}$.

j A translation of $\begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$.

EXERCISE 13E

1 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\dfrac{2\pi}{B} = 12$ $\therefore\ B = \frac{\pi}{6}$ $\{$assuming $B > 0\}$.

Amplitude, $A \doteqdot \dfrac{\text{max.} - \text{min.}}{2} \doteqdot \dfrac{27 - 14}{2} \doteqdot 6.5$

As the principal axis is midway between min. and max., then $D \doteqdot \dfrac{27 + 14}{2} \doteqdot 20.5$

When T is 20.5 (midway between min. and max.)

$$C \doteqdot \frac{2 + 7}{2} \doteqdot 4.5 \quad \{\text{average of } t \text{ values}\}$$

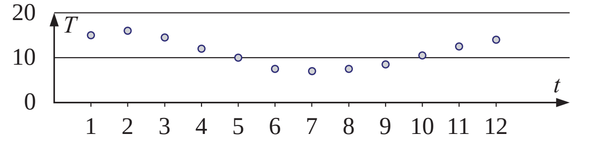
$$\therefore\ T \doteqdot 6.5 \sin \tfrac{\pi}{6}(t - 4.5) + 20.5 \quad (\textbf{Note:}\ \tfrac{\pi}{6} \doteqdot 0.524)$$

b Using technology, $T \doteqdot 6.14 \sin(0.575t - 2.70) + 20.4$

i.e., $T \doteqdot 6.14 \sin 0.575(t - 4.70) + 20.4$

2 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\dfrac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $\quad A \doteq \dfrac{\text{max.} - \text{min.}}{2} \doteq \dfrac{16-7}{2} \doteq 4.5$

As the principal axis is midway between min. and max. then $\quad D \doteq \dfrac{16+7}{2} \doteq 11.5$

At min., $t = 7$ and at max., $t = 2 + 12 = 14 \quad \therefore \quad C = \dfrac{7+14}{2} = 10.5$

So, $\quad T \doteq 4.5 \sin \frac{\pi}{6}(t - 10.5) + 11.5$

b Using technology, $\quad T \doteq 4.29 \sin(0.533t + 0.769) + 11.2$

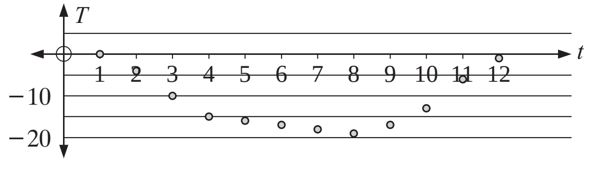
i.e., $\quad T \doteq 4.29 \sin 0.533(t + 1.44) + 11.2$

Note: (1) $\frac{\pi}{6} \doteq 0.524$ ✓

(2) $\frac{\pi}{6}(1.44 - (-10.5)) \doteq 6.25 \doteq 2\pi$

3

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	-4	-10	-15	-16	-17	-18	-19	-17	-13	-6	-1



The period is $\dfrac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $\quad A \doteq \dfrac{\text{max.} - \text{min.}}{2} \doteq \dfrac{0-(-19)}{2} \doteq 9.5$

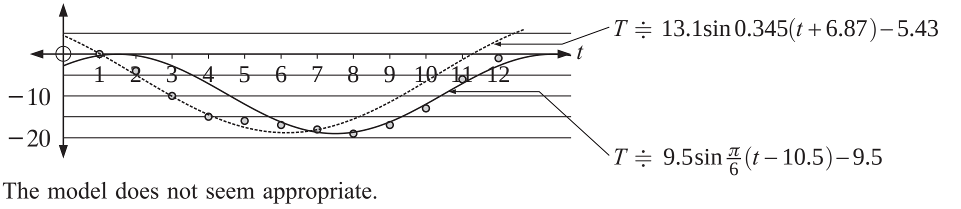
$D \doteq \dfrac{\text{max.} + \text{min.}}{2} \doteq \dfrac{0+(-19)}{2} \doteq -9.5$

At min., $t = 8$ and at max., $t = 1 + 12 = 13 \quad \therefore \quad C = \dfrac{8+13}{2} = 10.5$

So, $\quad T \doteq 9.5 \sin \frac{\pi}{6}(t - 10.5) - 9.5$ ...... (1)

From technology, $\quad T \doteq 13.1 \sin(0.345t + 2.37) - 5.43$

i.e., $\quad T \doteq 13.1 \sin 0.345(t + 6.87) - 5.43$ ...... (2)



The model does not seem appropriate.

5 **a** For the model $H = A\sin B(t-C)+D$

$$\text{period} = \frac{2\pi}{B} = 2\times 12.4 \text{ hours} \quad \therefore \quad B = \frac{2\pi}{24.8} \doteqdot 0.253$$

We let the principal axis be 0, i.e., $D=0$

$\therefore$ A, the amplitude $=7$, i.e., min. is -7, max. is $+7$

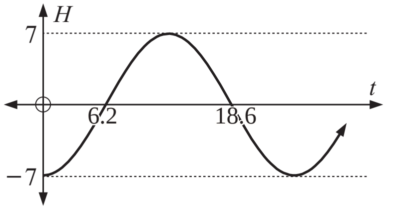
Let $t=0$ correspond to 'low tide' $\therefore$ $t=12.4$ corresponds to 'high tide'

$$\therefore \quad C = \frac{0+12.4}{2} = 6.2$$

So, $H \doteqdot 7\sin 0.253(t-6.2)+0$

i.e., $H \doteqdot 7\sin 0.253(t-6.2)$

b



6 Let the model be $H = A\sin B(t-C)+D$ metres

When $t=0$, $H=2$ and when $t=50$, $H=22$

$\uparrow$ min. $\quad\quad$ $\uparrow$ max.

$$\text{period} = \frac{2\pi}{B} = 100 \quad \therefore \quad B = \frac{2\pi}{100} = \frac{\pi}{50}$$

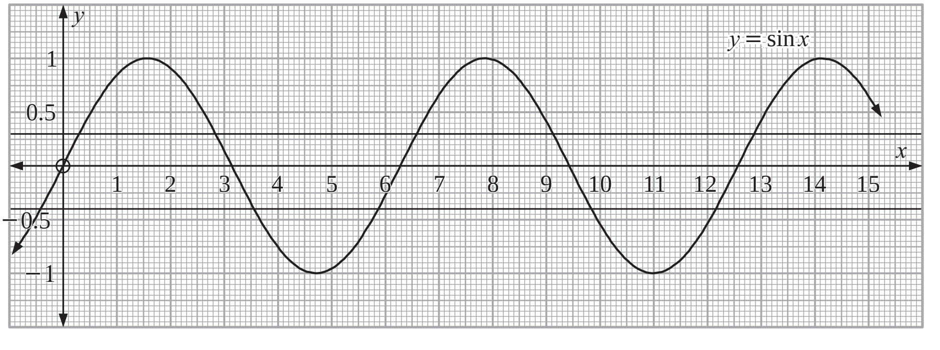
$$A = 10 \quad \{\text{from the diagram}\} \qquad D = \frac{\text{max.} + \text{min.}}{2} = \frac{24}{2} = 12$$

$$C = \frac{0+50}{2} = 25 \quad \{\text{values of } t \text{ at max. and min.}\}$$

$$\therefore \quad H = 10\sin \tfrac{\pi}{50}(t-25)+12$$

EXERCISE 13F.1

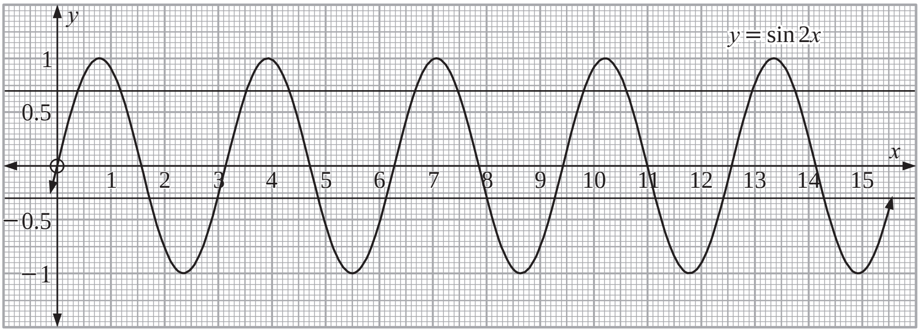
1



a When $\sin x = 0.3$, $x \doteqdot 0.3,\ 2.8,\ 6.6,\ 9.1,\ 12.9$

b When $\sin x = -0.4$, $x \doteqdot 5.9,\ 9.8,\ 12.2$

2



a When $\sin(2x) = 0.7$, $x \doteqdot 0.4,\ 1.2,\ 3.5,\ 4.3,\ 6.7,\ 7.5,\ 9.8,\ 10.6,\ 13.0,\ 13.8$

b When $\sin(2x) = -0.3$, $x \doteqdot 1.7,\ 3.0,\ 4.8,\ 6.1,\ 8.0,\ 9.3,\ 11.1,\ 12.4,\ 14.3$

EXERCISE 13F.2

1 **a** $\sin x = 0.414$
$\therefore\ x \doteqdot 0.4268,\ 2.715,\ 6.710$

b $\sin x = -0.673$
$\therefore\ x \doteqdot 3.880,\ 5.545$

c $\sin x = 1.289$
There are no solutions as all values of $\sin x$ lie between -1 and $+1$.

d $\sin 2x = 0.162$
$\therefore\ x \doteqdot 0.081\,36,\ 1.489,\ 3.223,\ 4.631,\ 6.365,\ 7.773$

e $\sin\left(\frac{x}{2}\right) = -0.606$
$\therefore\ x \doteqdot 7.585$

f $\sin(x+2) = 0.0652$
$\therefore\ x \doteqdot 1.076,\ 4.348,\ 7.360$

g $\sin(x - 1.3) = 0.866$
$\therefore\ x \doteqdot 2.347,\ 3.394$

h $\sin(x - \frac{\pi}{3}) = 0.7063$
$\therefore\ x \doteqdot 1.831,\ 3.405$

i $\sin\left(\frac{2x}{3}\right) = -0.9367 \quad \therefore\ x \doteqdot 6.532,\ 7.605$

EXERCISE 13F.3

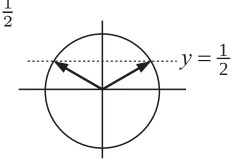
1 **a** $x = \frac{\pi}{6} + \frac{k12\pi}{6}$ and $0 \leqslant x \leqslant \frac{36\pi}{6}$
$\therefore\ x = \frac{\pi}{6},\ \frac{13\pi}{6},\ \frac{25\pi}{6}$

b $x = -\frac{\pi}{3} + \frac{k6\pi}{3}$ and $-\frac{6\pi}{3} \leqslant x \leqslant \frac{6\pi}{3}$
$\therefore\ x = -\frac{\pi}{3},\ \frac{5\pi}{3}$

c $x = -\frac{\pi}{2} + \frac{k2\pi}{2}$ and $-\frac{8\pi}{2} \leqslant x \leqslant \frac{8\pi}{2}$
$\therefore\ x = -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2},\ -\frac{3\pi}{2},\ -\frac{5\pi}{2},\ -\frac{7\pi}{2}$

d $x = \frac{5\pi}{6} + \frac{k3\pi}{6}$ and $0 \leqslant x \leqslant \frac{24\pi}{6}$
$\therefore\ x = \frac{2\pi}{6},\ \frac{5\pi}{6},\ \frac{8\pi}{6},\ \frac{11\pi}{6},\ \frac{14\pi}{6},\ \frac{17\pi}{6},\ \frac{20\pi}{6},\ \frac{23\pi}{6}$
i.e., $x = \frac{\pi}{3},\ \frac{5\pi}{6},\ \frac{4\pi}{3},\ \frac{11\pi}{6},\ \frac{7\pi}{3},\ \frac{17\pi}{6},\ \frac{10\pi}{3},\ \frac{23\pi}{6}$

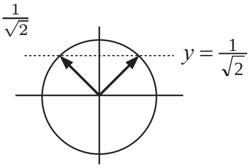
2 **a** $2\sin x = 1, \quad 0 \leqslant x \leqslant 6\pi$
$\therefore\ \sin x = \frac{1}{2}$



$\therefore\ x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$

$\therefore\ x = \frac{\pi}{6},\ \frac{13\pi}{6},\ \frac{25\pi}{6},\ \frac{5\pi}{6},\ \frac{17\pi}{6},\ \frac{29\pi}{6}$

b $\sqrt{2}\sin x = 1, \quad 0 \leqslant x \leqslant 4\pi$
$\therefore\ \sin x = \frac{1}{\sqrt{2}}$

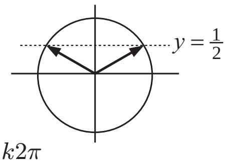


$\therefore\ x = \left.\begin{matrix}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{matrix}\right\} + k2\pi$

$\therefore\ x = \frac{\pi}{4},\ \frac{9\pi}{4},\ \frac{3\pi}{4},\ \frac{11\pi}{4}$

c $2\sin x - 1 = 0, \quad -2\pi \leqslant x \leqslant 2\pi$

$\therefore \quad \sin x = \frac{1}{2}$

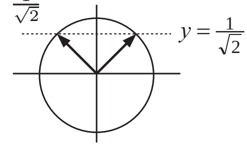


$\therefore \quad x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$

$\therefore \quad x = \frac{\pi}{6}, -\frac{11\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}$

d $\sqrt{2}\sin x - 1 = 0, \quad -4\pi \leqslant x \leqslant 0$

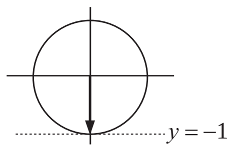
$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$



$\therefore \quad x = \left.\begin{matrix}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{matrix}\right\} + k2\pi$

$\therefore \quad x = -\frac{7\pi}{4}, -\frac{15\pi}{4}, -\frac{5\pi}{4}, -\frac{13\pi}{4}$

e $\sin x = -1, \quad 0 \leqslant x \leqslant 6\pi$

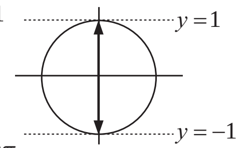


$\therefore \quad x = \frac{3\pi}{2} + k2\pi$

$\therefore \quad x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$

f $\sin^2 x = 1, \quad 0 \leqslant x \leqslant 4\pi$

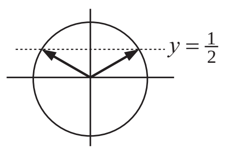
$\therefore \quad \sin x = \pm 1$



$\therefore \quad x = \frac{\pi}{2} + k\pi$

$\therefore \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

g $\sin 2x = \frac{1}{2}, \quad 0 \leqslant x \leqslant 3\pi$



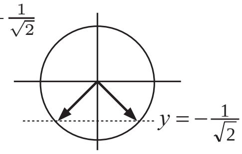
$\therefore \quad 2x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$

$\therefore \quad x = \left.\begin{matrix}\frac{\pi}{12}\\ \frac{5\pi}{12}\end{matrix}\right\} + k\pi$

$\therefore \quad x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}$

h $\sqrt{2}\sin 3x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin 3x = -\frac{1}{\sqrt{2}}$



$\therefore \quad 3x = \left.\begin{matrix}\frac{5\pi}{4}\\ \frac{7\pi}{4}\end{matrix}\right\} + k2\pi$

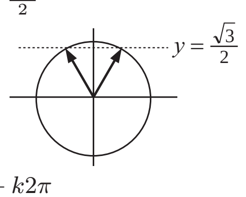
$\therefore \quad x = \left.\begin{matrix}\frac{5\pi}{12}\\ \frac{7\pi}{12}\end{matrix}\right\} + \frac{k2\pi}{3}$

$\therefore \quad x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{21\pi}{12}, \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$

i.e., $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

i $2\sin 2x - \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \quad \sin 2x = \frac{\sqrt{3}}{2}$



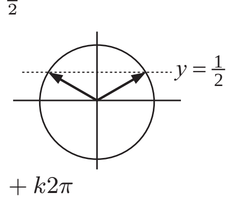
$\therefore \quad 2x = \left.\begin{matrix}\frac{\pi}{3}\\ \frac{2\pi}{3}\end{matrix}\right\} + k2\pi$

$\therefore \quad x = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{\pi}{3}\end{matrix}\right\} + k\pi$

$\therefore \quad x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$

j $2\sin(x + \frac{\pi}{3}) = 1, \quad -3\pi \leqslant x \leqslant 3\pi$

$\therefore \quad \sin(x + \frac{\pi}{3}) = \frac{1}{2}$



$\therefore \quad x + \frac{\pi}{3} = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$

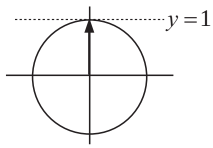
$\therefore \quad x = \left.\begin{matrix}-\frac{\pi}{6}\\ \frac{\pi}{2}\end{matrix}\right\} + k2\pi$

$\therefore \quad x = -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{13\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}$

3 **a**
$$\sin^2 x + \sin x - 2 = 0$$
$$\therefore \quad (\sin x - 1)(\sin x + 2) = 0$$
$$\therefore \quad \sin x = 1 \text{ or } -2$$

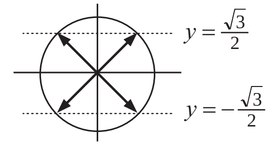
But values of sine can only lie between -1 and 1 inclusive.

$\therefore \quad \sin x = 1$



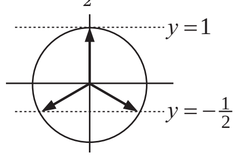
$\therefore \quad x = \frac{\pi}{2}$

b
$$4\sin^2 x = 3$$
$$\therefore \quad \sin^2 x = \tfrac{3}{4}$$
$$\therefore \quad \sin x = \pm\tfrac{\sqrt{3}}{2}$$



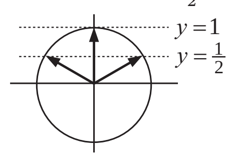
$\therefore \quad x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

c
$$2\sin^2 x = \sin x + 1$$
$$\therefore \quad 2\sin^2 x - \sin x - 1 = 0$$
$$\therefore \quad (2\sin x + 1)(\sin x - 1) = 0$$
$$\therefore \quad \sin x = -\tfrac{1}{2} \text{ or } 1$$



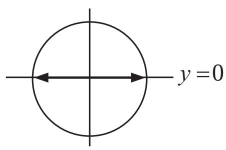
$\therefore \quad x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

d
$$2\sin^2 x + 1 = 3\sin x$$
$$\therefore \quad 2\sin^2 x - 3\sin x + 1 = 0$$
$$\therefore \quad (2\sin x - 1)(\sin x - 1) = 0$$
$$\therefore \quad \sin x = \tfrac{1}{2} \text{ or } 1$$



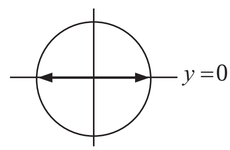
$\therefore \quad x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

4 **a** The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0 \quad (0 \leqslant x \leqslant \pi)$



$$\therefore \quad 2x = 0 + k\pi$$
$$\therefore \quad x = 0 + k\,\tfrac{\pi}{2}$$
$$\therefore \quad x = 0, \tfrac{\pi}{2}, \pi$$

b The zeros of $y = \sin(x - \frac{\pi}{4})$ are the solutions of $\sin(x - \frac{\pi}{4}) = 0 \quad (0 \leqslant x \leqslant 3\pi)$



$$\therefore \quad x - \tfrac{\pi}{4} = 0 + k\pi$$
$$\therefore \quad x = \tfrac{\pi}{4} + k\pi$$
$$\therefore \quad x = \tfrac{\pi}{4}, \tfrac{5\pi}{4}, \tfrac{9\pi}{4}$$

EXERCISE 13F.4

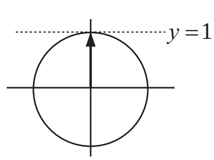
1 **a** $P(t) = 7500 + 3000\sin(\frac{\pi t}{8}), \quad 0 \leqslant t \leqslant 12$

i
$$\begin{aligned} P(0) &= 7500 + 3000\sin 0 \\ &= 7500 + 0 \\ &= 7500 \text{ grass-hoppers} \end{aligned}$$

ii
$$\begin{aligned} P(5) &= 7500 + 3000\sin(\tfrac{5\pi}{8}) \\ &\doteqdot 10\,271.63\,\ldots. \\ &\doteqdot 10\,300 \text{ grass-hoppers} \end{aligned}$$

b The greatest value of $P(t)$ occurs when $\sin\left(\frac{\pi t}{8}\right) = 1$

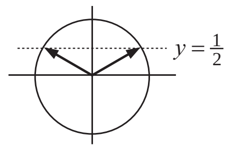
i.e., is $7500 + 3000 = 10\,500$ grass-hoppers when
$$\frac{\pi t}{8} = \tfrac{\pi}{2} + k2\pi$$
$$\therefore \quad \frac{t}{8} = \tfrac{1}{2} + 2k$$
$$\therefore \quad t = 4 + 16k$$
$$\therefore \quad t = 4 \quad \{\text{as } 0 \leqslant t \leqslant 12\}$$



i.e., after 4 weeks

c **i** When $P(t) = 9000$,

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) = 9000$$
$$\therefore\ 3000\sin\left(\tfrac{\pi t}{8}\right) = 1500$$
$$\therefore\ \sin\left(\tfrac{\pi t}{8}\right) = \tfrac{1}{2}$$

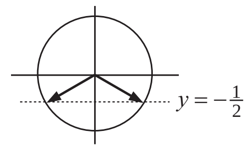


$$\therefore\ \frac{\pi t}{8} = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$$
$$\therefore\ \frac{t}{8} = \left.\begin{matrix}\frac{1}{6}\\ \frac{5}{6}\end{matrix}\right\} + k\,2$$
$$\therefore\ t = \left.\begin{matrix}\frac{4}{3}\\ \frac{20}{3}\end{matrix}\right\} + k\,16$$
$$\therefore\ t = 1\tfrac{1}{3} \text{ or } 6\tfrac{2}{3}$$

i.e., at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks

ii When $P(t) = 6000$,

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) = 6000$$
$$\therefore\ 3000\sin\left(\tfrac{\pi t}{8}\right) = -1500$$
$$\therefore\ \sin\left(\tfrac{\pi t}{8}\right) = -\tfrac{1}{2}$$

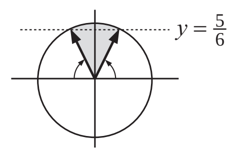


$$\therefore\ \frac{\pi t}{8} = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi$$
$$\therefore\ \frac{t}{8} = \left.\begin{matrix}\frac{7}{6}\\ \frac{11}{6}\end{matrix}\right\} + k\,2$$
$$\therefore\ t = \left.\begin{matrix}\frac{28}{3}\\ \frac{44}{3}\end{matrix}\right\} + k\,16$$
$$\therefore\ t = 9\tfrac{1}{3}$$

i.e., at $9\frac{1}{3}$ weeks

d If $P(t) > 10\,000$, then

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) > 10\,000$$
$$\therefore\ 3000\sin\left(\tfrac{\pi t}{8}\right) > 2500$$
$$\therefore\ \sin\left(\tfrac{\pi t}{8}\right) > \tfrac{5}{6}$$



Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$t \doteqdot 2.51$ or 5.49 So $2.51 \leqslant t \leqslant 5.49$ weeks.

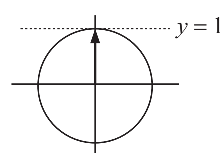
2 $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$

a $H(0) = 20 - 19(0)$
$= 20$ m

i.e., 20 m above the ground

b H is smallest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

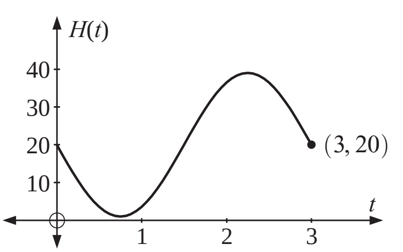
$$\therefore\ \frac{2\pi t}{3} = \tfrac{\pi}{2} + k2\pi$$
$$\therefore\ \frac{2t}{3} = \tfrac{1}{2} + k\,2$$
$$\therefore\ t = \tfrac{3}{4} + k\,3$$
$$\therefore\ t = \tfrac{3}{4} \text{ min} \quad \{\text{as } k = 0\}$$



c period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$ min

$\therefore$ one revolution takes 3 min

d



3 $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$ years

a $P(0) = 400 + 250(0)$
$= 400$ water buffalo

b **i** $P(\frac{1}{2}) = 400 + 250\sin\left(\frac{\pi(\frac{1}{2})}{2}\right)$
$= 400 + 250\sin(\frac{\pi}{4})$
$= 400 + 250 \times \frac{1}{\sqrt{2}}$
$\doteqdot 577$ water buffalo

ii $P(2) = 400 + 250\sin\pi$
$= 400 + 250(0)$
$= 400$ water buffalo

c $P(1) = 400 + 250\sin(\frac{\pi}{2})$
$= 400 + 250 \times 1$
$= 650$ water buffalo

This is the maximum herd size.

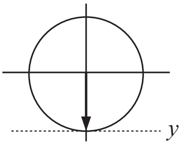
d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \quad \frac{t}{2} = \frac{3}{2} + k\,2$$

$$\therefore \quad t = 3 + 4k$$



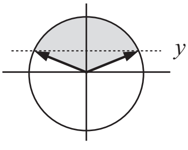
So, the first time is after 3 years.

e If $P(t) > 500$ then

$$400 + 250\sin\left(\frac{\pi t}{2}\right) > 500$$

$$\therefore \quad 250\sin\left(\frac{\pi t}{2}\right) > 100$$

$$\therefore \quad \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$$



$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5}$ when $\frac{\pi t}{2} = 0.4115$ or $\pi - 0.4115$

$\therefore \quad t \doteqdot 0.262$ or 1.74

So, for $\sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$, $0.26 < t < 1.74$

4 $C(t) = 9.2\sin\frac{\pi}{7}(t-4) + 107.8$ cents/L

a **i** 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$

i.e., $\underset{\text{min.}}{\underset{\uparrow}{98.6 \text{ cents/L}}}$ and $\underset{\text{max.}}{\underset{\uparrow}{117.0 \text{ cents/L}}}$

$\therefore$ the statement is true.

ii period $= \dfrac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2\sin\frac{\pi}{7}(3) + 107.8 \doteqdot 116.8$ cents/L

c When $C(t) = \$1.10/\text{L}$ then $9.2\sin\frac{\pi}{7}(t-4) + 107.8 = 110$

$$\therefore \quad \sin\tfrac{\pi}{7}(t-4) = \frac{2.2}{9.2} \doteqdot 0.239\,13$$

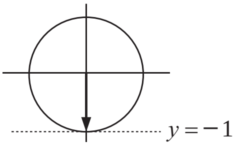
$$\therefore \quad \tfrac{\pi}{7}(t-4) \doteqdot 0.239\,13 \text{ or } \pi - 0.239\,13$$

$$\therefore \quad t - 4 \doteqdot 0.533 \text{ or } 6.467$$

$$\therefore \quad t \doteqdot 4.53 \text{ or } 10.47$$

i.e., on the 5th and 11th days

d The min. cost/litre is $-9.2 + 107.8 = 98.6$ cents when $\sin\frac{\pi}{7}(t-4) = -1$



$$\begin{aligned} \text{i.e.,}\quad \tfrac{\pi}{7}(t-4) &= \tfrac{3\pi}{2} \\ \therefore\quad \frac{t-4}{7} &= \frac{3}{2} \\ \therefore\quad 2t-8 &= 21 \\ \therefore\quad 2t &= 29 \\ \therefore\quad t &= 14.5 \end{aligned}$$

i.e., on the 15th day

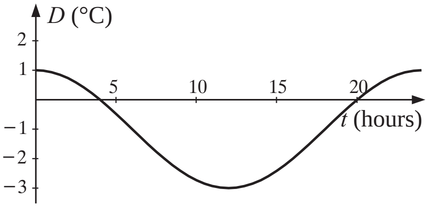
5 **a** $S(10) = 3\sin\frac{\pi}{12}(4) + 23$
$= 3\sin(\frac{\pi}{3}) + 23$
$\doteqdot 25.6^o\text{C}$

So, the inside temperature is 25.6^oC.

$T(10) = 5\sin\frac{\pi}{12}(10) + 24$
$= 5\sin(\frac{5\pi}{6}) + 24$
$\doteqdot 28.3^o\text{C}$

So, the outside temperature is 28.3^oC.

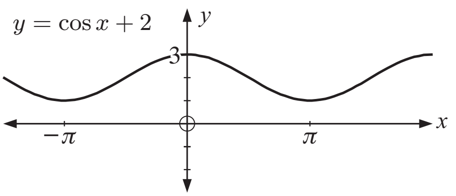
b $D = S - T$
$= 3\sin\frac{\pi}{12}(t-6) + 23 - 5\sin\frac{\pi}{12}(t-6) - 24$
$= -2\sin\frac{\pi}{12}(t-6) - 1\ \ ^o\text{C}$



c The outside and inside temperatures are the same when $D(t) = 0$
i.e., where the graph cuts the t-axis,
i.e., at about $t = 4$ and $t = 20$ i.e., 4 am and 8 pm.

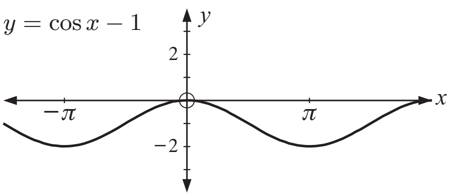
EXERCISE 13G

1 **a** $y = \cos x + 2$



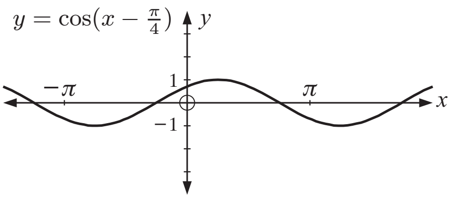
i.e., a vertical translation of $y = \cos x$ through $\begin{bmatrix}0\\2\end{bmatrix}$.

b $y = \cos x - 1$



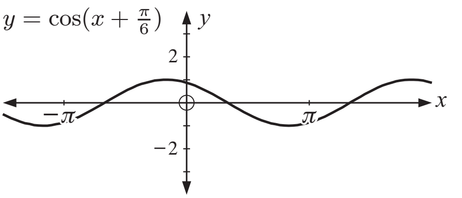
i.e., a vertical translation of $y = \cos x$ through $\begin{bmatrix}0\\-1\end{bmatrix}$.

c $y = \cos(x - \frac{\pi}{4})$



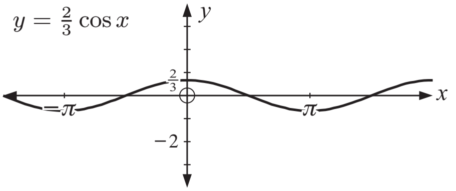
i.e., a horizontal translation of $y = \cos x$ through $\begin{bmatrix}\frac{\pi}{4}\\0\end{bmatrix}$.

d $y = \cos(x + \frac{\pi}{6})$



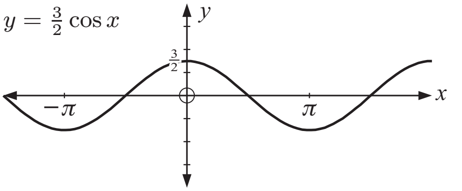
i.e., a horizontal translation of $y = \cos x$ through $\begin{bmatrix}-\frac{\pi}{6}\\0\end{bmatrix}$.

e $y = \frac{2}{3}\cos x$



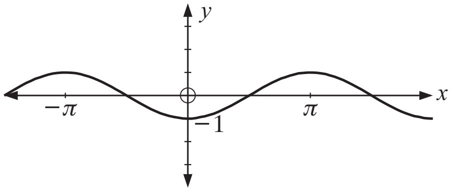
i.e., a vertical dilation of $y = \cos x$ with factor $\frac{2}{3}$.

f $y = \frac{3}{2}\cos x$



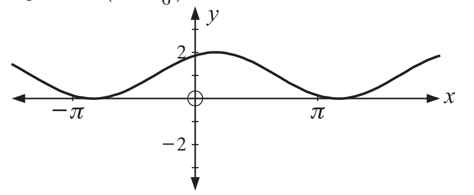
i.e., a vertical dilation of $y = \cos x$ with factor $\frac{3}{2}$.

g $y = -\cos x$



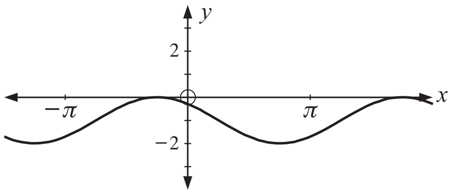
i.e., a reflection of $y = \cos x$ in the x-axis.

h $y = \cos(x - \frac{\pi}{6}) + 1$



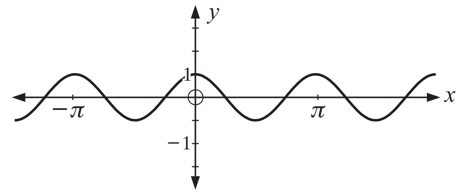
i.e., a translation of $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

i $y = \cos(x + \frac{\pi}{4}) - 1$



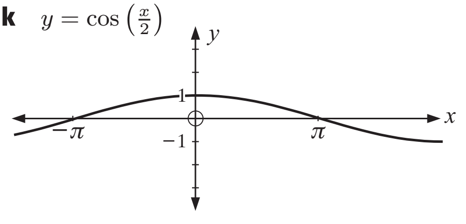
i.e., a translation of $\begin{bmatrix} -\frac{\pi}{4} \\ -1 \end{bmatrix}$.

j $y = \cos 2x$



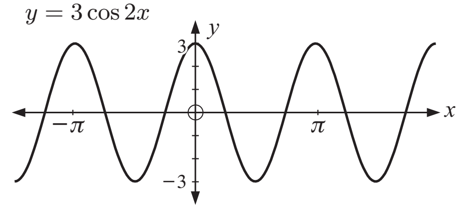
i.e., a horizontal dilation of factor $\frac{1}{2}$.

k $y = \cos\left(\frac{x}{2}\right)$



i.e., a horizontal dilation of factor 2.

l $y = 3\cos 2x$



i.e., a horizontal dilation of factor $\frac{1}{2}$ followed by a vertical dilation of factor 3.

2 **a** period $= \frac{2\pi}{3}$ **b** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period $= \dfrac{2\pi}{\frac{\pi}{50}} = 100$

3 A controls the amplitude B controls the period, period $= \dfrac{2\pi}{|B|}$

C controls the horizontal translation D controls the vertical translation

4 **a** If $y = A\cos B(x - C) + D$, then $A = 2$, $\pi = \dfrac{2\pi}{B}$ $\therefore$ $B = 2$

C and D are 0 as there is no horizontal or vertical shift. $\therefore$ $y = 2\cos 2x$

b If $y = A\cos B(x - C) + D$, then $A = 1$, $4\pi = \dfrac{2\pi}{B}$ $\therefore$ $B = \frac{1}{2}$

A vertical shift of 2 units, no horizontal shift $\therefore$ $D = 2$, $C = 0$.
So, $y = \cos\frac{1}{2}(x) + 2$ i.e., $y = \cos\left(\frac{x}{2}\right) + 2$.

c If $y = A\cos B(x - C) + D$, then $A = 3$, $8 = \dfrac{2\pi}{B}$ $\therefore$ $B = \frac{\pi}{4}$

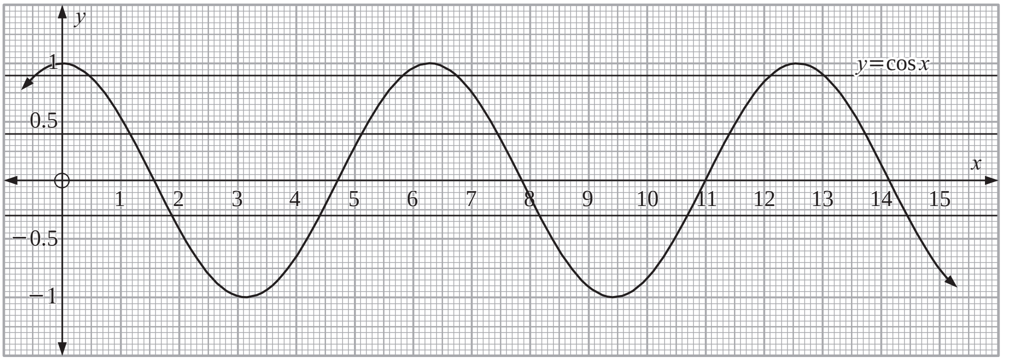
$C = D = 0$ {as no translation} $\therefore$ $y = 3\cos\left(\frac{\pi}{4}x\right)$

d If $y = A\cos B(x - C) + D$, then $A = -5$, $6 = \dfrac{2\pi}{B}$ $\therefore$ $B = \frac{\pi}{3}$

$C = D = 0$ {as no translation} $\therefore$ $y = -5\cos\left(\frac{\pi}{3}x\right)$

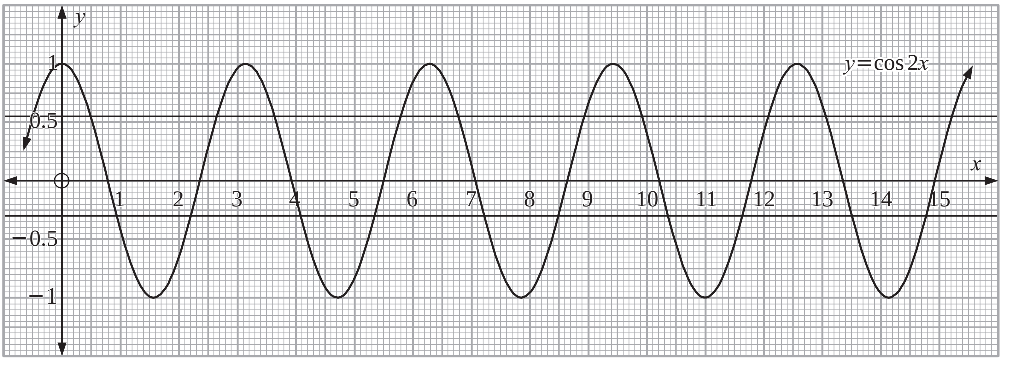
EXERCISE 13H

1



a $x \doteqdot 1.2,\ 5.1,\ 7.4$ **b** $x \doteqdot 4.4,\ 8.2,\ 10.7$ **c** $x \doteqdot 0.5,\ 5.8,\ 6.7$

2



a $x \doteqdot 0.5 + k\pi$ or $2.6 + k\pi$

i.e., $x \doteqdot \left.\begin{matrix} 0.5 \\ 2.6 \end{matrix}\right\} + k\pi$

b $2\cos 2x + 0.6 = 0$

$\therefore\ 2\cos 2x = -0.6$

$\therefore\ \cos 2x = -0.3$

$\therefore\ x \doteqdot \left.\begin{matrix} 0.9 \\ 2.2 \end{matrix}\right\} + k\pi$

3 **a** $\cos x = 0.561, \quad 0 \leqslant x \leqslant 10$

$\therefore\ x \doteqdot 0.975,\ 5.308,\ 7.258$

b $\cos 2x = 0.782, \quad 0 \leqslant x \leqslant 6$

$\therefore\ x \doteqdot 0.336,\ 2.805,\ 3.478,\ 5.947$

c $\cos(x - 1.3) = -0.609, \quad 0 \leqslant x \leqslant 12$

$\therefore\ x \doteqdot 3.526,\ 5.358,\ 9.809,\ 11.641$

d $4\cos 3x + 1 = 0, \quad 0 \leqslant x \leqslant 5$

$\therefore\ \cos 3x = -0.25, \quad 0 \leqslant x \leqslant 5$

$\therefore\ x \doteqdot 0.608,\ 1.487,\ 2.702,\ 3.581,\ 4.797$

e $5\cos 2x + 2 = 0$ for all x

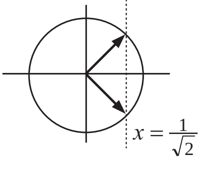
$\therefore\ \cos 2x = -\frac{2}{5} = -0.4$

$\therefore\ x = \left.\begin{matrix} 0.991 \\ 2.150 \end{matrix}\right\} + k\pi$

4 **a** $\cos x = \frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 4\pi$

$\therefore\ x = \left.\begin{matrix} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{matrix}\right\} + k2\pi$

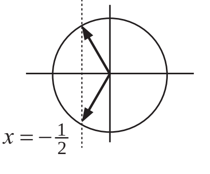
$\therefore\ x = \frac{\pi}{4},\ \frac{9\pi}{4},\ \frac{7\pi}{4},\ \frac{15\pi}{4}$



b $\cos x = -\frac{1}{2}, \quad 0 \leqslant x \leqslant 5\pi$

$\therefore\ x = \left.\begin{matrix} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{matrix}\right\} + k2\pi$

$\therefore\ x = \frac{2\pi}{3},\ \frac{8\pi}{3},\ \frac{14\pi}{3},\ \frac{4\pi}{3},\ \frac{10\pi}{3}$



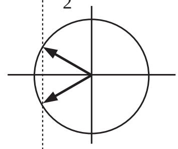
c $2\cos x + \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \cos x = -\frac{\sqrt{3}}{2}$

$\therefore x = \left.\begin{matrix}\frac{5\pi}{6}\\ \frac{7\pi}{6}\end{matrix}\right\} + k2\pi$

$\therefore x = \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{7\pi}{6}$

$x = -\frac{\sqrt{3}}{2}$



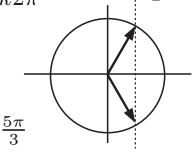
d $\cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

$\therefore x - \frac{2\pi}{3} = \left.\begin{matrix}\frac{\pi}{3}\\ \frac{5\pi}{3}\end{matrix}\right\} + k2\pi$

$\therefore x = \left.\begin{matrix}\pi\\ \frac{7\pi}{3}\end{matrix}\right\} + k2\pi$

$\therefore x = \pi, -\pi, \frac{\pi}{3}, -\frac{5\pi}{3}$

$x = \frac{1}{2}$



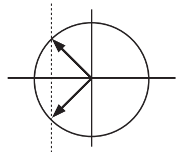
e $\sqrt{2}\cos(x - \frac{\pi}{4}) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \cos(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$\therefore x - \frac{\pi}{4} = \left.\begin{matrix}\frac{3\pi}{4}\\ \frac{5\pi}{4}\end{matrix}\right\} + k2\pi$

$\therefore x = \left.\begin{matrix}\pi\\ \frac{3\pi}{2}\end{matrix}\right\} + k2\pi$

$\therefore x = \pi, 3\pi, \frac{3\pi}{2}$



$x = -\frac{1}{\sqrt{2}}$

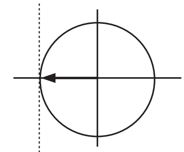
f $\cos 2x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \cos 2x = -1$

$\therefore 2x = \pi + k2\pi$

$\therefore x = \frac{\pi}{2} + k\pi$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$



$x = -1$

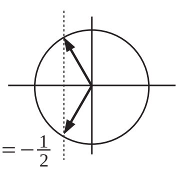
g $2\cos 3x + 1 = 0, \quad 0 \leqslant x \leqslant \pi$

$\therefore \cos 3x = -\frac{1}{2}$

$\therefore 3x = \left.\begin{matrix}\frac{2\pi}{3}\\ \frac{4\pi}{3}\end{matrix}\right\} + k2\pi$

$\therefore x = \left.\begin{matrix}\frac{2\pi}{9}\\ \frac{4\pi}{9}\end{matrix}\right\} + k\,\frac{2\pi}{3}$

$\therefore x = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{9}$



$x = -\frac{1}{2}$

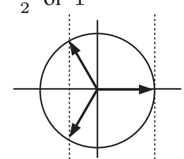
h $2\cos^2 x = \cos x + 1, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore 2\cos^2 x - \cos x - 1 = 0$

$\therefore (2\cos x + 1)(\cos x - 1) = 0$

$\therefore \cos x = -\frac{1}{2}$ or 1

$\therefore x = \left.\begin{matrix}0\\ \frac{2\pi}{3}\\ \frac{4\pi}{3}\end{matrix}\right\} + k2\pi$



$x = -\frac{1}{2}$ $\quad x = 1$

$\therefore x = 0, 2\pi, \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{4\pi}{3}$

5 **a** The period is 4 seconds.

$\therefore \frac{2\pi}{B} = 4$

$\therefore B = \frac{\pi}{2}$

Amplitude is 3

$\therefore A = 3$

$D = 1 + 3 = 4$

$C = 0$

$\therefore H(t) = 3\cos\frac{\pi}{2}(t - 0) + 4$ metres

i.e., $H(t) = 3\cos(\frac{\pi}{2}t) + 4$ metres

Check: When $t = 0$, $H(0) = 3\cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$\therefore 3\cos\left(\frac{\pi t}{2}\right) + 4 = 2$

$\therefore \cos\left(\frac{\pi t}{2}\right) = -\frac{2}{3}$

Using technology, $t \doteqdot 1.46$ sec

EXERCISE 13I

1 **a** $\sin\theta + \sin\theta$
$= 2\sin\theta$

b $2\cos\theta + \cos\theta$
$= 3\cos\theta$

c $3\sin\theta - \sin\theta$
$= 2\sin\theta$

d $3\sin\theta - 2\sin\theta$
$= \sin\theta$

e $\cos\theta - 3\cos\theta$
$= -2\cos\theta$

f $2\cos\theta - 5\cos\theta$
$= -3\cos\theta$

2 **a** $3\sin^2\theta + 3\cos^2\theta$
$= 3(\sin^2\theta + \cos^2\theta)$
$= 3(1)$
$= 3$

b $-2\sin^2\theta - 2\cos^2\theta$
$= -2(\sin^2\theta + \cos^2\theta)$
$= -2(1)$
$= -2$

c $-\cos^2\theta - \sin^2\theta$
$= -(\cos^2\theta + \sin^2\theta)$
$= -(1)$
$= -1$

d $3 - 3\sin^2\theta$
$= 3(1 - \sin^2\theta)$
$= 3\cos^2\theta$

e $4 - 4\cos^2\theta$
$= 4(1 - \cos^2\theta)$
$= 4\sin^2\theta$

f $\sin^3\theta + \sin\theta\cos^2\theta$
$= \sin\theta(\sin^2\theta + \cos^2\theta)$
$= \sin\theta(1)$
$= \sin\theta$

g $\cos^2\theta - 1$
$= 1 - \sin^2\theta - 1$
$= -\sin^2\theta$

h $\sin^2\theta - 1$
$= 1 - \cos^2\theta - 1$
$= -\cos^2\theta$

i $2\cos^2\theta - 2$
$= -2(1 - \cos^2\theta)$
$= -2\sin^2\theta$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$
$= \dfrac{\cos^2\theta}{\cos^2\theta}$
$= 1$

k $\dfrac{1 - \cos^2\theta}{\sin\theta}$
$= \dfrac{\sin^2\theta}{\sin\theta}$
$= \sin\theta$

l $\dfrac{\cos^2\theta - 1}{-\sin\theta}$
$= \dfrac{1 - \sin^2\theta - 1}{-\sin\theta}$
$= \dfrac{-\sin^2\theta}{-\sin\theta}$
$= \sin\theta$

3 **a** $(1 + \sin\theta)^2$
$= 1 + 2\sin\theta + \sin^2\theta$

b $(\sin\alpha - 2)^2$
$= \sin^2\alpha - 4\sin\alpha + 4$

c $(\cos\alpha - 1)^2$
$= \cos^2\alpha - 2\cos\alpha + 1$

d $(\sin\alpha + \cos\alpha)^2$
$= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha$
$= 1 + 2\sin\alpha\cos\alpha$

e $(\sin\beta - \cos\beta)^2$
$= \sin^2\beta - 2\sin\beta\cos\beta + \cos^2\beta$
$= 1 - 2\sin\beta\cos\beta$

f $-(2 - \cos\alpha)^2$
$= -[4 - 4\cos\alpha + \cos^2\alpha]$
$= -4 + 4\cos\alpha - \cos^2\alpha$

4 **a** $1 - \sin^2\theta$
$= (1 + \sin\theta)(1 - \sin\theta)$

b $\sin^2\alpha - \cos^2\alpha$
$= (\sin\alpha + \cos\alpha)(\sin\alpha - \cos\alpha)$

c $\cos^2\alpha - 1$
$= (\cos\alpha + 1)(\cos\alpha - 1)$

d $2\sin^2\beta - \sin\beta$
$= \sin\beta(2\sin\beta - 1)$

e $2\cos\phi + 3\cos^2\phi$
$= \cos\phi(2 + 3\cos\phi)$

f $3\sin^2\theta - 6\sin\theta$
$= 3\sin\theta(\sin\theta - 2)$

g $\sin^2\theta + 5\sin\theta + 6$
$= (\sin\theta + 2)(\sin\theta + 3)$

h $2\cos^2\theta + 7\cos\theta + 3$
$= (2\cos\theta + 1)(\cos\theta + 3)$

i $6\cos^2\alpha - \cos\alpha - 1$
$= (3\cos\alpha + 1)(2\cos\alpha - 1)$

5 **a** $\dfrac{1 - \sin^2\alpha}{1 - \sin\alpha}$
$= \dfrac{(1 + \sin\alpha)\cancelto{1}{(1 - \sin\alpha)}}{\cancelto{1}{1 - \sin\alpha}}$
$= 1 + \sin\alpha$

b $\dfrac{\cos^2\beta - 1}{\cos\beta + 1}$
$= \dfrac{\cancelto{1}{(\cos\beta + 1)}(\cos\beta - 1)}{\cancelto{1}{\cos\beta + 1}}$
$= \cos\beta - 1$

c $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi + \sin\phi}$
$= \dfrac{\cancelto{1}{(\cos\phi + \sin\phi)}(\cos\phi - \sin\phi)}{\cancelto{1}{\cos\phi + \sin\phi}}$
$= \cos\phi - \sin\phi$

d $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$
$= \dfrac{(\cos\phi + \sin\phi)\cancelto{1}{(\cos\phi - \sin\phi)}}{\cancelto{1}{\cos\phi - \sin\phi}}$
$= \cos\phi + \sin\phi$

e $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$

$= \dfrac{{}^1\cancel{\sin\alpha+\cos\alpha}}{{}_1\cancel{(\sin\alpha+\cos\alpha)}(\sin\alpha-\cos\alpha)}$

$= \dfrac{1}{\sin\alpha-\cos\alpha}$

f $\dfrac{3-3\sin^2\theta}{6\cos\theta} = \dfrac{3(1-\sin^2\theta)}{6\cos\theta}$

$= \dfrac{3\cos^2\theta}{6\cos\theta}$

$= \dfrac{\cos\theta}{2}$

6 a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2$

$= \cos^2\theta + \cancel{2\cos\theta\sin\theta} + \sin^2\theta$
$\quad + \cos^2\theta - \cancel{2\cos\theta\sin\theta} + \sin^2\theta$

$= 2\cos^2\theta + 2\sin^2\theta$

$= 2(\cos^2\theta+\sin^2\theta)$

$= 2(1)$

$= 2$

b $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2$

$= 4\sin^2\theta + \cancel{12\sin\theta\cos\theta} + 9\cos^2\theta$
$\quad + 9\sin^2\theta - \cancel{12\sin\theta\cos\theta} + 4\cos^2\theta$

$= 13\sin^2\theta + 13\cos^2\theta$

$= 13(\sin^2\theta+\cos^2\theta)$

$= 13(1)$

$= 13$

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)$

$= 1+\dfrac{1}{\cos\theta}-\cos\theta-1$

$= \dfrac{1}{\cos\theta}-\cos\theta$

$= \dfrac{1}{\cos\theta}-\cos\theta\left(\dfrac{\cos\theta}{\cos\theta}\right)$

$= \dfrac{1-\cos^2\theta}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta}$

d $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)$

$= \sin\theta-\sin^2\theta+1-\sin\theta$

$= 1-\sin^2\theta$

$= \cos^2\theta$

e $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}$

$= \dfrac{\sin^2\theta+(1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$

$= \dfrac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}$

$= \dfrac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}$

$= \dfrac{2\cancel{(1+\cos\theta)}^{\,1}}{\sin\theta\cancel{(1+\cos\theta)}_{\,1}}$

$= \dfrac{2}{\sin\theta}$

f $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}$

$= \dfrac{\sin\theta(1+\cos\theta)-\sin\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$= \dfrac{\cancel{\sin\theta}+\sin\theta\cos\theta-\cancel{\sin\theta}+\sin\theta\cos\theta}{1-\cos^2\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\sin^2\theta}$

$= \dfrac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}\sin\theta}$

$= \dfrac{2\cos\theta}{\sin\theta}$

g $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}$

$= \dfrac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$

$= \dfrac{2}{1-\sin^2\theta}$

$= \dfrac{2}{\cos^2\theta}$

EXERCISE 13J

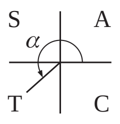
1 **a** $\sin 2A = 2\sin A\cos A$
$= 2(\frac{4}{5})(\frac{3}{5})$
$= \frac{24}{25}$

b $\cos 2A = \cos^2 A - \sin^2 A$
$= \frac{9}{25} - \frac{16}{25}$
$= -\frac{7}{25}$

2 $\cos 2A = 2\cos^2 A - 1$
$= 2(\frac{1}{3})^2 - 1$
$= 2 \times \frac{1}{9} - 1$
$= \frac{2}{9} - 1$
$= -\frac{7}{9}$

3 $\cos 2\phi = 1 - 2\sin^2\phi$
$= 1 - 2(-\frac{2}{3})^2$
$= 1 - 2(\frac{4}{9})$
$= 1 - \frac{8}{9}$
$= \frac{1}{9}$

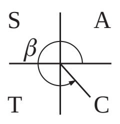
4 **a** $\sin\alpha = -\frac{2}{3}$
α is in Q3
$\therefore \cos\alpha < 0$



$\cos^2\alpha + \sin^2\alpha = 1$
$\therefore \cos^2\alpha + \frac{4}{9} = 1$
$\therefore \cos^2\alpha = \frac{5}{9}$
$\therefore \cos\alpha = -\frac{\sqrt{5}}{3}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$
$= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$
$= \frac{4\sqrt{5}}{9}$

b $\cos\beta = \frac{2}{5}$
β is in Q4
$\therefore \sin\beta < 0$



$\cos^2\beta + \sin^2\beta = 1$
$\therefore \frac{4}{25} + \sin^2\beta = 1$
$\therefore \sin^2\beta = \frac{21}{25}$
$\therefore \sin\beta = -\frac{\sqrt{21}}{5}$

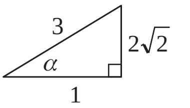
$\sin 2\beta = 2\sin\beta\cos\beta$
$= 2\left(-\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)$
$= -\frac{4\sqrt{21}}{25}$

5 α is acute $\therefore$ $\cos\alpha$ and $\sin\alpha$ are positive

a $\cos 2\alpha = 2\cos^2\alpha - 1$
$\therefore -\frac{7}{9} = 2\cos^2\alpha - 1$
$\therefore 2\cos^2\alpha = \frac{2}{9}$
$\therefore \cos^2\alpha = \frac{1}{9}$
$\therefore \cos\alpha = \frac{1}{3}$

b $\cos 2\alpha = 1 - 2\sin^2\alpha$
$\therefore -\frac{7}{9} = 1 - 2\sin^2\alpha$
$\therefore 2\sin^2\alpha = 1\frac{7}{9} = \frac{16}{9}$
$\therefore \sin^2\alpha = \frac{8}{9}$
$\therefore \sin\alpha = \frac{2\sqrt{2}}{3}$

or



$\sin\alpha = \frac{2\sqrt{2}}{3}$

6 **a** $2\sin\alpha\cos\alpha$
$= \sin 2\alpha$

b $4\cos\alpha\sin\alpha$
$= 2(2\sin\alpha\cos\alpha)$
$= 2\sin 2\alpha$

c $\sin\alpha\cos\alpha$
$= \frac{1}{2}(2\sin\alpha\cos\alpha)$
$= \frac{1}{2}\sin 2\alpha$

d $2\cos^2\beta - 1$
$= \cos 2\beta$

e $1 - 2\cos^2\phi$
$= -(2\cos^2\phi - 1)$
$= -\cos 2\phi$

f $1 - 2\sin^2 N$
$= \cos 2N$

g $2\sin^2 M - 1$
$= -(1 - 2\sin^2 M)$
$= -\cos 2M$

h $\cos^2\alpha - \sin^2\alpha$
$= \cos 2\alpha$

i $\sin^2\alpha - \cos^2\alpha$
$= -(\cos^2\alpha - \sin^2\alpha)$
$= -\cos 2\alpha$

j $2\sin 2A\cos 2A$
$= \sin 2(2A)$
$= \sin 4A$

k $2\cos 3\alpha\sin 3\alpha$
$= \sin 2(3\alpha)$
$= \sin 6\alpha$

l $2\cos^2 4\theta - 1$
$= \cos 2(4\theta)$
$= \cos 8\theta$

m $1 - 2\cos^2 3\beta$
$= -(2\cos^2 3\beta - 1)$
$= -\cos 2(3\beta)$
$= -\cos 6\beta$

n $1 - 2\sin^2 5\alpha$
$= \cos 2(5\alpha)$
$= \cos 10\alpha$

o $2\sin^2 3D - 1$
$= -(1 - 2\sin^2 3D)$
$= -\cos 2(3D)$
$= -\cos 6D$

p
$$\begin{aligned}&\cos^2 2A - \sin^2 2A\\ &= \cos 2(2A)\\ &= \cos 4A\end{aligned}$$

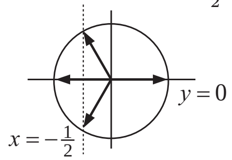
q
$$\begin{aligned}&\cos^2(\tfrac{\alpha}{2}) - \sin^2(\tfrac{\alpha}{2})\\ &= \cos 2(\tfrac{\alpha}{2})\\ &= \cos \alpha\end{aligned}$$

r
$$\begin{aligned}&2\sin^2 3P - 2\cos^2 3P\\ &= -2[\cos^2 3P - \sin^2 3P]\\ &= -2\cos 2(3P)\\ &= -2\cos 6P\end{aligned}$$

7 **a**
$$\begin{aligned}&(\sin\theta + \cos\theta)^2\\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= \underbrace{\sin^2\theta + \cos^2\theta} + 2\sin\theta\cos\theta\\ &= 1 + \sin 2\theta\end{aligned}$$

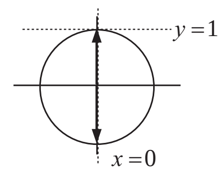
b
$$\begin{aligned}&\cos^4\theta - \sin^4\theta\\ &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)\\ &= 1 \times \cos 2\theta\\ &= \cos 2\theta\end{aligned}$$

8 **a**
$$\begin{aligned}\sin 2x + \sin x &= 0\\ \therefore\ 2\sin x\cos x + \sin x &= 0\\ \therefore\ \sin x(2\cos x + 1) &= 0\\ \therefore\ \sin x = 0 \quad\text{or}\quad \cos x &= -\tfrac{1}{2}\end{aligned}$$



$\therefore\ x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$

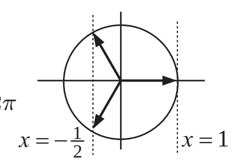
b
$$\begin{aligned}\sin 2x - 2\cos x &= 0\\ \therefore\ 2\sin x\cos x - 2\cos x &= 0\\ \therefore\ 2\cos x(\sin x - 1) &= 0\\ \therefore\ \cos x = 0 \quad\text{or}\quad \sin x &= 1\end{aligned}$$



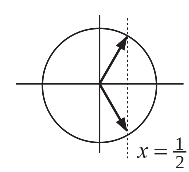
$\therefore\ x = \frac{\pi}{2}, \frac{3\pi}{2}$

c
$$\begin{aligned}\cos 2x - \cos x &= 0\\ \therefore\ 2\cos^2 x - 1 - \cos x &= 0\\ \therefore\ 2\cos^2 x - \cos x - 1 &= 0\\ \therefore\ (2\cos x + 1)(\cos x - 1) &= 0\\ \therefore\ \cos x &= -\tfrac{1}{2} \text{ or } 1\end{aligned}$$

$\therefore\ x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

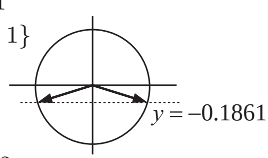


d
$$\begin{aligned}\cos 2x + 3\cos x &= 1\\ \therefore\ 2\cos^2 x - 1 + 3\cos x - 1 &= 0\\ \therefore\ 2\cos^2 x + 3\cos x - 2 &= 0\\ \therefore\ (2\cos x - 1)(\cos x + 2) &= 0\\ \therefore\ \cos x &= \tfrac{1}{2} \text{ or } -2\\ \therefore\ \cos x &= \tfrac{1}{2} \quad \{\text{as } -1 \leqslant \cos x \leqslant 1\}\end{aligned}$$



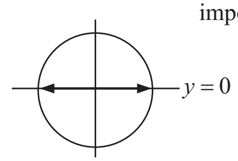
$\therefore\ x = \frac{\pi}{3}, \frac{5\pi}{3}$

e
$$\begin{aligned}\cos 2x + 5\sin x &= 0\\ \therefore\ 1 - 2\sin^2 x + 5\sin x &= 0\\ \therefore\ 2\sin^2 x - 5\sin x - 1 &= 0\\ \therefore\ \sin x &= \frac{5 \pm \sqrt{25 - 4(2)(-1)}}{4}\\ &= \frac{5 \pm \sqrt{33}}{4}\\ &\doteqdot 2.6861 \text{ or } -0.1861\\ \therefore\ \sin x &= -0.1861\\ &\{\text{as } -1 \leqslant \sin x \leqslant 1\}\end{aligned}$$



$$\begin{aligned}\therefore\ x &\doteqdot \pi + 0.1872\\ &\text{or } 2\pi - 0.1872\\ \therefore\ x &\doteqdot 3.33 \text{ or } 6.10\end{aligned}$$

f
$$\begin{aligned}\sin 2x + 3\sin x &= 0\\ \therefore\ 2\sin x\cos x + 3\sin x &= 0\\ \therefore\ \sin x(2\cos x + 3) &= 0\\ \therefore\ \sin x = 0 \quad\text{or}\quad \cos x &= \underset{\substack{\uparrow\\ \text{impossible}}}{-\tfrac{3}{2}}\end{aligned}$$



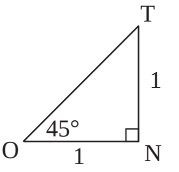
$\therefore\ x = 0, \pi, 2\pi$

EXERCISE 13K.1

1 **a** $\tan 0^o$ $= 0$ **b** $\tan 15^o$ $\doteqdot 0.27$ **c** $\tan 20^o$ $\doteqdot 0.36$ **d** $\tan 25^o$ $\doteqdot 0.47$

e $\tan 35^o$ $\doteqdot 0.70$ **f** $\tan 45^o$ $= 1.00$ **g** $\tan 50^o$ $\doteqdot 1.19$ **h** $\tan 55^o$ $\doteqdot 1.43$

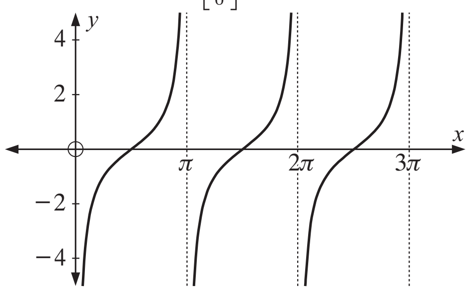
3 In ΔTON, ON $=$ NT $= 1$ (Δ is isosceles} $\tan 45^o = \dfrac{\text{NT}}{\text{ON}} = \frac{1}{1} = 1$



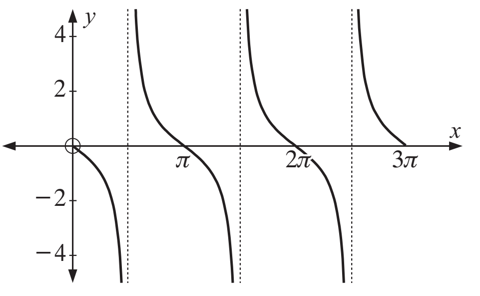
4 The graph does not extend far enough to find $\tan 85^o$.
$\tan 85^o \doteqdot 11.43$

EXERCISE 13K.2

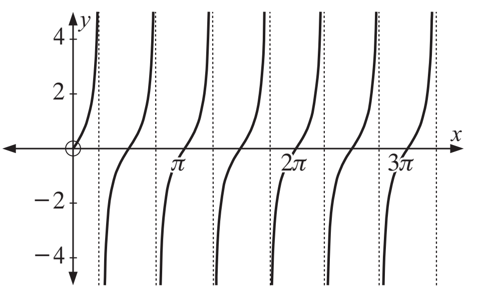
1 **a** **i** $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$ translated $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$.



ii $y = -\tan x$ is $y = \tan x$ reflected in the x-axis.

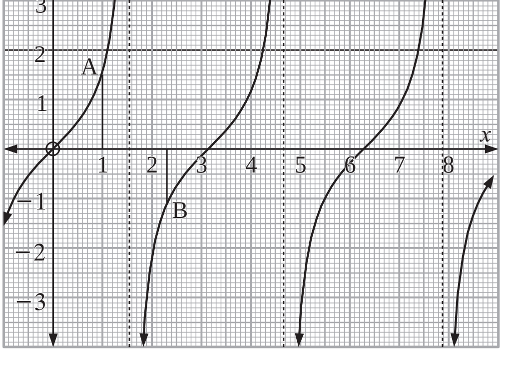


iii $y = \tan 2x$ comes from $y = \tan x$ under a horizontal dilation of factor $\frac{1}{2}$.



2 **a** translation through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **b** reflection in x-axis **c** horizontal dilation, factor $k = 2$

3



a **i** $\tan 1 \doteqdot 1.6$ {point A}
ii $\tan 2.3 \doteqdot -1.1$ {point B}

b **i** $\tan 1 \doteqdot 1.557$
ii $\tan 2.3 \doteqdot -1.119$

c **i** When $\tan x = 2$, $x \doteqdot 1.1, 4.2, 7.4$
ii When $\tan x = -1.4$, $x \doteqdot 2.2, 5.3$

4 **a** period $= \frac{\pi}{1} = \pi$ **b** period $= \frac{\pi}{2}$ **c** period $= \frac{\pi}{n}$

EXERCISE 13L.1

1 $X = \tan^{-1} 2 \quad \therefore \quad X \doteqdot 1.107 + k\pi$

a If $\tan 2x = 2$
then $2x = 1.107 + k\pi$
$\therefore \quad x = 0.554 + k(\frac{\pi}{2})$

b $\tan\left(\frac{x}{3}\right) = 2$
$\therefore \quad \dfrac{x}{3} = 1.107 + k\pi$
$\therefore \quad x = 3.32 + k3\pi$

c $\tan(x + 1.2) = 2$
$\therefore \quad x + 1.2 = 1.107 + k\pi$
$\therefore \quad x = -0.0929 + k\pi$

2 $X = \tan^{-1}(-3) \doteqdot -1.249 + k\pi$

a $\tan(x - 2) = -3$
$\therefore \quad x - 2 \doteqdot -1.249 + k\pi$
$\therefore \quad x \doteqdot 0.751 + k\pi$

b $\tan(3x) = -3$
$\therefore \quad 3x = -1.249 + k\pi$
$\therefore \quad x \doteqdot -0.416 + \dfrac{k\pi}{3}$

c $\tan\left(\frac{x}{2}\right) = -3$
$\therefore \quad \dfrac{x}{2} = -1.249 + k\pi$
$\therefore \quad x = -2.50 + k2\pi$

3 $X = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} + k\pi$

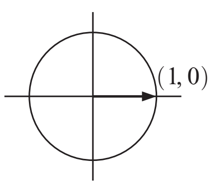
a $\tan(x - \frac{\pi}{6}) = \sqrt{3}$
$\therefore \quad x - \frac{\pi}{6} = \frac{\pi}{3} + k\pi$
$\therefore \quad x = \frac{\pi}{2} + k\pi$

b $\tan 4x = \sqrt{3}$
$\therefore \quad 4x = \frac{\pi}{3} + k\pi$
$\therefore \quad x = \frac{\pi}{12} + \dfrac{k\pi}{4}$

c $\tan^2 x = 3$
$\therefore \quad \tan x = \pm\sqrt{3}$
$\therefore \quad x = \left.\begin{matrix} \frac{\pi}{3} \\ -\frac{\pi}{3} \end{matrix}\right\} + k\pi$

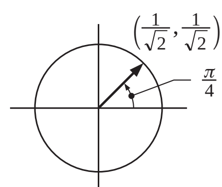
EXERCISE 13L.2

1 **a**



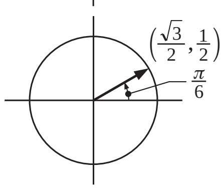
$\tan 0$
$= \frac{0}{1}$
$= 0$

b



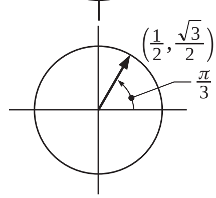
$\tan(\frac{\pi}{4})$
$= \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$
$= 1$

c



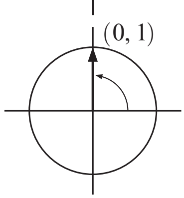
$\tan(\frac{\pi}{6})$
$= \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$
$= \frac{1}{\sqrt{3}}$

d



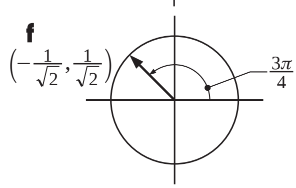
$\tan(\frac{\pi}{3})$
$= \frac{\sqrt{3}}{2} \div \frac{1}{2}$
$= \sqrt{3}$

e



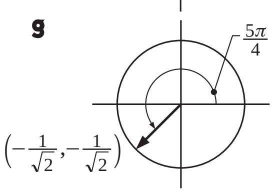
$\tan(\frac{\pi}{2})$
$= \frac{1}{0}$
i.e., undefined

f



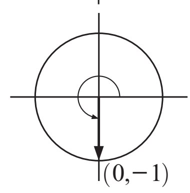
$\tan(\frac{3\pi}{4})$
$= \frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}}$
$= -1$

g



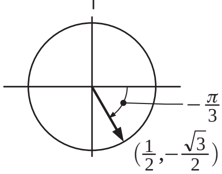
$\tan(\frac{5\pi}{3})$
$= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$
$= -\sqrt{3}$

h



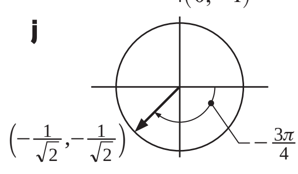
$\tan(\frac{3\pi}{2})$
$= -\frac{1}{0}$
i.e., undefined

i



$\tan(-\frac{\pi}{3})$
$= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$
$= -\sqrt{3}$

j



$\tan(-\frac{3\pi}{4})$
$= -\frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}}$
$= 1$

2 **a** $\frac{\pi}{4}, \frac{\pi}{4}+\pi$
i.e., $\frac{\pi}{4}, \frac{5\pi}{4}$

b $\frac{3\pi}{4}, \frac{3\pi}{4}+\pi$
i.e., $\frac{3\pi}{4}, \frac{7\pi}{4}$

c $\frac{\pi}{3}, \frac{\pi}{3}+\pi$
i.e., $\frac{\pi}{3}, \frac{4\pi}{3}$

d $0, 0+\pi, 0+2\pi$
i.e., $0, \pi, 2\pi$

e $\frac{\pi}{6}, \frac{\pi}{6}+\pi$
i.e., $\frac{\pi}{6}, \frac{7\pi}{6}$

f $\frac{5\pi}{3}, \frac{5\pi}{3}-\pi$
i.e., $\frac{5\pi}{3}, \frac{2\pi}{3}$

3 **a**
$$\begin{aligned}&3\tan x - \tan x\\ &= 2\tan x\end{aligned}$$

b
$$\begin{aligned}&\tan x - 4\tan x\\ &= -3\tan x\end{aligned}$$

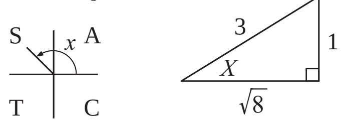
c
$$\begin{aligned}&\tan x\cos x\\ &= \frac{\sin x}{\cos x}\times\cos x\\ &= \sin x\end{aligned}$$

d
$$\begin{aligned}&\frac{\sin x}{\tan x}\\ &= \sin x \div \frac{\sin x}{\cos x}\\ &= \sin \times \frac{\cos x}{\sin x}\\ &= \cos x\end{aligned}$$

e
$$\begin{aligned}&3\sin x + 2\cos x\tan x\\ &= 3\sin x + 2\cos x\frac{\sin x}{\cos x}\\ &= 3\sin x + 2\sin x\\ &= 5\sin x\end{aligned}$$

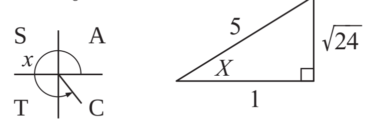
f
$$\begin{aligned}&\frac{2\tan x}{\sin x}\\ &= 2\left(\frac{\sin x}{\cos x}\right)\div\frac{\sin x}{1}\\ &= \frac{2\sin x}{\cos x}\times\frac{1}{\sin x}\\ &= \frac{2}{\cos x}\end{aligned}$$

4 **a** $\sin x = \frac{1}{3}$



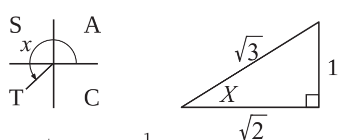
$\therefore\quad \tan x = -\frac{1}{\sqrt{8}} = -\frac{1}{2\sqrt{2}}$

b $\cos x = \frac{1}{5}$



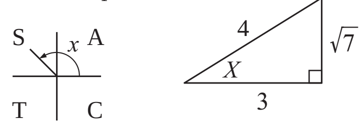
$\therefore\quad \tan x = -\frac{\sqrt{24}}{1} = -2\sqrt{6}$

c $\sin x = -\frac{1}{\sqrt{3}}$



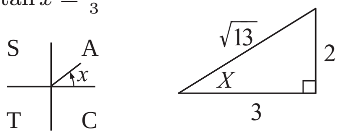
$\therefore\quad \tan x = \frac{1}{\sqrt{2}}$

d $\cos x = -\frac{3}{4}$



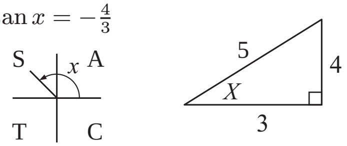
$\therefore\quad \tan x = -\frac{\sqrt{7}}{3}$

5 **a** $\tan x = \frac{2}{3}$



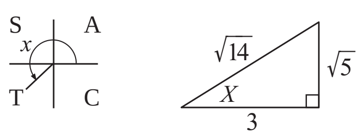
$\therefore\quad \sin x = \frac{2}{\sqrt{13}},\quad \cos x = \frac{3}{\sqrt{13}}$

b $\tan x = -\frac{4}{3}$



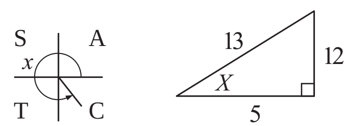
$\therefore\quad \sin x = \frac{4}{5},\quad \cos x = -\frac{3}{5}$

c $\tan x = \frac{\sqrt{5}}{3}$



$\therefore\quad \sin x = -\frac{\sqrt{5}}{\sqrt{14}},\quad \cos x = -\frac{3}{\sqrt{14}}$

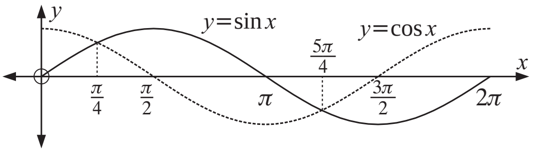
d $\tan x = -\frac{12}{5}$



$\therefore\quad \sin x = -\frac{12}{13},\quad \cos x = \frac{5}{13}$

EXERCISE 13M

1 **a**



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

c If $\sin x = \cos x$ then $\frac{\sin x}{\cos x} = 1$

$\therefore \tan x = 1$

$\therefore x = \frac{\pi}{4} + k\pi$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

2 **a** $\sin x = -\cos x$

$\therefore \frac{\sin x}{\cos x} = \frac{-\cos x}{\cos x}$

$\therefore \tan x = -1$

$\therefore x = \frac{3\pi}{4} + k\pi$

$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}$

b $\sin(3x) = \cos(3x)$

$\therefore \frac{\sin(3x)}{\cos(3x)} = 1$

$\therefore \tan(3x) = 1$

$\therefore 3x = \frac{\pi}{4} + k\pi$

$\therefore x = \frac{\pi}{12} + \frac{k\pi}{3}$

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

c $\sin(2x) = \sqrt{3}\cos(2x)$

$\therefore \frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$

$\therefore \tan(2x) = \sqrt{3}$

$\therefore 2x = \frac{\pi}{3} + k\pi$

$\therefore x = \frac{\pi}{6} + \frac{k\pi}{2}$

$\therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \frac{10\pi}{6}$

i.e., $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

4 **a** $\sin x = 5\cos x$

$\therefore \tan x = 5$

$\therefore x = \tan^{-1}(5)$

$\therefore x = 1.373 + k\pi$

$\therefore x \doteqdot 1.37, 4.51, 7.66$

b $4\sin x + 3\cos x = 0$

$\therefore 4\sin x = -3\cos x$

$\therefore \frac{\sin x}{\cos x} = -\frac{3}{4}$

$\therefore \tan x = -\frac{3}{4}$

$\therefore x = \tan^{-1}(-\frac{3}{4})$

$\therefore x \doteqdot -1.081 + k\pi$

$\therefore x \doteqdot 2.50, 5.64, 8.78$

REVIEW SET 13A

1 **a** 120^o
$= 2 \times 60^o$
$= 2 \times \frac{\pi}{3}$
$= \frac{2\pi}{3}$

b 225^o
$= 5 \times 45^o$
$= 5 \times \frac{\pi}{4}$
$= \frac{5\pi}{4}$

c 150^o
$= 5 \times 30^o$
$= 5 \times \frac{\pi}{6}$
$= \frac{5\pi}{6}$

d 540^o
$= 3 \times 180^o$
$= 3\pi$

2 **a** 71^o
$= 71 \times \frac{\pi}{180}$ radians
$\doteqdot 1.239^c$

b 124.6^o
$= 124.6 \times \frac{\pi}{180}^c$
$\doteqdot 2.175^c$

c -142^o
$= -142 \times \frac{\pi}{180}^c$
$\doteqdot -2.478^c$

d -25.3^o
$= -25.3 \times \frac{\pi}{180}^c$
$\doteqdot -0.4416^c$

3 **a** $\frac{2\pi}{5} = \frac{2(180^o)}{5} = 72^o$

b $\frac{5\pi}{4} = \frac{5(180^o)}{4} = 225^o$

c $\frac{7\pi}{9} = \frac{7(180^o)}{9} = 140^o$

d $\frac{11\pi}{6} = \frac{11(180^o)}{6} = 330^o$

4 **a** 3
$= 3 \times \left(\frac{180}{\pi}\right)^o$
$\doteqdot 171.89^o$

b 1.46
$= 1.46 \times \left(\frac{180}{\pi}\right)^o$
$\doteqdot 83.65^o$

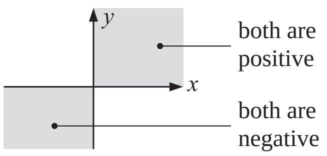
c 0.435
$= 0.435 \times \left(\frac{180}{\pi}\right)^o$
$\doteqdot 24.92^o$

d -5.271
$= -5.271 \times \left(\frac{180}{\pi}\right)^o$
$\doteqdot -302.01^o$

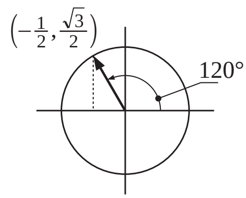
5 **a** point is $(\cos 320^o, \sin 320^o)$
i.e., $(0.766, -0.643)$

b point is $(\cos 163^o, \sin 163^o)$
i.e., $(-0.956, 0.292)$

6



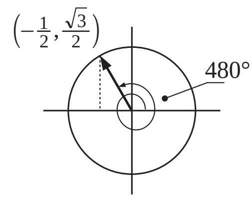
7 **a**



$\cos 120^o = -\frac{1}{2}$

$\sin 120^o = \frac{\sqrt{3}}{2}$

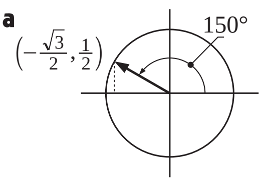
b



$\cos 480^o = -\frac{1}{2}$

$\sin 480^o = \frac{\sqrt{3}}{2}$

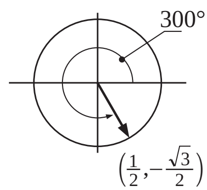
8 **a**



$\cos 150^o = -\frac{\sqrt{3}}{2}$

$\sin 150^o = \frac{1}{2}$

b



$\cos 300^o = \frac{1}{2}$

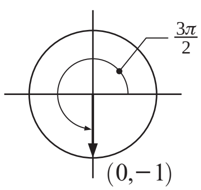
$\sin 300^o = -\frac{\sqrt{3}}{2}$

9 Look for points where the y-coordinate is the negative of the x-coordinate

or notice that if $\cos\theta = -\sin\theta$ then $\dfrac{\sin\theta}{\cos\theta} = -1$ i.e., $\tan\theta = -1$.

REVIEW SET 13B

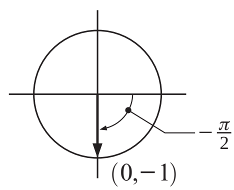
1 **a**



$\cos(\frac{3\pi}{2}) = 0$

$\sin(\frac{3\pi}{2}) = -1$

b



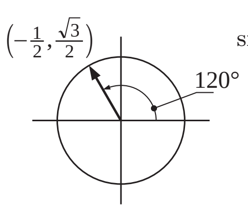
$\cos(-\frac{\pi}{2}) = 0$

$\sin(-\frac{\pi}{2}) = -1$

2
$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore \quad \tfrac{9}{16} + \sin^2\theta &= 1 \\ \therefore \quad \sin^2\theta &= \tfrac{7}{16} \\ \therefore \quad \sin\theta &= \pm\tfrac{\sqrt{7}}{4} \end{aligned}$$

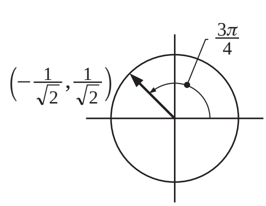
3
$$\begin{aligned} \cos\theta &= -\tfrac{3}{4} \\ \cos^2\theta + \sin^2\theta &= 1 \\ \therefore \quad \tfrac{9}{16} + \sin^2\theta &= 1 \\ \therefore \quad \sin^2\theta &= \tfrac{7}{16} \\ \therefore \quad \sin\theta &= \tfrac{\pm\sqrt{7}}{4}, \quad \text{but } \sin\theta > 0 \\ \therefore \quad \sin\theta &= \tfrac{\sqrt{7}}{4} \end{aligned}$$

4 **a**



$$\begin{aligned} \sin 120^o &= \tfrac{\sqrt{3}}{2} \\ \sin^2 120^o &= \left(\tfrac{\sqrt{3}}{2}\right)^2 \\ &= \tfrac{3}{4} \end{aligned}$$

b

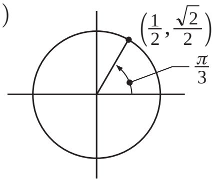


$$\begin{aligned} &\cos(\tfrac{3\pi}{4}) - \sin(\tfrac{3\pi}{4}) \\ &= -\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} \\ &= -\tfrac{2}{\sqrt{2}} \\ &= -\sqrt{2} \end{aligned}$$

5 **a** $2\sin(\frac{\pi}{3})\cos(\frac{\pi}{3})$

$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2}$

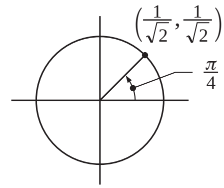


b $\sin^2(\frac{\pi}{4}) - 1$

$= \left(\frac{1}{\sqrt{2}}\right)^2 - 1$

$= \frac{1}{2} - 1$

$= -\frac{1}{2}$

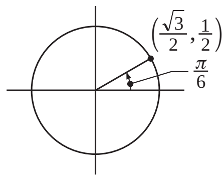


c $\cos^2(\frac{\pi}{6}) - \sin^2(\frac{\pi}{6})$

$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{1}{2}$



or $\cos^2(\frac{\pi}{6}) - \sin^2(\frac{\pi}{6})$

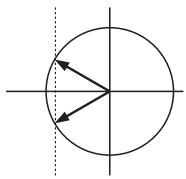
$= \cos 2(\frac{\pi}{6})$

$= \cos(\frac{\pi}{3})$

$= \frac{1}{2}$

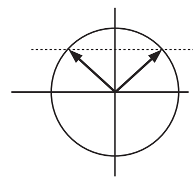
6 **a** If $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = 150^o,\ 210^o$

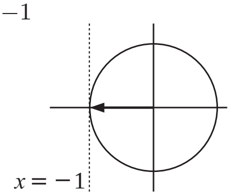


b If $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = 45^o,\ 135^o$



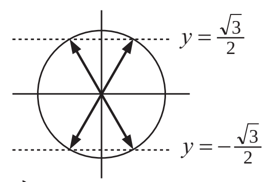
7 **a** $\cos\theta = -1$



$\therefore\ \theta = \pi + k2\pi$

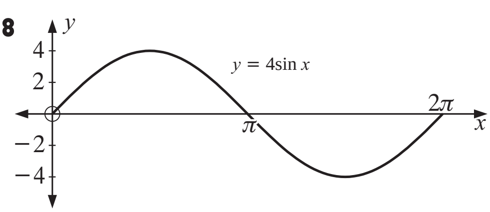
b $\sin^2\theta = \frac{3}{4}$

$\therefore\ \sin\theta = \pm\frac{\sqrt{3}}{2}$

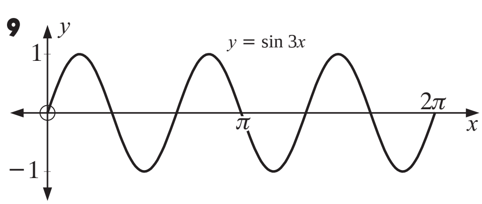


$\therefore\ \theta = \left.\begin{matrix}\frac{\pi}{3} \\ \frac{2\pi}{3}\end{matrix}\right\} + k\pi$

8



9

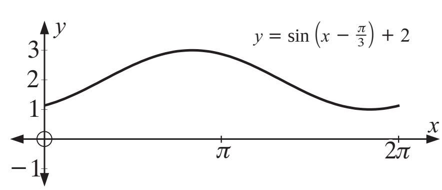


REVIEW SET 13C

1 **a** period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$

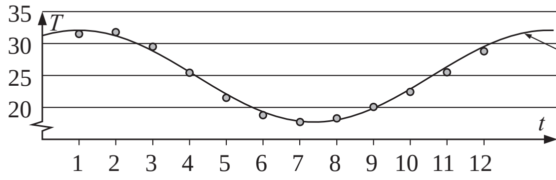
b period $= \frac{2\pi}{4} = \frac{\pi}{2}$

2



3

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



$T = A\sin B(t-C)+D$ period $= \dfrac{2\pi}{B} = 12$, $\therefore$ $B = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$

max. $= 31.8$ $\therefore$ $A = \dfrac{\text{max.} - \text{min.}}{2} \doteqdot \dfrac{31.8-17.7}{2} \doteqdot 7.05$

min. $= 17.7$ $D = \dfrac{\text{max.} + \text{min.}}{2} \doteqdot \dfrac{31.8+17.7}{2} \doteqdot 24.75$

$C = \dfrac{7+14}{2} = 10.5$ {values of t at min. and max.}

So, $T \doteqdot 7.05\sin\frac{\pi}{6}(t-10.5)+24.75$

From technology, $T \doteqdot 7.21\sin(0.488t+1.082)+24.75$

$\doteqdot 7.21\sin 0.488(t+2.22)+24.75$ **Note:** $0.488(2.22 - -10.5) \doteqdot 6.21 \doteqdot 2\pi$

4 **a** $\sin x = 0.382$

$\therefore$ $x \doteqdot 0.392,\ 2.750,\ 6.675$

b $\sin\left(\frac{x}{2}\right) = -0.458$

$\therefore$ $x \doteqdot 7.235$

5 **a** $\sin(x-2.4) = 0.754$

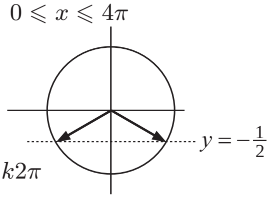
$\therefore$ $x \doteqdot 3.254,\ 4.687$

b $\sin\left(x+\frac{\pi}{3}\right) = 0.6049$

$\therefore$ $x \doteqdot 1.445,\ 5.89,\ 7.73$

6 **a** $2\sin x = -1$, $0 \leqslant x \leqslant 4\pi$

$\therefore$ $\sin x = -\frac{1}{2}$

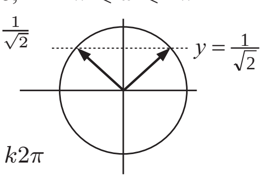


$\therefore$ $x = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi$

$\therefore$ $x = \frac{7\pi}{6},\ \frac{19\pi}{6},\ \frac{11\pi}{6},\ \frac{23\pi}{6}$

b $\sqrt{2}\sin x - 1 = 0$, $-2\pi \leqslant x \leqslant 2\pi$

$\therefore$ $\sin x = \frac{1}{\sqrt{2}}$

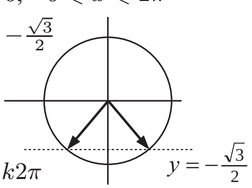


$\therefore$ $x = \left.\begin{matrix}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{matrix}\right\} + k2\pi$

$\therefore$ $x = \frac{\pi}{4},\ \frac{3\pi}{4},\ -\frac{7\pi}{4},\ -\frac{5\pi}{4}$

7 **a** $2\sin 3x + \sqrt{3} = 0$, $0 \leqslant x \leqslant 2\pi$

$\therefore$ $\sin 3x = -\frac{\sqrt{3}}{2}$



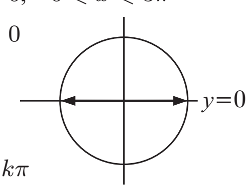
$\therefore$ $3x = \left.\begin{matrix}\frac{4\pi}{3}\\ \frac{5\pi}{3}\end{matrix}\right\} + k2\pi$

$\therefore$ $x = \left.\begin{matrix}\frac{4\pi}{9}\\ \frac{5\pi}{9}\end{matrix}\right\} + k\,\frac{2\pi}{3}$

$\therefore$ $x = \frac{4\pi}{9},\ \frac{10\pi}{9},\ \frac{16\pi}{9},\ \frac{5\pi}{9},\ \frac{11\pi}{9},\ \frac{17\pi}{9}$

b $\sqrt{2}\sin(x+\frac{\pi}{4}) = 0$, $0 \leqslant x \leqslant 3\pi$

$\therefore$ $\sin(x+\frac{\pi}{4}) = 0$



$\therefore$ $x+\frac{\pi}{4} = 0 + k\pi$

$\therefore$ $x = -\frac{\pi}{4} + k\pi$

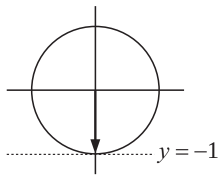
$\therefore$ $x = \frac{3\pi}{4},\ \frac{7\pi}{4},\ \frac{11\pi}{4}$

8 **a**
$$\sin^2 x - \sin x - 2 = 0$$
$$\therefore \quad (\sin x - 2)(\sin x + 1) = 0$$
$$\therefore \quad \sin x = 2 \text{ or } -1$$

But $\sin x$ values lie between -1 and 1 inclusive

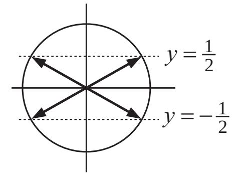
$\therefore \quad \sin x = -1$

$\therefore \quad x = \frac{3\pi}{2} + k2\pi$



b
$$4\sin^2 x = 1$$
$$\therefore \quad \sin^2 x = \tfrac{1}{4}$$
$$\therefore \quad \sin x = \pm\tfrac{1}{2}$$

$$\therefore \quad x = \left.\begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix}\right\} + k\pi$$



9 $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right), \quad 0 \leqslant t \leqslant 8$

a $P(0) = 5 + 2\sin 0$
$\quad = 5$

i.e., 5000 water beetles

b Smallest $P = 5 + 2(-1) = 3$

Largest $P = 5 + 2(1) = 7$

$\therefore$ smallest is 3000 water beetles
largest is 7000 water beetles

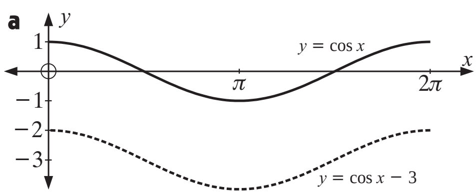
c If population is > 6000,
then $P(t) > 6$

$$\therefore \quad 5 + 2\sin\left(\tfrac{\pi t}{3}\right) > 6$$
$$\therefore \quad 2\sin\left(\tfrac{\pi t}{3}\right) > 1$$
$$\therefore \quad \sin\left(\tfrac{\pi t}{3}\right) > \tfrac{1}{2}$$

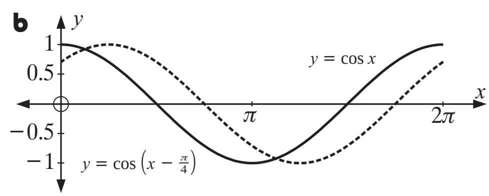
Using technology,
$0.5 < t < 2.5$ and $6.5 < t < 8$

REVIEW SET 13D

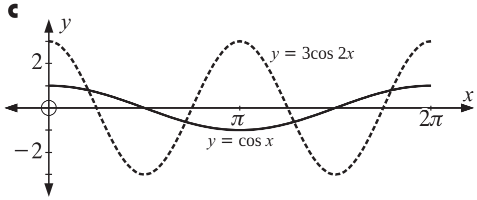
1 **a**



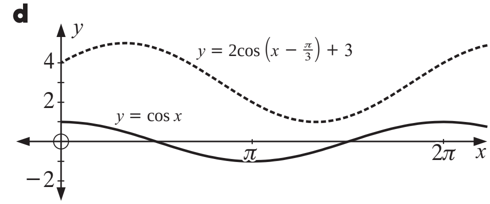
b



c



d

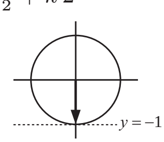


2 $P(t) = 40 + 12\sin\frac{2\pi}{7}\left(t - \frac{37}{12}\right) \ \text{m}^3$

a $P(t)$ is a minimum of $40 + 12(-1)$
$\quad = 28 \text{ mg/m}^3$

b when $\sin\frac{2\pi}{7}(t - \frac{37}{12}) = -1$

$$\therefore \quad \tfrac{2\pi}{7}(t - \tfrac{37}{12}) = \tfrac{3\pi}{2} + k2\pi$$
$$\therefore \quad \tfrac{2}{7}(t - \tfrac{37}{12}) = \tfrac{3}{2} + k\,2$$



So, $t - \frac{37}{12} = \frac{21}{4} + k\,7$

$\therefore \quad t = 8\frac{1}{3} + k\,7$

$\therefore \quad t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}$, etc.

$\therefore$ on Mondays at 8.00 am

$\{1\frac{1}{3}$ days after midnight Sat.$\}$

3 **a** If $y = A\cos B(t - C) + D$

then $A = -4, \quad \dfrac{2\pi}{B} = \pi$

$\therefore \quad B = 2$

$C = D = 0$

$\therefore \quad y = -4\cos 2x$

b If $y = A\cos B(x - C) + D$

then $A = 1, \quad \dfrac{2\pi}{B} = 8 \quad \therefore \quad B = \frac{\pi}{4}$

$$D = \frac{\text{max.} + \text{min.}}{2} = \frac{3 + 1}{2} = 2$$

$C = 0$

So, $y = \cos\left(\frac{\pi}{4}x\right) + 2$

4 **a** $\cos x = 0.4379, \quad 0 \leqslant x \leqslant 10$

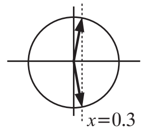
$\therefore \quad x \doteqdot 1.12,\ 5.17,\ 7.40$

b $\cos(x - 2.4) = -0.6014, \quad 0 \leqslant x \leqslant 6$

$\therefore \quad x \doteqdot 0.184,\ 4.62$

5 **a** $\cos 4x = 0.3, \quad \text{for all } x$

$\therefore \quad 4x = \cos^{-1}(0.3)$



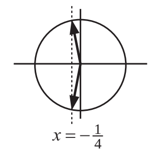
$$\therefore \quad 4x = \left.\begin{array}{c} 1.2661 \\ 2\pi - 1.2661 \end{array}\right\} + k2\pi$$

$$\therefore \quad x = \left.\begin{array}{c} 0.317 \\ 1.254 \end{array}\right\} + k(\tfrac{\pi}{2})$$

b $4\cos 2x + 1 = 0, \quad 0 \leqslant x \leqslant 5$

$\therefore \quad \cos 2x = -\frac{1}{4}$

$$\therefore \quad 2x = \left.\begin{array}{c} \pi - 1.318 \\ \pi + 1.318 \end{array}\right\} + k2\pi$$

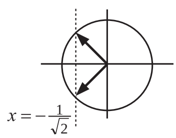


$$\therefore \quad x = \left.\begin{array}{c} 0.912 \\ 2.230 \end{array}\right\} + k\pi$$

$\therefore \quad x = 0.912,\ 4.05,\ 2.23$

6 **a** $\cos x = -\frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 4\pi$

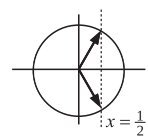
$$x = \left.\begin{array}{c} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{array}\right\} + k2\pi$$



$\therefore \quad x = \frac{3\pi}{4},\ \frac{11\pi}{4},\ \frac{5\pi}{4},\ \frac{13\pi}{4}$

b $\cos\left(x + \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

$$x + \tfrac{2\pi}{3} = \left.\begin{array}{c} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{array}\right\} + k2\pi$$



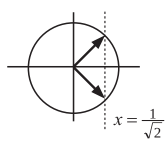
$$\therefore \quad x = \left.\begin{array}{c} -\frac{\pi}{3} \\ \pi \end{array}\right\} + k2\pi$$

$\therefore \quad x = -\frac{\pi}{3},\ \frac{5\pi}{3},\ \pi,\ -\pi$

7 **a** $\sqrt{2}\cos(x + \frac{\pi}{4}) - 1 = 0, \quad 0 \leqslant x \leqslant 4\pi$

$\therefore \quad \cos(x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\therefore \quad x + \tfrac{\pi}{4} = \left.\begin{array}{c} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{array}\right\} + k2\pi$$



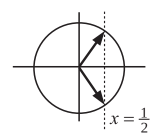
$$\therefore \quad x = \left.\begin{array}{c} 0 \\ \frac{3\pi}{2} \end{array}\right\} + k2\pi$$

$\therefore \quad x = 0,\ 2\pi,\ 4\pi,\ \frac{3\pi}{2},\ \frac{7\pi}{2}$

b $2\cos 2x - 1 = 0, \quad \text{for all } x$

$\therefore \quad \cos 2x = \frac{1}{2}$

$$\therefore \quad 2x = \left.\begin{array}{c} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{array}\right\} + k2\pi$$



$$\therefore \quad x = \left.\begin{array}{c} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{array}\right\} + k\pi$$

8 **a**
$$\begin{aligned} &\cos^3\theta + \sin^2\theta\cos\theta \\ &= \cos\theta(\cos^2\theta + \sin^2\theta) \\ &= \cos\theta(1) \\ &= \cos\theta \end{aligned}$$

b
$$\begin{aligned} &\frac{\cos^2\theta - 1}{\sin\theta} \\ &= \frac{-(1 - \cos^2\theta)}{\sin\theta} \\ &= -\frac{\sin^2\theta}{\sin\theta} \\ &= -\sin\theta \end{aligned}$$

c
$$\begin{aligned} &3\cos\theta - \cos\theta \\ &= 2\cos\theta \end{aligned}$$

d
$$\begin{aligned} &5 - 5\sin^2\theta \\ &= 5(1 - \sin^2\theta) \\ &= 5\cos^2\theta \end{aligned}$$

e
$$\begin{aligned} &\frac{\sin^2\theta - 1}{\cos\theta} \\ &= -\frac{(1 - \sin^2\theta)}{\cos\theta} \\ &= -\frac{\cos^2\theta}{\cos\theta} \\ &= -\cos\theta \end{aligned}$$

9 **a**
$$\begin{aligned}&(2\sin\alpha - 1)^2\\ &= 4\sin^2\alpha - 4\sin\alpha + 1\end{aligned}$$

b
$$\begin{aligned}&(\cos\alpha - \sin\alpha)^2\\ &= \cos^2\alpha - 2\sin\alpha\cos\alpha + \sin^2\alpha\\ &= \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha\\ &= 1 - \sin 2\alpha\end{aligned}$$

REVIEW SET 13E

1 **a**
$$\begin{aligned}&\frac{1-\cos^2\theta}{1+\cos\theta}\\ &= \frac{{}^{1}\cancel{(1+\cos\theta)}(1-\cos\theta)}{\cancel{1+\cos\theta}_{\,1}}\\ &= 1-\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}\\ &= \frac{\cancel{\sin\alpha - \cos\alpha}^{\,1}}{(\sin\alpha + \cos\alpha)\cancel{(\sin\alpha - \cos\alpha)}_{\,1}}\\ &= \frac{1}{\sin\alpha + \cos\alpha}\end{aligned}$$

c
$$\begin{aligned}&\frac{4\sin^2\alpha - 4}{8\cos\alpha}\\ &= \frac{-4(1-\sin^2\alpha)}{8\cos\alpha}\\ &= \frac{-4\cos^2\alpha}{8\cos\alpha}\\ &= -\tfrac{1}{2}\cos\alpha\end{aligned}$$

2 **a**
$$\begin{aligned}&\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta}\\ &= \frac{\cos^2\theta + (1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}\\ &= \frac{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}{(1+\sin\theta)\cos\theta}\\ &= \frac{2+2\sin\theta}{(1+\sin\theta)\cos\theta} \qquad \{\cos^2\theta + \sin^2\theta = 1\}\\ &= \frac{2\cancel{(1+\sin\theta)}^{\,1}}{{}_{1}\cancel{(1+\sin\theta)}\cos\theta}\\ &= \frac{2}{\cos\theta}\end{aligned}$$

b
$$\begin{aligned}&\left(1+\frac{1}{\cos\theta}\right)\left(\cos\theta - \cos^2\theta\right)\\ &= \cancel{\cos\theta} - \cos^2\theta + 1 - \cancel{\cos\theta}\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

3 **a**
$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2(\tfrac{5}{13})(\tfrac{12}{13})\\ &= \tfrac{120}{169}\end{aligned}$$

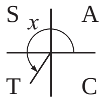
b
$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= (\tfrac{12}{13})^2 - (\tfrac{5}{13})^2\\ &= \tfrac{144-25}{169}\\ &= \tfrac{119}{169}\end{aligned}$$

4

$$\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1\\ \therefore\quad \cos^2\alpha + \tfrac{9}{16} &= 1\\ \therefore\quad \cos^2\alpha &= \tfrac{7}{16}\\ \therefore\quad \cos\alpha &= \pm\tfrac{\sqrt{7}}{4}\\ \text{But in Q3,}\quad \cos\alpha &< 0\\ \therefore\quad \cos\alpha &= -\tfrac{\sqrt{7}}{4}\end{aligned}$$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2(-\tfrac{3}{4})(-\tfrac{\sqrt{7}}{4})\\ &= \tfrac{3\sqrt{7}}{8}\end{aligned}$$

5



$$\begin{aligned}
\cos 2A &= 1 - 2\sin^2 A \\
\therefore\ \cos x &= 1 - 2\sin^2\left(\tfrac{x}{2}\right) \quad \{\text{letting } 2A = x,\ A = \tfrac{x}{2}\} \\
\therefore\ -\tfrac{3}{4} &= 1 - 2\sin^2\left(\tfrac{x}{2}\right) \\
\therefore\ 2\sin^2\left(\tfrac{x}{2}\right) &= \tfrac{7}{4} \\
\therefore\ \sin^2\left(\tfrac{x}{2}\right) &= \tfrac{7}{8} \\
\therefore\ \sin\left(\tfrac{x}{2}\right) &= \pm\tfrac{\sqrt{7}}{2\sqrt{2}}
\end{aligned}$$

But $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$ (in Q2)

$$\therefore\ \sin\left(\tfrac{x}{2}\right) = \tfrac{\sqrt{7}}{2\sqrt{2}}$$

6 **a** **i** $\tan x = 4$
$\therefore\ x \doteqdot 1.326 + k\pi$
i.e., $x \doteqdot 1.33 + k\pi$

ii $\tan\left(\frac{x}{4}\right) = 4$
$\therefore\ \frac{x}{4} \doteqdot 1.326 + k\pi$
$\therefore\ x \doteqdot 5.30 + k4\pi$

iii $\tan(x - 1.5) = 4$
$\therefore\ x - 1.5 \doteqdot 1.326 + k2\pi$
$\therefore\ x \doteqdot 2.83 + k\pi$

b **i** $\tan(x + \frac{\pi}{6}) = -\sqrt{3}$
$\therefore\ x + \frac{\pi}{6} = \frac{2\pi}{3} + k\pi$
$\therefore\ x = \frac{\pi}{2} + k\pi$

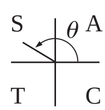
ii $\tan 2x = -\sqrt{3}$
$\therefore\ 2x = \frac{2\pi}{3} + k\pi$
$\therefore\ x = \frac{\pi}{3} + \frac{k\pi}{2}$

iii $\tan^2 x - 3 = 0$
$\therefore\ \tan x = \pm\sqrt{3}$
$\therefore\ x = \left.\begin{matrix}\frac{\pi}{3} \\ \frac{2\pi}{3}\end{matrix}\right\} + k\pi$

c

$$\begin{aligned}
3\tan(x - 1.2) &= -2 \\
\therefore\ \tan(x - 1.2) &= -\tfrac{2}{3} \\
\therefore\ x - 1.2 &\doteqdot -0.588 + k\pi \\
\therefore\ x &\doteqdot 0.612 + k\pi
\end{aligned}$$

7 $\tan\theta = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$



$$\begin{aligned}
\therefore\ \frac{\sin\theta}{\cos\theta} &= -\frac{2}{3} \\
\therefore\ \sin\theta &= -2k, \quad \cos\theta = 3k \\
\text{but}\quad \cos^2\theta + \sin^2\theta &= 1 \\
\therefore\ 9k^2 + 4k^2 &= 1 \\
\therefore\ 13k^2 &= 1 \\
\therefore\ k &= \pm\tfrac{1}{\sqrt{13}}
\end{aligned}$$

But in Q2,
$\sin\theta > 0$, $\cos\theta < 0$
$\therefore\ k = -\frac{1}{\sqrt{13}}$
$\therefore\ \sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = -\frac{3}{\sqrt{13}}$

8

$$\begin{aligned}
\frac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1} &= \frac{2\sin\alpha\cos\alpha - \sin\alpha}{2\cos^2\alpha - 1 - \cos\alpha + 1} \\
&= \frac{\sin\alpha\cancelto{1}{(2\cos\alpha - 1)}}{\cos\alpha\cancelto{1}{(2\cos\alpha - 1)}} \\
&= \frac{\sin\alpha}{\cos\alpha} \\
&= \tan\alpha
\end{aligned}$$

Chapter 14
MATRICES

EXERCISE 14A

1 **a** 1 row and 4 columns $\therefore$ 1×4 **b** 2 rows and 1 column $\therefore$ 2×1
c 2 rows and 2 columns $\therefore$ 2×2 **d** 3 rows and 3 columns $\therefore$ 3×3

2 $\begin{bmatrix} 2 & 1 & 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{bmatrix}$ **c** $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.15) + (1 \times 0.95)$ represents the total cost of the groceries.

3

$$\begin{array}{cl} \begin{array}{ccc} 200\text{ g} & 300\text{ g} & 500\text{ g} \end{array} & \\ \begin{bmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{bmatrix} & \begin{array}{l} \text{week 1} \\ \text{week 2} \\ \text{week 3} \\ \text{week 4} \end{array} \end{array}$$

4

$$\begin{array}{cl} \begin{array}{cccc} \text{pies} & \text{pasties} & \text{rolls} & \text{buns} \end{array} & \\ \begin{bmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{bmatrix} & \begin{array}{l} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \\ \text{Monday} \end{array} \end{array}$$

EXERCISE 14B

1 **a** $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$$

b $\mathbf{A} + \mathbf{B} + \mathbf{C}$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$$

c $\mathbf{B} + \mathbf{C}$

$$= \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -6 & -1 \end{bmatrix}$$

d $\mathbf{C} + \mathbf{B} - \mathbf{A}$

$$= \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -11 & -3 \end{bmatrix}$$

2 **a** $\mathbf{P} + \mathbf{Q}$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{bmatrix}$$

b $\mathbf{P} - \mathbf{Q}$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} - \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{bmatrix}$$

c $$\mathbf{Q} - \mathbf{P} = \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{bmatrix}$$

3 **a**

$$\begin{array}{cc} \text{Friday} & \text{Saturday} \\ \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} & \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} \end{array}$$

b Total for Friday and Saturday $= \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} + \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} = \begin{bmatrix} 187 \\ 229 \\ 101 \end{bmatrix}$

4 **a** **i** Cost price $\begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix}$ **ii** Selling price $\begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix}$

b In order to find David's profit/loss matrix we subtract the cost price matrix from the selling price matrix.

c Profit/Loss matrix $= \begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix} - \begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{bmatrix}$

5 **a** **i** $\begin{array}{cc} \text{Lou} & \text{Rose} \end{array}$ $\begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix}$ **ii** $\begin{array}{cc} \text{Lou} & \text{Rose} \end{array}$ $\begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix}$

iii Total sales for November and December

$$= \begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} + \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} = \begin{bmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{bmatrix}$$

b If **F** represents the furniture in one flat, 12**F** represents the furniture in all 12 flats.

6 **a** $\begin{bmatrix} x & x^2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} y & 4 \\ 3 & y+1 \end{bmatrix}$

Equating corresponding elements:

$x = y$, $x^2 = 4$ and $-1 = y+1$

$\therefore$ $y = -2$ and $x = \pm 2$

But $x = y$ $\therefore$ $x = y = -2$

b $\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} -y & x \\ x & -y \end{bmatrix}$

Equating corresponding elements:

$$\left.\begin{aligned} x &= -y \\ y &= x \end{aligned}\right\} \quad \therefore \quad y = 0,\ x = 0$$

7 **a**

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2+(-1) & 1+2 \\ 3+2 & -1+3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{B}+\mathbf{A} &= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1+2 & 2+1 \\ 2+3 & 3+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \end{aligned}$$

b $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ for all 2×2 matrices **A** and **B** because addition of numbers is commutative.

8 **a**

$$\begin{aligned} (\mathbf{A}+\mathbf{B})+\mathbf{C} &= \left(\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}\right) + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}+(\mathbf{B}+\mathbf{C}) &= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \left(\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

b Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$\therefore$ $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$

$$= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) + \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$= \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix}$$

$\mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix}$$

$$= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix}$$

$$= (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

EXERCISE 14C

1 **a** $2\mathbf{B}$

$$= 2\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix}$$

b $\frac{1}{3}\mathbf{B}$

$$= \frac{1}{3}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix}$$

c $\frac{1}{12}\mathbf{B}$

$$= \frac{1}{12}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$$

d $-\frac{1}{2}\mathbf{B}$

$$= -\frac{1}{2}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ -12 & -3 \end{bmatrix}$$

2 **a** $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3+2 & 5+1 \\ 1+1 & 6+2 & 4+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{bmatrix}$$

b $\mathbf{A} - \mathbf{B}$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3-2 & 5-1 \\ 1-1 & 6-2 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix}$$

c $2\mathbf{A} + \mathbf{B}$

$$= 2\begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 10 \\ 2 & 12 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 6+2 & 10+1 \\ 2+1 & 12+2 & 8+3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{bmatrix}$$

d $3\mathbf{A} - \mathbf{B}$

$$= 3\begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 15 \\ 3 & 18 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-1 & 9-2 & 15-1 \\ 3-1 & 18-2 & 12-3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{bmatrix}$$

3 **a**

$$2\mathbf{H} = 2\begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 24 \\ 120 \\ 60 \end{bmatrix}$$

b

$$\frac{1}{2}\mathbf{H} = \frac{1}{2}\begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \\ 30 \\ 15 \end{bmatrix}$$

c

$$150\% \text{ of } \mathbf{H} = 1.5\begin{bmatrix} 6 \\ 12 \\ 60 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 18 \\ 90 \\ 45 \end{bmatrix}$$

4 **a** Increase of 15% $= 1.15\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{bmatrix}$ rounded to the nearest whole number.

b Decrease of 15% $= 0.85\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{bmatrix}$ rounded to the nearest whole number.

5 **a** Weekdays Weekends

$$\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} \quad \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} \quad \begin{matrix} \text{VHS} \\ \text{DVD} \\ \text{games} \end{matrix}$$

b $\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} + \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} = \begin{bmatrix} 211 \\ 70 \\ 231 \end{bmatrix}$

c The sum matrix of **b** represents total weekly average hirings.

6 The matrix is $12\mathbf{F} = 12\begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 48 \\ 24 \\ 12 \end{bmatrix}$

EXERCISE 14D

1 **a** $\mathbf{A} + 2\mathbf{A} = 3\mathbf{A}$ **b** $3\mathbf{B} - 3\mathbf{B} = \mathbf{O}$ **c** $\mathbf{C} - 2\mathbf{C} = -\mathbf{C}$

d $-\mathbf{B} + \mathbf{B} = \mathbf{O}$ **e** $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$ **f** $-(\mathbf{A} + \mathbf{B}) = -\mathbf{A} - \mathbf{B}$

g $-(2\mathbf{A} - \mathbf{C})$
$= -2\mathbf{A} + \mathbf{C}$

h $3\mathbf{A} - (\mathbf{B} - \mathbf{A})$
$= 3\mathbf{A} - \mathbf{B} + \mathbf{A}$
$= 4\mathbf{A} - \mathbf{B}$

i $\mathbf{A} + 2\mathbf{B} - (\mathbf{A} - \mathbf{B})$
$= \mathbf{A} + 2\mathbf{B} - \mathbf{A} + \mathbf{B}$
$= 3\mathbf{B}$

2 **a**
$$\begin{aligned} \text{if} \quad \mathbf{X} + \mathbf{B} &= \mathbf{A} \\ \text{then} \quad \mathbf{X} + \mathbf{B} + (-\mathbf{B}) &= \mathbf{A} + (-\mathbf{B}) \\ \therefore \quad \mathbf{X} + \mathbf{O} &= \mathbf{A} - \mathbf{B} \\ \therefore \quad \mathbf{X} &= \mathbf{A} - \mathbf{B} \end{aligned}$$

b
$$\begin{aligned} \text{if} \quad \mathbf{B} + \mathbf{X} &= \mathbf{C} \\ \text{then} \quad \mathbf{B} + \mathbf{X} + (-\mathbf{B}) &= \mathbf{C} + (-\mathbf{B}) \\ \therefore \quad \mathbf{O} + \mathbf{X} &= \mathbf{C} - \mathbf{B} \\ \therefore \quad \mathbf{X} &= \mathbf{C} - \mathbf{B} \end{aligned}$$

c
$$\begin{aligned} \text{if} \quad 4\mathbf{B} + \mathbf{X} &= 2\mathbf{C} \\ \text{then} \quad 4\mathbf{B} + \mathbf{X} + (-4\mathbf{B}) &= 2\mathbf{C} + (-4\mathbf{B}) \\ \therefore \quad \mathbf{O} + \mathbf{X} &= 2\mathbf{C} - 4\mathbf{B} \\ \therefore \quad \mathbf{X} &= 2\mathbf{C} - 4\mathbf{B} \end{aligned}$$

d
$$\begin{aligned} \text{if} \quad 2\mathbf{X} &= \mathbf{A} \\ \text{then} \quad \tfrac{1}{2}(2\mathbf{X}) &= \tfrac{1}{2}\mathbf{A} \\ \therefore \quad 1\mathbf{X} &= \tfrac{1}{2}\mathbf{A} \quad \text{i.e.,} \quad \mathbf{X} = \tfrac{1}{2}\mathbf{A} \end{aligned}$$

e
$$\begin{aligned} \text{if} \quad 3\mathbf{X} &= \mathbf{B} \\ \text{then} \quad \tfrac{1}{3}(3\mathbf{X}) &= \tfrac{1}{3}\mathbf{B} \\ \therefore \quad 1\mathbf{X} &= \tfrac{1}{3}\mathbf{B} \quad \text{i.e.,} \quad \mathbf{X} = \tfrac{1}{3}\mathbf{B} \end{aligned}$$

f
$$\begin{aligned} \text{if} \quad \mathbf{A} - \mathbf{X} &= \mathbf{B} \\ \text{then} \quad \mathbf{A} - \mathbf{X} + \mathbf{X} &= \mathbf{B} + \mathbf{X} \\ \therefore \quad \mathbf{A} + \mathbf{O} &= \mathbf{B} + \mathbf{X} \\ \therefore \quad \mathbf{A} &= \mathbf{B} + \mathbf{X} \\ \text{and} \quad \mathbf{A} + (-\mathbf{B}) &= \mathbf{B} + \mathbf{X} + (-\mathbf{B}) \\ \therefore \quad \mathbf{A} - \mathbf{B} &= \mathbf{X} + \mathbf{O} \\ \therefore \quad \mathbf{A} - \mathbf{B} &= \mathbf{X} \\ \therefore \quad \mathbf{X} &= \mathbf{A} - \mathbf{B} \end{aligned}$$

g
$$\begin{aligned} \text{if} \quad \tfrac{1}{2}\mathbf{X} &= \mathbf{C} \\ \text{then} \quad 2(\tfrac{1}{2}\mathbf{X}) &= 2\mathbf{C} \\ \therefore \quad 1\mathbf{X} &= 2\mathbf{C} \quad \text{i.e.,} \quad \mathbf{X} = 2\mathbf{C} \end{aligned}$$

h
$$\begin{aligned} \text{if} \quad 2(\mathbf{X} + \mathbf{A}) &= \mathbf{B} \\ \text{then} \quad \tfrac{1}{2}\,[2(\mathbf{X} - \mathbf{A})] &= \tfrac{1}{2}\mathbf{B} \\ \therefore \quad 1(\mathbf{X} + \mathbf{A}) &= \tfrac{1}{2}\mathbf{B} \\ \therefore \quad \mathbf{X} + \mathbf{A} &= \tfrac{1}{2}\mathbf{B} \\ \therefore \quad \mathbf{X} + \mathbf{A} + (-\mathbf{A}) &= \tfrac{1}{2}\mathbf{B} + (-\mathbf{A}) \\ \therefore \quad \mathbf{X} + \mathbf{O} &= \tfrac{1}{2}\mathbf{B} - \mathbf{A} \\ \therefore \quad \mathbf{X} &= \tfrac{1}{2}\mathbf{B} - \mathbf{A} \end{aligned}$$

i
$$\begin{aligned} \text{if} \quad \mathbf{A} - 4\mathbf{X} &= \mathbf{C} \\ \text{then} \quad \mathbf{A} - 4\mathbf{X} + 4\mathbf{X} &= \mathbf{C} + 4\mathbf{X} \\ \therefore \quad \mathbf{A} + \mathbf{O} &= \mathbf{C} + 4\mathbf{X} \\ \therefore \quad \mathbf{A} &= \mathbf{C} + 4\mathbf{X} \\ \text{and} \quad \mathbf{A} - \mathbf{C} &= 4\mathbf{X} \\ \therefore \quad \tfrac{1}{4}(\mathbf{A} - \mathbf{C}) &= \tfrac{1}{4}(4\mathbf{X}) \\ \therefore \quad \mathbf{X} &= \tfrac{1}{4}(\mathbf{A} - \mathbf{C}) \end{aligned}$$

3 **a**

$$\text{if } \tfrac{1}{3}\mathbf{X} = \mathbf{M}$$
$$\text{then } 3\left(\tfrac{1}{3}\mathbf{X}\right) = 3\mathbf{M}$$
$$\therefore \mathbf{X} = 3\mathbf{M} = 3\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix}$$

b if $4\mathbf{X} = \mathbf{N}$ then $\frac{1}{4}(4\mathbf{X}) = \frac{1}{4}\mathbf{N}$

$$\therefore \mathbf{X} = \tfrac{1}{4}\mathbf{N}$$
$$\therefore \mathbf{X} = \tfrac{1}{4}\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$$

c

$$\text{if } \mathbf{A} - 2\mathbf{X} = 3\mathbf{B}$$
$$\text{then } \mathbf{A} - 2\mathbf{X} + 2\mathbf{X} = 3\mathbf{B} + 2\mathbf{X}$$
$$\therefore \mathbf{A} = 3\mathbf{B} - 2\mathbf{X}$$
$$\therefore \mathbf{A} + (-3\mathbf{B}) = 3\mathbf{B} - 2\mathbf{X} + (-3\mathbf{B})$$
$$\therefore \mathbf{A} - 3\mathbf{B} = 2\mathbf{X}$$
$$\therefore \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \tfrac{1}{2}(2\mathbf{X})$$
$$\therefore \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B}) = 1\mathbf{X}$$
$$\therefore \mathbf{X} = \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B})$$
$$= \tfrac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\right)$$
$$= \tfrac{1}{2}\begin{bmatrix} -2 & -12 \\ 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

EXERCISE 14E.1

1 **a**

$$\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$= [3 \times 5 + (-1) \times 4]$$
$$= [15 - 4]$$
$$= [11]$$

b

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$$
$$= [1 \times 5 + 3 \times 1 + 2 \times 7]$$
$$= [5 + 3 + 14]$$
$$= [22]$$

c

$$\begin{bmatrix} 6 & -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = [6 \times 1 + -1 \times 0 + 2 \times -1 + 3 \times 4]$$
$$= [6 + 0 - 2 + 12]$$
$$= [16]$$

2

$$\begin{bmatrix} w & x & y & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} w + x + y + z \end{bmatrix}$$

$\therefore$ $\frac{1}{4}(w + x + y + z)$, which is the average of w, x, y and z, can be represented as $\begin{bmatrix} w & x & y & z \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

3 **a** $\mathbf{Q} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ $\mathbf{P} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix}$

b total cost $= \mathbf{PQ} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$$= [27 \times 4 + 35 \times 3 + 39 \times 2]$$
$$= [291] \quad \therefore \text{ total cost is \$291}$$

4 **a** $\mathbf{P} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix}$

$\mathbf{N} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$

b total points $= \mathbf{PN} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$

$$= [10 \times 3 + 6 \times 2 + 3 \times 4 + 1 \times 2]$$
$$= [30 + 12 + 12 + 2]$$
$$= [56]$$ So, the number of points awarded is 56.

EXERCISE 14E.2

1 $\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$ which is 1 row $\times$ 3 columns

$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ which is 2 rows $\times$ 3 columns

AB cannot be found because the number of columns in **A** does not equal the number of rows in **B**.

2 **A** is $2 \times n$ and **B** is $m \times 3$.

a We can find **AB** if the number of columns in **A** equals the number of rows in **B**, i.e., $n = m$.

b If **AB** can be found its order is 2×3.

c **BA** cannot be found because the number of columns in **B** does not equal the number of rows in **A**.

3 **a** **B** is 1×2 and **A** is 2×2

$\therefore$ **BA** is 1×2

$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 2 + 6 \times 3 & 5 \times 1 + 6 \times 4 \end{bmatrix} \\ &= \begin{bmatrix} 10 + 18 & 5 + 24 \end{bmatrix} \\ &= \begin{bmatrix} 28 & 29 \end{bmatrix} \end{aligned}$$

b **i** **A** is 1×3 and **B** is 3×1, $\therefore$ **AB** is 1×1 and

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = [2 \times 1 + 0 \times 4 + 3 \times 2] = [2 + 0 + 6] = [8]$$

ii **B** is 3×1 and **A** is 1×3, $\therefore$ **BA** is 3×3 and

$$\mathbf{BA} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 & 1 \times 0 & 1 \times 3 \\ 4 \times 2 & 4 \times 0 & 4 \times 3 \\ 2 \times 2 & 2 \times 0 & 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{bmatrix}$$

4 **a** $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ is 1×3 by 3×3 $\therefore$ resultant matrix is 1×3

$$\begin{aligned} &= \begin{bmatrix} 1 \times 2 + 2 \times 0 + 1 \times 1 & 1 \times 3 + 2 \times 1 + 1 \times 0 & 1 \times 1 + 2 \times 0 + 1 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 0 + 1 & 3 + 2 + 0 & 1 + 0 + 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 & 3 \end{bmatrix} \end{aligned}$$

b $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is 3×3 by 3×1 $\therefore$ resultant matrix is 3×1

$$\begin{aligned} &= \begin{bmatrix} 1 \times 2 + 0 \times 3 + (-1) \times 4 \\ (-1) \times 2 + 1 \times 3 + 0 \times 4 \\ 0 \times 2 + (-1) \times 3 + 1 \times 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 0 - 4 \\ -2 + 3 + 0 \\ 0 - 3 + 4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

5 **a**

$$\mathbf{C} = \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} \qquad \mathbf{N} = \begin{array}{c} \begin{array}{cc} \text{adults} & \text{children} \end{array} \\ \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{first day} \\ \text{second day} \end{array}$$

b $\mathbf{N}$ is 2×2 and $\mathbf{C}$ is 2×1

$\therefore$ $\mathbf{NC}$ is 2×1

$$\begin{aligned} \mathbf{NC} &= \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix} \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} \\ &= \begin{bmatrix} 2375 \times 12.5 + 5156 \times 9.5 \\ 2502 \times 12.5 + 3612 \times 9.5 \end{bmatrix} \\ &= \begin{bmatrix} 29\,687.5 + 48\,982 \\ 31\,275 + 34\,314 \end{bmatrix} \\ &= \begin{bmatrix} 78\,669.5 \\ 65\,589 \end{bmatrix} \begin{array}{l} \text{income from day 1} \\ \text{income from day 2} \end{array} \end{aligned}$$

c Total income $= \$78\,669.50 + \$65\,589 = \$144\,258.50$

6 **a**

$$\mathbf{R} = \begin{array}{c} \begin{array}{cc} \text{me} & \text{friend} \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{hammers} \\ \text{screwdrivers} \\ \text{cans of paint} \end{array}$$

b

$$\mathbf{P} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{array}{l} \text{store A} \\ \text{store B} \end{array}$$

c $\mathbf{P}$ is 2×3 and $\mathbf{R}$ is 3×2, $\therefore$ $\mathbf{PR}$ is 2×2

$$\mathbf{PR} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 \times 1 + 3 \times 1 + 19 \times 2 & 7 \times 1 + 3 \times 2 + 19 \times 3 \\ 6 \times 1 + 2 \times 1 + 22 \times 2 & 6 \times 1 + 2 \times 2 + 22 \times 3 \end{bmatrix}$$

$$\therefore \ \mathbf{PR} = \begin{bmatrix} 7+3+38 & 7+6+57 \\ 6+2+44 & 6+4+66 \end{bmatrix} = \begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix}$$

d My costs at Store A are \$48; my friend's costs at Store B are \$76.

e My costs at Store B are \$52. Therefore I should shop at Store A, which is cheaper.

7

$$\text{Fruit matrix } \mathbf{F} = \begin{array}{c} \begin{array}{ccc} \text{apples} & \text{bananas} & \text{oranges} \end{array} \\ \begin{bmatrix} 6 & 7 & 9 \\ 5 & 8 & 4 \\ 4 & 7 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{first day} \\ \text{second day} \\ \text{third day} \end{array} \qquad \text{cost matrix } \mathbf{C} = \begin{bmatrix} 18 \\ 15 \\ 13 \end{bmatrix} \begin{array}{l} \text{apples} \\ \text{bananas} \\ \text{oranges} \end{array}$$

The total cost of the three types of fruit over the three-day period is represented by $\mathbf{FC}$.

$\mathbf{F}$ is 3×3 and

$\mathbf{C}$ is 3×1,

$\therefore$ $\mathbf{FC}$ is 3×1.

$$\mathbf{FC} = \begin{bmatrix} 6 & 7 & 9 \\ 5 & 8 & 4 \\ 4 & 7 & 2 \end{bmatrix} \begin{bmatrix} 18 \\ 15 \\ 13 \end{bmatrix} = \begin{bmatrix} 6 \times 18 + 7 \times 15 + 9 \times 13 \\ 5 \times 18 + 8 \times 15 + 4 \times 13 \\ 4 \times 18 + 7 \times 15 + 2 \times 13 \end{bmatrix}$$

$$\therefore \ \mathbf{FC} = \begin{bmatrix} 108 + 105 + 117 \\ 90 + 120 + 52 \\ 72 + 105 + 26 \end{bmatrix} = \begin{bmatrix} 330 \\ 262 \\ 203 \end{bmatrix}$$

$\therefore$ total cost $= \$330 + \$262 + \$203 = \$795$

EXERCISE 14F

1 **a** $\begin{bmatrix} 16 & 18 & 15 \\ 13 & 21 & 16 \\ 10 & 22 & 24 \end{bmatrix}$

b $\begin{bmatrix} 10 & 6 & -7 \\ 9 & 3 & 0 \\ 4 & -4 & -10 \end{bmatrix}$

c $\begin{bmatrix} 22 & 0 & 132 & 176 & 198 \\ 44 & 154 & 88 & 110 & 0 \\ 176 & 44 & 88 & 88 & 132 \end{bmatrix}$ **d** $\begin{bmatrix} 115 \\ 136 \\ 46 \\ 106 \end{bmatrix}$

2 **a**

$$\text{Numbers matrix } \mathbf{N} = \begin{array}{c} \begin{array}{ccc} \text{nights} & \text{breakfasts} & \text{dinners} \end{array} \\ \begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \end{array}$$

b

$$\text{Prices matrix } \quad \mathbf{P} = \begin{array}{cc} \begin{array}{ccc} \text{Bay View} & \text{Terrace} & \text{Staunton Star} \end{array} & \\ \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} & \begin{array}{l} \text{room} \\ \text{breakfast} \\ \text{dinner} \end{array} \end{array}$$

c Total prices for each venue $=$ numbers matrix $\times$ prices matrix $= \mathbf{NP}$

$\mathbf{N}$ is 1×3 and $\mathbf{P}$ is 3×3, $\therefore$ $\mathbf{NP}$ is 1×3

$$\mathbf{NP} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ \$657 for Bay View, \$730 for Terrace, \$670 for Staunton Star.

d

$$\text{Total prices} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 369 & 420 & 385 \end{bmatrix}, \quad \{\text{using technology}\}$$

i.e., \$369 for Bay View, \$420 for Terrace, \$385 for Staunton Star.

e To include both scenarios we calculate

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \\ 369 & 420 & 385 \end{bmatrix}, \quad \text{using technology}$$

3

$$\text{Prices matrix} = \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix}$$

To find total income we calculate:

prices matrix $\times$ numbers matrix.

$$= \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix} \begin{bmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{bmatrix}$$

$$= \begin{bmatrix} 51\,145 \\ 60\,655 \\ 61\,575 \\ 51\,285 \end{bmatrix} \quad \text{using technology}$$

$\therefore$ total income $= \$51\,145 + \$60\,655 + \$61\,575 + \$51\,285 = \$224\,660$

4

$$\text{Numbers matrix } \mathbf{N} = \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix}$$

$$\text{Cost price matrix } \mathbf{C} = \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix}$$

$$\text{Selling price matrix } \mathbf{S} = \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix}$$

$$\begin{aligned}\text{Profit per day} &= (\text{selling price per drink}) \times (\text{number of drinks sold}) \\ &\quad - (\text{cost price per drink}) \times (\text{number of drinks sold}) \\ &= \mathbf{SN} - \mathbf{CN} \\ &= \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix} \\ &\quad - \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix} \\ &= \begin{bmatrix} 515.35 & 421.30 & 414.55 & 575.45 & 702.05 & 769.40 \end{bmatrix} \quad \text{using technology}\end{aligned}$$

$\therefore$ the profit for the week $= \$515.35 + ... + \$769.40 = \$3398.10$

A quicker way to solve this problem is to create a 'profit/loss per drink' matrix,

i.e., a selling price matrix $-$ cost price matrix

$$\begin{aligned} &= \mathbf{S} - \mathbf{C} \\ &= \begin{bmatrix} 2.55 & 4.40 & 3.50 & 1.80 \end{bmatrix} - \begin{bmatrix} 1.95 & 2.10 & 1.45 & 0.95 \end{bmatrix} \\ &= \begin{bmatrix} 0.60 & 2.30 & 2.05 & 0.85 \end{bmatrix}\end{aligned}$$

We then calculate $\begin{bmatrix} 0.60 & 2.30 & 2.05 & 0.85 \end{bmatrix} \begin{bmatrix} 225 & 195 & 215 & 240 & 352 & 321 \\ 75 & 62 & 50 & 92 & 80 & 97 \\ 62 & 54 & 55 & 72 & 102 & 112 \\ 95 & 60 & 68 & 85 & 115 & 146 \end{bmatrix}$

Using technology the result checks with the result obtained above.

5 **a** Income matrix $\mathbf{I} = \begin{bmatrix} 125 & 195 & 225 \end{bmatrix}$

Cost matrix $\mathbf{C} = \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$

Numbers (bookings) matrix $\mathbf{N} = \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$

$$\begin{aligned}\text{profit per day} &= (\text{income from room}) \times (\text{bookings per day}) \\ &\quad - (\text{maintenance cost per room}) \times (\text{bookings per day}) \\ &= \mathbf{IN} - \mathbf{CN} \\ &= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1185 & 800 & 1350 & 970 & 845 & 1130 & 845 \end{bmatrix}, \text{ using technology}\end{aligned}$$

$\therefore$ profit for the week $= \$1185 + \$800 + \$1350 + \$970 + \$845 + \$1130 + \$845$
$= \$7125$

b If the hotel maintained every room every day we would need to calculate

(income from room) × (bookings per day) − (maintenance costs per room) × (number of rooms)

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \times \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \times \begin{bmatrix} 20 & 20 & 20 & 20 & 20 & 20 & 20 \\ 15 & 15 & 15 & 15 & 15 & 15 & 15 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$\therefore$ using technology, profit per day $= \begin{bmatrix} -820 & -1840 & -455 & -1485 & -1725 & -920 & -1785 \end{bmatrix}$

$\therefore$ the profit per week would be $(-\$820) + \ldots.. + (-\$1785) = -\$9030$, i.e., a loss of \$9030.

c Profit per room matrix $=$ income per room matrix − cost per room matrix

$$= \mathbf{I} - \mathbf{C}$$

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 75 & 95 \end{bmatrix}$$

$\therefore$ the result in **a** can be obtained if we calculate:

$$\begin{bmatrix} 40 & 75 & 95 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

This checks using technology.

EXERCISE 14G

1 $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1+0 & 1+0 \\ -1+0 & 1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 7 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 0+2 \\ 0+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 6 \end{bmatrix}$

$\mathbf{AB} \neq \mathbf{BA}$ $\therefore$ in the general case **AB** does not necessarily equal **BA**.

2 $\mathbf{AO} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

$\mathbf{OA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$ $\therefore$ $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$

3 **a** Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, say

A(B + C)

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 4+4 \\ 9+8 & 12+8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix}$$

AB + AC

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{bmatrix} + \begin{bmatrix} 2+2 & 3+2 \\ 6+4 & 9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 10 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix}$$

$$= \mathbf{A(B + C)}$$

b $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} = \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix}$$

$$\begin{aligned}\mathbf{AB}+\mathbf{AC} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}+\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}\\ &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}+\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}\\ &= \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix} = \mathbf{A}(\mathbf{B}+\mathbf{C})\end{aligned}$$

c Using the matrices in **a**,

$$\begin{aligned}(\mathbf{AB})\mathbf{C} &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} 6+3 & 9+3 \\ 14+7 & 21+7 \end{bmatrix}\\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\right)\\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2+1 & 3+1 \\ 2+1 & 3+1 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\\ &= \begin{bmatrix} 3+6 & 4+8 \\ 9+12 & 12+16 \end{bmatrix}\\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} = (\mathbf{AB})\mathbf{C}\end{aligned}$$

d Using the matrices in **b**,

$$\begin{aligned}(\mathbf{AB})\mathbf{C} &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}\\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left[\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right]\\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{bmatrix}\\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} = (\mathbf{AB})\mathbf{C}\end{aligned}$$

4 **a** If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore$ equating coefficients of corresponding elements: $w=1$, $y=0$, $x=0$, $z=1$

which checks with the coefficients in the second line. $\therefore$ $w=z=1$, $x=y=0$ i.e., $\mathbf{X}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b In **a** we showed that $\mathbf{AX}=\mathbf{A}$ where $\mathbf{X}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ where $\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5 **a** $\mathbf{A}^2 = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$

$= \begin{bmatrix} 4+3 & 2+(-2) \\ 6+(-6) & 3+4 \end{bmatrix}$

$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

b $\mathbf{A}^3 = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 25+(-2) & -5+(-4) \\ 10+8 & -2+16 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 23 & -9 \\ 18 & 14 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 115+(-18) & -23+(-36) \\ 90+28 & -18+56 \end{bmatrix}$

$= \begin{bmatrix} 97 & -59 \\ 118 & 38 \end{bmatrix}$

6 **a** $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\therefore$ $\mathbf{A}^2$ is 3×2 by 3×2

$2 \neq 3$ so $\mathbf{A}^2$ does not exist.

b We can square a matrix when the number of columns equals the number of rows, i.e., if it is a square matrix.

7 $\mathbf{I}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

$\mathbf{I}^3 = \mathbf{I}\mathbf{I}^2 = \mathbf{I}\mathbf{I} = \mathbf{I}^2 = \mathbf{I}$

8 **a** $\mathbf{A}(\mathbf{A}+\mathbf{I})$
$= \mathbf{A}^2 + \mathbf{A}\mathbf{I}$
$= \mathbf{A}^2 + \mathbf{A}$

b $(\mathbf{B}+2\mathbf{I})\mathbf{B}$
$= \mathbf{B}^2 + 2\mathbf{I}\mathbf{B}$
$= \mathbf{B}^2 + 2\mathbf{B}$

c $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})$
$= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}\mathbf{I}$
$= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}$

d $\mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I})$
$= \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A}\mathbf{I}$
$= \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A}$

e $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$
$= (\mathbf{A}+\mathbf{B})\mathbf{C} + (\mathbf{A}+\mathbf{B})\mathbf{D}$
$= \mathbf{AC} + \mathbf{BC} + \mathbf{AD} + \mathbf{BD}$

f $(\mathbf{A}+\mathbf{B})^2$
$= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$
$= (\mathbf{A}+\mathbf{B})\mathbf{A} + (\mathbf{A}+\mathbf{B})\mathbf{B}$
$= \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$

g $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$
$= (\mathbf{A}+\mathbf{B})\mathbf{A} - (\mathbf{A}+\mathbf{B})\mathbf{B}$
$= \mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} - \mathbf{B}^2$

h $(\mathbf{A}+\mathbf{I})^2$
$= (\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I})$
$= (\mathbf{A}+\mathbf{I})\mathbf{A} + (\mathbf{A}+\mathbf{I})\mathbf{I}$
$= \mathbf{A}^2 + \mathbf{IA} + \mathbf{AI} + \mathbf{I}^2$
$= \mathbf{A}^2 + \mathbf{A} + \mathbf{A} + \mathbf{I}$
$= \mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}$

i $(3\mathbf{I}-\mathbf{B})^2$
$= (3\mathbf{I}-\mathbf{B})(3\mathbf{I}-\mathbf{B})$
$= (3\mathbf{I}-\mathbf{B})3\mathbf{I} - (3\mathbf{I}-\mathbf{B})\mathbf{B}$
$= 9\mathbf{I}^2 - 3\mathbf{BI} - 3\mathbf{IB} + \mathbf{B}^2$
$= 9\mathbf{I} - 3\mathbf{B} - 3\mathbf{B} + \mathbf{B}^2$
$= 9\mathbf{I} - 6\mathbf{B} + \mathbf{B}^2$

9 **a** $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$ $\therefore$ $\mathbf{A}^3 = \mathbf{A} \times \mathbf{A}^2$
$= \mathbf{A}(2\mathbf{A} - \mathbf{I})$
$= 2\mathbf{A}^2 - \mathbf{AI}$
$= 2(2\mathbf{A} - \mathbf{I}) - \mathbf{A}$
$= 4\mathbf{A} - 2\mathbf{I} - \mathbf{A}$
$= 3\mathbf{A} - 2\mathbf{I}$

and $\mathbf{A}^4 = \mathbf{A} \times \mathbf{A}^3$
$= \mathbf{A}(3\mathbf{A} - 2\mathbf{I})$
$= 3\mathbf{A}^2 - 2\mathbf{AI}$
$= 3(2\mathbf{A} - \mathbf{I}) - 2\mathbf{A}$
$= 6\mathbf{A} - 3\mathbf{I} - 2\mathbf{A}$
$= 4\mathbf{A} - 3\mathbf{I}$

b $\mathbf{B}^2 = 2\mathbf{I} - \mathbf{B}$

$$\therefore \quad \mathbf{B}^3 = \mathbf{B} \times \mathbf{B}^2 = \mathbf{B}(2\mathbf{I} - \mathbf{B}) = 2\mathbf{BI} - \mathbf{B}^2 = 2\mathbf{B} - (2\mathbf{I} - \mathbf{B}) = 2\mathbf{B} - 2\mathbf{I} + \mathbf{B} = 3\mathbf{B} - 2\mathbf{I}$$

and $\mathbf{B}^4 = \mathbf{B} \times \mathbf{B}^3 = \mathbf{B}(3\mathbf{B} - 2\mathbf{I}) = 3\mathbf{B}^2 - 2\mathbf{BI} = 3(2\mathbf{I} - \mathbf{B}) - 2\mathbf{B} = 6\mathbf{I} - 3\mathbf{B} - 2\mathbf{B} = 6\mathbf{I} - 5\mathbf{B}$

and $\mathbf{B}^5 = \mathbf{B} \times \mathbf{B}^4 = \mathbf{B}(6\mathbf{I} - 5\mathbf{B}) = 6\mathbf{BI} - 5\mathbf{B}^2 = 6\mathbf{B} - 5(2\mathbf{I} - \mathbf{B}) = 6\mathbf{B} - 10\mathbf{I} + 5\mathbf{B} = 11\mathbf{B} - 10\mathbf{I}$

c $\mathbf{C}^2 = 4\mathbf{C} - 3\mathbf{I}$

$$\mathbf{C}^3 = \mathbf{C} \times \mathbf{C}^2 = \mathbf{C}(4\mathbf{C} - 3\mathbf{I}) = 4\mathbf{C}^2 - 3\mathbf{CI} = 4(4\mathbf{C} - 3\mathbf{I}) - 3\mathbf{C} = 16\mathbf{C} - 12\mathbf{I} - 3\mathbf{C} = 13\mathbf{C} - 12\mathbf{I}$$

$$\mathbf{C}^5 = \mathbf{C}^2 \times \mathbf{C}^3 = (4\mathbf{C} - 3\mathbf{I})(13\mathbf{C} - 12\mathbf{I}) = (4\mathbf{C} - 3\mathbf{I})13\mathbf{C} - (4\mathbf{C} - 3\mathbf{I})12\mathbf{I} = 52\mathbf{C}^2 - 39\mathbf{IC} - 48\mathbf{CI} + 36\mathbf{I}^2 = 52(4\mathbf{C} - 3\mathbf{I}) - 39\mathbf{C} - 48\mathbf{C} + 36\mathbf{I} = 208\mathbf{C} - 156\mathbf{I} - 87\mathbf{C} + 36\mathbf{I} = 121\mathbf{C} - 120\mathbf{I}$$

10 a If $\mathbf{A}^2 = \mathbf{I}$:

i $\mathbf{A}(\mathbf{A} + 2\mathbf{I}) = \mathbf{A}^2 + 2\mathbf{AI} = \mathbf{I} + 2\mathbf{A}$

ii $(\mathbf{A} - \mathbf{I})^2 = (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I}) = (\mathbf{A} - \mathbf{I})\mathbf{A} - (\mathbf{A} - \mathbf{I})\mathbf{I} = \mathbf{A}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{I}^2 = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{I} = 2\mathbf{I} - 2\mathbf{A}$

iii $\mathbf{A}(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{A}(\mathbf{A} + 3\mathbf{I})(\mathbf{A} + 3\mathbf{I}) = \mathbf{A}[(\mathbf{A} + 3\mathbf{I})\mathbf{A} + (\mathbf{A} + 3\mathbf{I})3\mathbf{I}] = \mathbf{A}\left[\mathbf{A}^2 + 3\mathbf{IA} + 3\mathbf{AI} + 9\mathbf{I}^2\right] = \mathbf{A}[\mathbf{I} + 3\mathbf{A} + 3\mathbf{A} + 9\mathbf{I}] = \mathbf{A}[10\mathbf{I} + 6\mathbf{A}] = 10\mathbf{AI} + 6\mathbf{A}^2 = 10\mathbf{A} + 6\mathbf{I}$

b If $\mathbf{A}^3 = \mathbf{I}$, $\mathbf{A}^2(\mathbf{A} + \mathbf{I})^2 = \mathbf{A}^2(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) = \mathbf{A}^4 + 2\mathbf{A}^3 + \mathbf{A}^2\mathbf{I} = \mathbf{A}(\mathbf{A}^3) + 2\mathbf{A}^3 + \mathbf{A}^2\mathbf{I} = \mathbf{AI} + 2\mathbf{I} + \mathbf{A}^2 = \mathbf{A}^2 + \mathbf{A} + 2\mathbf{I}$

c If $\mathbf{A}^2 = \mathbf{O}$:

i $\mathbf{A}(2\mathbf{A} - 3\mathbf{I}) = 2\mathbf{A}^2 - 3\mathbf{AI} = 2\mathbf{O} - 3\mathbf{A} = -3\mathbf{A}$

ii $\mathbf{A}(\mathbf{A} + 2\mathbf{I})(\mathbf{A} - \mathbf{I}) = \mathbf{A}[(\mathbf{A} + 2\mathbf{I})\mathbf{A} - (\mathbf{A} + 2\mathbf{I})\mathbf{I}] = \mathbf{A}(\mathbf{A}^2 + 2\mathbf{IA} - \mathbf{AI} - 2\mathbf{I}^2) = \mathbf{A}(\mathbf{O} + \mathbf{A} - 2\mathbf{I}) = \mathbf{A}^2 - 2\mathbf{AI} = \mathbf{O} - 2\mathbf{A} = -2\mathbf{A}$

iii $\mathbf{A}(\mathbf{A} + \mathbf{I})^3 = \mathbf{A}(\mathbf{A} + \mathbf{I})(\mathbf{A} + \mathbf{I})^2 = \mathbf{A}(\mathbf{A} + \mathbf{I})(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) = (\mathbf{A}^2 + \mathbf{AI})(\mathbf{O} + 2\mathbf{A} + \mathbf{I}) = (\mathbf{O} + \mathbf{A})(2\mathbf{A} + \mathbf{I}) = 2\mathbf{A}^2 + \mathbf{AI} = 2\mathbf{O} + \mathbf{A} = \mathbf{A}$

11 **a** $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \mathbf{O}$

b $\mathbf{A}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$= \mathbf{A}$

c
$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \\ \therefore\ \mathbf{A}^2 - \mathbf{A} &= \mathbf{O} \\ \therefore\ \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{O} \\ \therefore\ \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O} \\ \therefore\ \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{I} \end{aligned}$$

The argument contains a false step. As the example in **a** illustrates, $\mathbf{AB} = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

This is a property of real numbers that does not hold for matrices. Therefore it is false to say that if $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I}$ equals $\mathbf{O}$.

d Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\mathbf{A}^2 = \mathbf{A}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Equating corresponding elements:

$$\begin{aligned} a^2+bc &= a & \therefore\quad bc &= a(1-a) & \text{...... (1)} \\ ab+bd &= b & \therefore\quad b(a+d-1) &= 0 & \text{...... (2)} \\ ac+cd &= c & \therefore\quad c(a+d-1) &= 0 & \text{...... (3)} \\ bc+d^2 &= d & \therefore\quad bc &= d(1-d) & \text{...... (4)} \end{aligned}$$

<u>If $a+d-1 \neq 0$</u> then from (2) and (3), $b = c = 0$.

$\therefore$ from (1) and (4), $a = 0$ or 1 and $d = 0$ or 1

$\therefore$ $a = 0,\ d = 0$ or $a = 0,\ d = 1$ or $a = 1,\ d = 0$ or $a = 1,\ d = 1$

where the last two cases are not possible as $a + d \neq 1$.

So, if $a = 0,\ d = 0$ then $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and if $a = 1,\ d = 1$ then $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>If $a+d-1 = 0$</u> then $d = 1-a$ and $c = \dfrac{a-a^2}{b}$

So $\mathbf{A}$ is $\begin{bmatrix} a & b \\ \dfrac{a-a^2}{b} & 1-a \end{bmatrix}$ provided $b \neq 0$

12 Choose $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, then $\mathbf{A}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+(-1) & -1+1 \\ 1+(-1) & -1+1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $\mathbf{A}^2 = \mathbf{O}$, but $\mathbf{A} \neq \mathbf{O}$ so "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a false statement.

13 **a** Since $\mathbf{A}^2 = a\mathbf{A}+b\mathbf{I}$, $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+(-2) & 2+4 \\ -1+(-2) & -2+4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ -a & 2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ -a & 2a+b \end{bmatrix}$$

$\therefore \quad a+b = -1 \quad$ and $\quad 2a = 6$

i.e., $\quad a = 3 \quad$ and $\quad b = -4$

Checking for consistency: $\quad -a = -3, \quad 2a+b = 6+(-4) = 2 \quad \checkmark$

$\therefore \quad \mathbf{A}^2 = 3\mathbf{A} - 4\mathbf{I}$

b Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} = a\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 9+2 & 3+(-2) \\ 6+(-4) & 2+4 \end{bmatrix} = \begin{bmatrix} 3a & a \\ 2a & -2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3a+b & a \\ 2a & -2a+b \end{bmatrix}$$

$\therefore \quad 3a+b = 11 \quad$ and $\quad a = 1$

$\therefore \quad a = 1 \quad$ and $\quad b = 8$

Checking for consistency $\quad 2a = 2(1) = 2, \quad -2a+b = -2(1)+8 = 6 \quad \checkmark$

$\therefore \quad \mathbf{A}^2 = \mathbf{A} + 8\mathbf{I}$

14 If $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = p\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+(-2) & 2+(-6) \\ -1+3 & -2+9 \end{bmatrix} = \begin{bmatrix} p & 2p \\ -p & -3p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} p+q & 2p \\ -p & -3p+q \end{bmatrix}$$

$\therefore \quad p+q = -1 \quad$ and $\quad 2p = -4$

$\therefore \quad p = -2 \quad$ and $\quad q = 1$

Checking for consistency $\quad -p = -(-2) = 2, \quad -3p+q = -3(-2)+1 = 7 \quad \checkmark$

$\therefore \quad \mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$

a
$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\ &= \mathbf{A}(-2\mathbf{A} + \mathbf{I}) \\ &= -2\mathbf{A}^2 + \mathbf{A}\mathbf{I} \\ &= -2(-2\mathbf{A} + \mathbf{I}) + \mathbf{A} \\ &= 4\mathbf{A} - 2\mathbf{I} + \mathbf{A} \\ &= 5\mathbf{A} - 2\mathbf{I} \end{aligned}$$

b
$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\ &= \mathbf{A}(5\mathbf{A} - 2\mathbf{I}) \\ &= 5\mathbf{A}^2 - 2\mathbf{A}\mathbf{I} \\ &= 5(-2\mathbf{A} + \mathbf{I}) - 2\mathbf{A} \\ &= -10\mathbf{A} + 5\mathbf{I} - 2\mathbf{A} \\ &= -12\mathbf{A} + 5\mathbf{I} \end{aligned}$$

EXERCISE 14H

1 **a** $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= 3\mathbf{I}$

$\therefore \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} = \mathbf{I}$

$\therefore \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$

b $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

$= 10\mathbf{I}$

$\therefore \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \times \frac{1}{10}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \mathbf{I}$

$\therefore \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{bmatrix}$

c $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\mathbf{I}$

$\therefore \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \mathbf{I}$

and so $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}$

2 **a** $|\mathbf{A}| = 12 - 14$
$= -2$

b $|\mathbf{A}| = 2 - 3$
$= -1$

c $|\mathbf{A}| = 0 - 0$
$= 0$

d $|\mathbf{A}| = 1 - 0$
$= 1$

3 **a** $\det\mathbf{B}$
$= 12 - -14$
$= 26$

b $\det\mathbf{B} = 6 - 0$
$= 6$

c $\det\mathbf{B} = 0 - 1$
$= -1$

d $\det\mathbf{B} = a^2 - -a$
$= a^2 + a$

4 $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

a $|\mathbf{A}| = 2(-1) - (-1)(-1) = -3$

b $\mathbf{A}^2 = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4+1 & -2+1 \\ -2+1 & 1+1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}$

$\therefore \left|\mathbf{A}^2\right| = 5(2) - (-1)(-1) = 9$

c $2\mathbf{A} = 2\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix}$

$\therefore |2\mathbf{A}| = 4(-2) - (-2)(-2) = -12$

5 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $k\mathbf{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ and $|k\mathbf{A}| = ka(kd) - kb(kc)$

$= k^2(ad - bc)$

$= k^2\,|\mathbf{A}|$

6 $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

a $|\mathbf{A}| = ad - bc$
and $|\mathbf{B}| = wz - xy$

b $$\mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$$
$\therefore\ |\mathbf{AB}| = (aw+by)(cx+dz) - (ax+bz)(cw+dy)$

c Expanding brackets,
$$\begin{aligned} |\mathbf{AB}| &= awcx + awdz + bycx + bydz - axcw - axdy - bzcw - bzdy \\ &= wz(ad-bc) - xy(ad-bc) \\ &= (ad-bc)(wz-xy) \\ &= |\mathbf{A}|\,|\mathbf{B}| \end{aligned}$$

7 $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

a **i** $$\begin{aligned} |\mathbf{A}| &= 1(4) - 2(3) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$
ii $$\begin{aligned} |2\mathbf{A}| &= 2^2\,|\mathbf{A}| \\ &= 4(-2) \\ &= -8 \end{aligned}$$
iii $$\begin{aligned} |-\mathbf{A}| &= (-1)^2\,|\mathbf{A}| \\ &= 1(-2) \\ &= -2 \end{aligned}$$
iv $$\begin{aligned} |\mathbf{B}| &= (-1)(1) - 2(0) \\ &= -1 \end{aligned}$$ $$\therefore\ \begin{aligned} |-3\mathbf{B}| &= (-3)^2\,|\mathbf{B}| \\ &= 9(-1) \\ &= -9 \end{aligned}$$
v $$\begin{aligned} |\mathbf{AB}| &= |\mathbf{A}|\,|\mathbf{B}| \\ &= (-2)(-1) \\ &= 2 \end{aligned}$$

b *Checking:*

ii $2\mathbf{A} = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$
$|2\mathbf{A}| = 2(8) - 4(6) = -8$ ✓

iii $|-\mathbf{A}| = \begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix}$
$= (-1)(-4) - (-2)(-3) = -2$ ✓

iv $-3\mathbf{B} = -3\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & -3 \end{bmatrix}$
$|-3\mathbf{B}| = (3)(-3) - (-6)(0)$
$= -9$ ✓

v $\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1+0 & 2+2 \\ -3+0 & 6+4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -3 & 10 \end{bmatrix}$
$|\mathbf{AB}| = (-1)(10) - 4(-3) = 2$ ✓

8 **a** $$\begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{2(5) - 4(-1)}\begin{bmatrix} 5 & -4 \\ -(-1) & 2 \end{bmatrix} = \tfrac{1}{14}\begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}$$

b $$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{1(-1) - 0(1)}\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

c $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}^{-1}$ does not exist, since $ad - bc = 2(2) - 4(1) = 0$

d $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{1(1) - 0(0)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

e $\begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}^{-1}$ does not exist, since $ad - bc = 3(-10) - 5(-6) = 0$

f $\begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} = \dfrac{1}{(-1)(7)-2(4)} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix}$

g $\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \dfrac{1}{3(2)-(-1)(4)} \begin{bmatrix} 2 & -4 \\ -(-1) & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$

h $\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \dfrac{1}{(-1)3-(-1)2} \begin{bmatrix} 3 & -(-1) \\ -2 & -1 \end{bmatrix} = -\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

EXERCISE 14I

1 **a** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+3b \\ a-4b \end{bmatrix}$

2 **a** $\left.\begin{aligned} 3x-y &= 8 \\ 2x+3y &= 6 \end{aligned}\right\}$ can be written as $\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

b $\left.\begin{aligned} 4x-3y &= 11 \\ 3x+2y &= -5 \end{aligned}\right\}$ can be written as $\begin{bmatrix} 4 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$

c $\left.\begin{aligned} 3a-b &= 6 \\ 2a+7b &= -4 \end{aligned}\right\}$ can be written as $\begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

3 **a** $\begin{aligned} 2x-y &= 6 \\ x+3y &= 14 \end{aligned}$ In matrix form, the system is: $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 18+14 \\ -6+28 \end{bmatrix} = \begin{bmatrix} \frac{32}{7} \\ \frac{22}{7} \end{bmatrix}$ and so $x = \frac{32}{7}$, $y = \frac{22}{7}$

b $\begin{aligned} 5x-4y &= 5 \\ 2x+3y &= -13 \end{aligned}$ In matrix form, the system is: $\begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -13 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -13 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 15+(-52) \\ -10+(-65) \end{bmatrix} = \begin{bmatrix} -\frac{37}{23} \\ -\frac{75}{23} \end{bmatrix}$ and so $x = -\frac{37}{23}$, $y = -\frac{75}{23}$

c $\begin{aligned} x-2y &= 7 \\ 5x+3y &= -2 \end{aligned}$ In matrix form, the system is: $\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 21+(-4) \\ -35+(-2) \end{bmatrix} = \begin{bmatrix} \frac{17}{13} \\ -\frac{37}{13} \end{bmatrix}$ and so $x = \frac{17}{13}$, $y = -\frac{37}{13}$

d $\begin{aligned} 3x+5y &= 4 \\ 2x-y &= 11 \end{aligned}$ In matrix form, the system is: $\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \tfrac{1}{-13}\begin{bmatrix} -1 & -5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\tfrac{1}{13}\begin{bmatrix} -4+(-55) \\ -8+33 \end{bmatrix} = \begin{bmatrix} \frac{59}{13} \\ \frac{-25}{13} \end{bmatrix} \quad \text{and so} \quad x = \tfrac{59}{13}, \quad y = -\tfrac{25}{13}$$

e $4x - 7y = 8$
$3x - 5y = 0$ In matrix form, the system is: $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 0 \end{bmatrix} = 1\begin{bmatrix} -5 & 7 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -40+0 \\ -24+0 \end{bmatrix} \quad \text{and so} \quad x = -40, \quad y = -24$$

f $7x + 11y = 18$
$11x - 7y = -11$ In matrix form, the system is: $\begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -11 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{-170}\begin{bmatrix} -7 & -11 \\ -11 & 7 \end{bmatrix}\begin{bmatrix} 18 \\ -11 \end{bmatrix} = -\tfrac{1}{170}\begin{bmatrix} -126+121 \\ -198-77 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\tfrac{1}{170}\begin{bmatrix} -5 \\ -275 \end{bmatrix} = \begin{bmatrix} \frac{1}{34} \\ \frac{55}{34} \end{bmatrix} \quad \text{and so} \quad x = \tfrac{1}{34}, \quad y = \tfrac{55}{34}$$

4 **a** If $\mathbf{AX} = \mathbf{B}$
then $\mathbf{A}^{-1}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ {pre-mult by $\mathbf{A}^{-1}$}
$\therefore \mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$
$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

If $\mathbf{XA} = \mathbf{B}$
then $\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$ {post-mult by $\mathbf{A}^{-1}$}
$\therefore \mathbf{XI} = \mathbf{BA}^{-1}$
$\therefore \mathbf{X} = \mathbf{BA}^{-1}$

b **i** $\mathbf{X}\begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}$

$$\therefore \mathbf{X} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}\tfrac{1}{-11}\begin{bmatrix} -1 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= -\tfrac{1}{11}\begin{bmatrix} -14+25 & -28-5 \\ -22+0 & -44+0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$$

ii $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \tfrac{1}{-7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= -\tfrac{1}{7}\begin{bmatrix} -1+(-12) & 3+(-6) \\ -2+4 & 6+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13}{7} & \frac{3}{7} \\ -\frac{2}{7} & -\frac{8}{7} \end{bmatrix}$$

5 **a** $\mathbf{A} = \begin{bmatrix} k & 1 \\ -6 & 2 \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \dfrac{1}{2k+6}\begin{bmatrix} 2 & -1 \\ 6 & k \end{bmatrix}$, provided that $2k + 6 \neq 0$, i.e., $k \neq -3$

b $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & k \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \dfrac{1}{3k}\begin{bmatrix} k & 1 \\ 0 & 3 \end{bmatrix}$, provided that $k \neq 0$

c $\mathbf{A} = \begin{bmatrix} k+1 & 2 \\ 1 & k \end{bmatrix} \quad \therefore \quad \mathbf{A}^{-1} = \dfrac{1}{k(k+1)-2}\begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix},$

$= \dfrac{1}{(k+2)(k-1)}\begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix},$ provided that $k \neq -2$ or 1

6 **a** $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×2 $\quad\therefore\quad \mathbf{AB}$ is 2×2

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+0+2 & 2+0+(-2) \\ 1+(-4)+3 & -2+6+(-3) \end{bmatrix}$$

$$\therefore \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

b $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 2×3 $\therefore$ $\mathbf{BA}$ is 3×3 whereas $\mathbf{AB}$ is 2×2. Hence $\mathbf{BA} \neq \mathbf{AB}$, and $\mathbf{A}$ and $\mathbf{B}$ are not inverses. The inverse of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$. This requires that $\mathbf{A}$ has the same number of rows as columns, i.e., that $\mathbf{A}$ is a square matrix.

7 $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$

Since $\mathbf{AXB} = \mathbf{C}$,

then $\mathbf{A}^{-1}\mathbf{AXB} = \mathbf{A}^{-1}\mathbf{C}$ {premult. by $\mathbf{A}^{-1}$}

$\therefore\ \mathbf{IXB} = \mathbf{A}^{-1}\mathbf{C}$

$\therefore\ \mathbf{XB} = \mathbf{A}^{-1}\mathbf{C}$

and

$\mathbf{XBB}^{-1} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$ {postmult. by $\mathbf{B}^{-1}$}

$\therefore\ \mathbf{XI} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$

$\therefore\ \mathbf{X} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$

$$\therefore \quad \mathbf{X} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}\tfrac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 0+(-1) & 3+(-2) \\ 0+2 & 0+4 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{X} = \tfrac{1}{4}\begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 0+1 & 2+1 \\ 0+4 & -4+4 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{bmatrix}$$

8 **a** **i** $\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$

and $|\mathbf{A}| = -2 - -12$
$= -2+12$
$= 10$

ii As $|\mathbf{A}| \neq 0$, the system has a unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \tfrac{1}{10}\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \tfrac{1}{10}\begin{bmatrix} 25 \\ -10 \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ -1 \end{bmatrix}$$

$\therefore \quad x = 2\frac{1}{2}, \quad y = -1$

b **i** $\begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$

and $|\mathbf{A}| = -2 - 4k$

ii The system has a unique solution if
$-2-4k \neq 0$ i.e., $k \neq -\frac{1}{2}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \frac{1}{-2-4k}\begin{bmatrix} -1 & -k \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \frac{1}{-2-4k}\begin{bmatrix} -8-11k \\ -10 \end{bmatrix}$$

$\therefore \quad x = \dfrac{8+11k}{2+4k}, \quad y = \dfrac{5}{1+2k}, \quad k \neq -\frac{1}{2}$

is the unique solution

iii When $k = -\frac{1}{2}$, the equations are

$\begin{cases} 2x - \frac{1}{2}y = 8 \\ 4x - y = 11 \end{cases}$ i.e., $\begin{matrix} 4x - y = 16 \\ 4x - y = 11 \end{matrix}$

So, we have no solutions (as the lines are parallel and so do not meet).

9 **a** If $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{AA} = \mathbf{AA}^{-1} = \mathbf{I}$

b If $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse, then $\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore\ a^2+b^2 = 1$ and $2ab = 0$

If $2ab = 0$, then $a = 0$ or $b = 0$

then $a = 0$ and $b^2 = 1$ $\therefore\ a = 0$ and $b = \pm 1$

or $b = 0$ and $a^2 = 1$ $\therefore\ a = \pm 1$ and $b = 0$

This gives four possible combinations: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

10 **a** $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ $\therefore\ \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$\therefore\ (\mathbf{A}^{-1})^{-1} = \frac{1}{\frac{1}{2}}\begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \mathbf{A}$$

b If $\mathbf{A}^{-1} = \mathbf{B}$

then $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$

and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{BB}^{-1} = \mathbf{I}$

c We can deduce from **b** that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ and $(\mathbf{A}^{-1})^{-1}$ is the inverse of $\mathbf{A}^{-1}$.

11 **a** $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$

i $\mathbf{A}^{-1} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

ii $\mathbf{B}^{-1} = \frac{1}{-2}\begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

iii $(\mathbf{AB})^{-1}$

$$= \left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 1+(-3) \\ 0+(-2) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \text{ i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

iv $(\mathbf{BA})^{-1}$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 0+(-1) \\ 2+(-6) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}^{-1}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \text{ i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

v $\mathbf{A}^{-1}\mathbf{B}^{-1}$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6}+\frac{1}{3} & \frac{1}{6}+0 \\ \frac{6}{6}+\left(-\frac{1}{3}\right) & \frac{2}{6}+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

vi $\mathbf{B}^{-1}\mathbf{A}^{-1}$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6}+\frac{2}{6} & \frac{3}{6}+\left(-\frac{1}{6}\right) \\ \frac{1}{3}+0 & \frac{1}{3}+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

b Choose appropriate vectors and repeat question **a**.

c The results of **a** and **b** suggest that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

d
$$\begin{aligned} &(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) \\ &= (\mathbf{AB})(\mathbf{AB})^{-1} \quad \{\text{from } \mathbf{c}\} \\ &= \mathbf{I} \end{aligned} \qquad \text{and} \qquad \begin{aligned} &(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) \\ &= (\mathbf{AB})^{-1}(\mathbf{AB}) \quad \{\text{from } \mathbf{c}\} \\ &= \mathbf{I} \end{aligned}$$

$\therefore$ $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$ i.e., $\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses.

12 $(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = k \times \frac{1}{k}(\mathbf{AA}^{-1}) = \mathbf{I}$ also $\left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \frac{1}{k} \times k(\mathbf{A}^{-1}\mathbf{A}) = \mathbf{I}$

$\therefore$ $(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$ i.e., $k\mathbf{A}$ and $\frac{1}{k}\mathbf{A}^{-1}$ are inverses.

13 $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$

a $\mathbf{X} = \mathbf{AY} = \mathbf{A}(\mathbf{BZ}) = \mathbf{ABZ}$

b $(\mathbf{AB})^{-1}\mathbf{X} = (\mathbf{AB})^{-1}\mathbf{ABZ}$ $\{$premult by $(\mathbf{AB})^{-1}\}$

$(\mathbf{AB})^{-1}\mathbf{X} = \mathbf{IZ}$

$\therefore$ $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X}$ $\{$as $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\}$

14 In each example we premultiply by $\mathbf{A}^{-1}$.

a
$$\begin{aligned} \mathbf{A}^2 &= 4\mathbf{A} - \mathbf{I} \\ \therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(4\mathbf{A} - \mathbf{I}) \\ \therefore\ \mathbf{A}^{-1}\mathbf{AA} &= 4\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \\ \therefore\ \mathbf{IA} &= 4\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{A} - 4\mathbf{I} &= -\mathbf{A}^{-1} \\ \therefore\ \mathbf{A}^{-1} &= 4\mathbf{I} - \mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 5\mathbf{A} &= \mathbf{I} - \mathbf{A}^2 \\ \therefore\ \mathbf{A}^{-1}5\mathbf{A} &= \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}^2) \\ \therefore\ 5\mathbf{A}^{-1}\mathbf{A} &= \mathbf{A}^{-1}\mathbf{I} - \mathbf{A}^{-1}\mathbf{AA} \\ \therefore\ 5\mathbf{I} &= \mathbf{A}^{-1} - \mathbf{IA} \\ \therefore\ 5\mathbf{I} &= \mathbf{A}^{-1} - \mathbf{A} \\ \therefore\ \mathbf{A}^{-1} &= 5\mathbf{I} + \mathbf{A} \end{aligned}$$

c
$$\begin{aligned} 2\mathbf{I} &= 3\mathbf{A}^2 - 4\mathbf{A} \\ \therefore\ \mathbf{A}^{-1}2\mathbf{I} &= \mathbf{A}^{-1}3\mathbf{A}^2 - \mathbf{A}^{-1}4\mathbf{A} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{A}^{-1}\mathbf{AA} - 4\mathbf{A}^{-1}\mathbf{A} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{IA} - 4\mathbf{I} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{A} - 4\mathbf{I} \\ \therefore\ \mathbf{A}^{-1} &= \tfrac{3}{2}\mathbf{A} - 2\mathbf{I} \end{aligned}$$

15 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$, let $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\therefore \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} = p\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 9+(-4) & 6+(-2) \\ -6+2 & -4+1 \end{bmatrix} = \begin{bmatrix} 3p & 2p \\ -2p & -p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 3p+q & 2p \\ -2p & -p+q \end{bmatrix}$$

$\therefore$ $5 = 3p + q$ and $4 = 2p$

$\therefore$ $p = 2$ and $q = -1$ and $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$

Checking for consistency: $-2p = -2(2) = -4$ ✓

$-p + q = -2 + (-1) = -3$ ✓

$$\begin{aligned}
\text{Now}\quad \mathbf{A}^2 &= 2\mathbf{A} - \mathbf{I}\\
\therefore\quad \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}2\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \qquad \{\text{premultiplying by } \mathbf{A}^{-1}\}\\
\therefore\quad \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\\
\therefore\quad \mathbf{I}\mathbf{A} &= 2\mathbf{I} - \mathbf{A}^{-1}\\
\therefore\quad \mathbf{A} &= 2\mathbf{I} - \mathbf{A}^{-1}\\
\therefore\quad \mathbf{A}^{-1} &= 2\mathbf{I} - \mathbf{A}
\end{aligned}$$

16 If $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$,

$$\begin{aligned}
\text{then}\quad \mathbf{A}^2 &= \mathbf{AA}\\
&= (\mathbf{AB})\mathbf{A}\\
&= \mathbf{A}(\mathbf{BA}) \qquad \{\text{associative rule}\}\\
&= \mathbf{AB}\\
\therefore\quad \mathbf{A}^2 &= \mathbf{A}
\end{aligned}$$

$ab = ac$ implies that $b = c$ for real numbers, but this property does not hold for matrices.

Thus from $\mathbf{AB} = \mathbf{AI} = \mathbf{A}$ it does not follow that $\mathbf{B} = \mathbf{I}$.

17

$$\begin{aligned}
\text{If}\quad \mathbf{AB} &= \mathbf{AC}\\
\text{then}\quad \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{AC} \qquad \{\text{premultiplying by } \mathbf{A}^{-1}\}\\
\therefore\quad \mathbf{IB} &= \mathbf{IC}\\
\therefore\quad \mathbf{B} &= \mathbf{C}
\end{aligned}$$

i.e., if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$ if $\mathbf{A}^{-1}$ exists, i.e., if $ad - bc \neq 0$.

18 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$

$$\begin{aligned}
\text{then}\quad \mathbf{X}^3 &= (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})\\
&= (\mathbf{P}^{-1}\mathbf{A})(\mathbf{PP}^{-1})\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} \qquad \{\text{associative rule}\}\\
&= \mathbf{P}^{-1}\mathbf{AIAIAP}\\
&= \mathbf{P}^{-1}\mathbf{AAAP}\\
&= \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P}\\
&= \mathbf{P}^{-1}\mathbf{IP}\\
&= \mathbf{P}^{-1}\mathbf{P} \qquad \text{and so}\quad \mathbf{X}^3 = \mathbf{I}
\end{aligned}$$

19

$$\begin{aligned}
\text{If}\quad a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} &= \mathbf{O} \quad\text{and}\quad \mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}\\
\text{then}\quad a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} &= a(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}\\
&= a\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}\\
&= a\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}\\
&= \mathbf{P}^{-1}(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{P}\\
&= \mathbf{P}^{-1}\mathbf{OP}\\
&= \mathbf{O}
\end{aligned}$$

EXERCISE 14J

1 **a**

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & 0\\ -1 & 2 & 1\\ 2 & 0 & 5 \end{vmatrix} &= 2\begin{vmatrix} 2 & 1\\ 0 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & -1\\ 5 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 2\\ 2 & 0 \end{vmatrix}\\
&= 2(10 - 0) + 3(2 - -5) + 0\\
&= 41
\end{aligned}$$

b

$$\begin{aligned}
\begin{vmatrix} -1 & 2 & -3\\ 1 & 0 & 0\\ -1 & 2 & 1 \end{vmatrix} &= -1\begin{vmatrix} 0 & 0\\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1\\ 1 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 0\\ -1 & 2 \end{vmatrix}\\
&= -1(0 - 0) + 2(0 - 1) - 3(2 - 0)\\
&= -8
\end{aligned}$$

c $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$

$= 2(3-2) + 1(4 - -3) + 3(-1-2)$
$= 0$

d $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 1(6-0) = 6$

e $\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = 2(0-3) = -6$

f $\begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 4\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}$

$= 4(0-2) + 1(-2 - -1) + 3(-1-0)$
$= -12$

g $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}$

$= 3(3 - -1) - 1(-1-0) - 2(0 - -1)$
$= 11$

h $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 2 & 6 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 2 & 6 \end{vmatrix}$

$= 1(8-6) + 3(2 - -4) + 2(-6-4)$
$= 0$

i $\begin{vmatrix} 0 & 3 & 0 \\ 1 & 2 & 5 \\ 6 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 1 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} = 3(30-1) = 87$

2 **a** $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ c & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = a(bc-0) = abc$

b $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & z \\ -z & 0 \end{vmatrix} + x\begin{vmatrix} z & -x \\ 0 & -y \end{vmatrix} + y\begin{vmatrix} -x & 0 \\ -y & -z \end{vmatrix}$

$= x(-zy-0) + y(xz-0)$
$= -xyz + xyz$
$= 0$

c $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} + b\begin{vmatrix} a & b \\ b & c \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$

$= a(cb-a^2) + b(ac-b^2) + c(ba-c^2)$
$= abc - a^3 + abc - b^3 + abc - c^3$
$= 3abc - a^3 - b^3 - c^3$

3 $\begin{cases} x+2y-3z = 5 \\ 2x-y-z = 8 \\ kx+y+2z = 14 \end{cases}$ has matrix equation $\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 5 \\ 8 \\ 14 \end{bmatrix}}_{\mathbf{B}}$

$\therefore$ $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ has a unique solution if $|\mathbf{A}| \neq 0$.

Now $\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 2 & k \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ k & 1 \end{vmatrix}$

$= 1(-2 - -1) + 2(-k-4) - 3(2 - -k)$

$= -1 - 2k - 8 - 6 - 3k$

$= -15 - 5k$ i.e., a unique solution for all k provided $k \neq -3$.

4 $\begin{cases} 2x-y-4z = 8 \\ 3x-ky+z = 1 \\ 5x-y+kz = -2 \end{cases}$ has matrix equation $\underbrace{\begin{bmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}}_{\mathbf{B}}$

This has a unique solution if $|\mathbf{A}| \neq 0$.

Now $\begin{vmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{vmatrix} = 2\begin{vmatrix} -k & 1 \\ -1 & k \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3 \\ k & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -k \\ 5 & -1 \end{vmatrix}$

$= 2(-k^2 - -1) - 1(5-3k) - 4(-3 - -5k)$

$= -2k^2 + 2 - 5 + 3k + 12 - 20k$

$= -2k^2 - 17k + 9$

$= -(2k^2 + 17k - 9)$

$= -(2k-1)(k+9)$

$\therefore$ there is a unique solution for all k provided $k \neq \frac{1}{2}$ and $k \neq -9$.

5 **a**

$$\begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7$$

$$\therefore \quad 1\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + k\begin{vmatrix} -1 & k \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 3 & 4 \end{vmatrix} = 7$$

$$\therefore \quad 1(2 - -4) + k(-3-2k) + 3(4k-3) = 7$$

$$\therefore \quad 6 - 3k - 2k^2 + 12k - 9 = 7$$

$$\therefore \quad 2k^2 - 9k + 10 = 0$$

$$\therefore \quad (2k-5)(k-2) = 0 \qquad \text{and so } k = \tfrac{5}{2} \text{ or } 2$$

b

$$\begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{vmatrix} = 0$$

$$\therefore \quad k\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ k & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 0$$

$$\therefore \quad k(k^2-4) + 2(2-2k) + (4-k) = 0$$

$$\therefore \quad k^3 - 4k + 4 - 4k + 4 - k = 0$$

$$\therefore \quad k^3 - 9k + 8 = 0$$

Using technology there is one rational zero, $k = 1$

$$\therefore \quad (k-1)(k^2+k-8) = 0$$

$$\therefore \quad k = 1 \quad \text{or} \quad k = \frac{-1 \pm \sqrt{1 - 4(1)(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$$

6 Using technology,

a $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{vmatrix} = 16$

b $\begin{vmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{vmatrix} = -34$

7 **a** Using Excel, det $= -17$ and

$$\text{inverse} = \begin{bmatrix} -\frac{5}{17} & \frac{2}{17} & \frac{14}{17} & -\frac{1}{17} \\ -\frac{11}{17} & \frac{1}{17} & \frac{7}{17} & \frac{8}{17} \\ -\frac{3}{17} & \frac{8}{17} & \frac{5}{17} & -\frac{4}{17} \\ \frac{10}{17} & -\frac{4}{17} & -\frac{11}{17} & \frac{2}{17} \end{bmatrix}$$

b Using Excel, det $= 2$ and

$$\text{inverse} = \begin{bmatrix} \frac{7}{2} & -4 & -\frac{5}{2} & \frac{11}{2} & 0 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 2 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ -3 & 3 & 2 & -4 & 1 \end{bmatrix}$$

8 **a** Let o, a, p, c, l represent (respectively) the cost (in dollars) of each orange, apple, pear, cabbage and lettuce. The system, in matrix form, becomes:

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{bmatrix}} \underset{\mathbf{X}}{\begin{bmatrix} o \\ a \\ p \\ c \\ l \end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{bmatrix}}$$

b $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Using technology, $|\mathbf{A}| = 0$, $\therefore$ $\mathbf{A}^{-1}$ does not exist and $\mathbf{X}$ cannot be found using this information.

c If the last line is amended, the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \det\mathbf{A} = 6$$

$$\therefore \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.7 \\ 2 \\ 1.5 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ oranges cost 50 cents, apples cost 80 cents, pears cost 70 cents, cabbages cost \$2.00, and lettuces cost \$1.50

EXERCISE 14K

1 **a** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix}$

b $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}$

2 **a**

$$\mathbf{B}^{-1} = \begin{bmatrix} 0.050\,23 & -0.011\,48 & -0.066\,34 \\ 4.212 \times 10^{-4} & 0.013\,53 & 0.027\,75 \\ -0.029\,90 & 0.039\,33 & 0.030\,06 \end{bmatrix} \doteqdot \begin{bmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{bmatrix}$$

b

$$\mathbf{B}^{-1} = \begin{bmatrix} 1.596 & -0.9964 & -0.1686 \\ -3.224 & 1.925 & 0.6291 \\ 2.000 & -1.086 & -0.3958 \end{bmatrix} \doteqdot \begin{bmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{bmatrix}$$

EXERCISE 14L

1 **a** $\left.\begin{aligned} x-y-z &= 2\\ x+y+3z &= 7\\ 9x-y-3z &= -1 \end{aligned}\right\}$ has matrix equation $\begin{bmatrix} 1 & -1 & -1\\ 1 & 1 & 3\\ 9 & -1 & -3 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 2\\ 7\\ -1 \end{bmatrix}$

b $\left.\begin{aligned} 2x+y-z &= 3\\ y+2z &= 6\\ x-y+z &= 13 \end{aligned}\right\}$ has matrix equation $\begin{bmatrix} 2 & 1 & -1\\ 0 & 1 & 2\\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 3\\ 6\\ 13 \end{bmatrix}$

c $\left.\begin{aligned} a+b-c &= 7\\ a-b+c &= 6\\ 2a+b-3c &= -2 \end{aligned}\right\}$ has matrix equation $\begin{bmatrix} 1 & 1 & -1\\ 1 & -1 & 1\\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} 7\\ 6\\ -2 \end{bmatrix}$

2 $$\mathbf{AB} = \begin{bmatrix} 1 & -1 & 0\\ -1 & 0 & 1\\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1\\ 1 & 1 & 1\\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-1+0 & 1-1+0 & 1-1+0\\ -2+0+2 & -1+0+2 & -1+0+1\\ 0+2-2 & 0+2-2 & 0+2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \quad \text{and so } \mathbf{AB} = \mathbf{I}$$

$$\mathbf{BA} = \begin{bmatrix} 2 & 1 & 1\\ 1 & 1 & 1\\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0\\ -1 & 0 & 1\\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2-1+0 & -2+0+2 & 0+1-1\\ 1-1+0 & -1+0+2 & 0+1-1\\ 2-2+0 & -2+0+2 & 0+2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore\ \mathbf{BA} = \mathbf{I}$

i.e., $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ $\therefore$ $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other.

3 $$\mathbf{AB} = \begin{bmatrix} 2 & 1 & -1\\ -1 & 2 & 1\\ 0 & 6 & 1 \end{bmatrix}\begin{bmatrix} 4 & 7 & -3\\ -1 & -2 & 1\\ 6 & 12 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-1-6 & 14-2-12 & -6+1+5\\ -4-2+6 & -7-4+12 & 3+2-5\\ 0-6+6 & 0-12+12 & 0+6-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \qquad \therefore\ \mathbf{AB} = \mathbf{I} \quad \text{and so} \quad \mathbf{A} = \mathbf{B}^{-1}$$

$\left.\begin{aligned} 4a+7b-3c &= -8\\ -a-2b+c &= 3\\ 6a+12b-5c &= -15 \end{aligned}\right\}$ has matrix equation $\begin{bmatrix} 4 & 7 & -3\\ -1 & -2 & 1\\ 6 & 12 & -5 \end{bmatrix}\begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} -8\\ 3\\ -15 \end{bmatrix}$

$$\therefore\ \begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} 4 & 7 & -3\\ -1 & -2 & 1\\ 6 & 12 & -5 \end{bmatrix}^{-1}\begin{bmatrix} -8\\ 3\\ -15 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1\\ -1 & 2 & 1\\ 0 & 6 & 1 \end{bmatrix}\begin{bmatrix} -8\\ 3\\ -15 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} a\\ b\\ c \end{bmatrix} = \begin{bmatrix} -16+3+15\\ 8+6-15\\ 0+18-15 \end{bmatrix} = \begin{bmatrix} 2\\ -1\\ 3 \end{bmatrix} \qquad \therefore\ a=2,\quad b=-1,\quad c=3$$

4 $\mathbf{MN} = \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 15+3-14 & 10-3-7 & 15+6-21 \\ -3-3+6 & -2+3+3 & -3-6+9 \\ -9-1+10 & -6+1+5 & -9-2+15 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad \therefore \ \mathbf{MN} = 4\mathbf{I}$

$\therefore \ \left(\tfrac{1}{4}\mathbf{M}\right)\mathbf{N} = \mathbf{I}$

$\therefore \ \tfrac{1}{4}\mathbf{M} = \mathbf{N}^{-1}$

Now $\left.\begin{aligned} 3u+2v+3w &= 18 \\ u-v+2w &= 6 \\ 2u+v+3w &= 16 \end{aligned}\right\}$ has matrix equation $\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$

$\therefore \ \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$

$\therefore \ \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 90+18-112 \\ -18-18+48 \\ -54-6+80 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} -4 \\ 12 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ $\quad \therefore \ u=-1, \quad v=3, \quad w=5$

5 **a** $\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}$

$\therefore \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}$

Using technology, $x = \frac{23}{10}, \quad y = \frac{13}{10}, \quad z = -\frac{9}{2}$

b $\begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$

$\therefore \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$

Using technology, $x = -\frac{1}{3}, \quad y = -\frac{95}{21}, \quad z = \frac{2}{21}$

c $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$

$\therefore \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$

Using technology, $x=2, \quad y=4, \quad z=-1$

6 **a** $x = 2, \quad y = -1, \quad z = 5$ **b** $x = 4, \quad y = -2, \quad z = 1$

c $x = 4, \quad y = -3, \quad z = 2$ **d** $x = 4, \quad y = 6, \quad z = -7$

e $x = 3, \quad y = 11, \quad z = -7$ **f** $x = 0.326, \quad y = 7.652, \quad z = 4.156$

7 **a** Using technology, $x = 14, \quad y = 11, \quad z = 17.$

b Let x be the cost of a cricket ball in dollars,
y be the cost of softball in dollars, and
z be the cost of a netball in dollars.

c Cost of 4 cricket balls and 5 softballs is $4x + 5y = 4(14) + 5(11) = 111$, i.e., \$111

$\therefore$ amount left for netballs is \$315 – \$111 = \$204

Number of netballs bought $= \dfrac{204}{17} = 12$ i.e., 12 netballs are bought.

8 **a** System of equations is:

$$\begin{aligned} 2x + 3y + 8z &= 352 \\ x + 5y + 4z &= 274 \\ x + 2y + 11z &= 351 \end{aligned}$$

b Using technology, $x = 42, \quad y = 28, \quad z = 23,$
i.e., the salaries are: manager \$42 000, clerk \$28 000 and labourer \$23 000.

c Salary bill is

$$\begin{aligned} &3x + 8y + 37z \\ &= 3(42) + 8(28) + 37(23) \\ &= 1201 \quad \text{(thousands of dollars)} \end{aligned}$$

i.e., \$1 201 000

9 Let the quadratic function have form $f(x) = ax^2 + bx + c.$

$(-2, -15)$ is a point on the curve $\therefore \quad 4a - 2b + c = -15$

$(1, -3)$ is a point on the curve $\therefore \quad a + b + c = -3$

$(3, -5)$ is a point on the curve $\therefore \quad 9a + 3b + c = -5$

Using technology to solve these equations, $a = -1, \quad b = 3$ and $c = -5.$

$\therefore \quad f(x) = -x^2 + 3x - 5$

$$\begin{aligned} \text{Now} \quad f(-3) &= -(-3)^2 + 3(-3) - 5 = -23 \\ f(-1) &= -(-1)^2 + 3(-1) - 5 = -9 \\ f(0) &= -5 \\ f(2) &= -2^2 + 3(2) - 5 = -3 \end{aligned}$$

The table is

x	-3	-2	-1	0	1	2	3
y	-23	-15	-9	-5	-3	-3	-5

10 Let x be the cost in dollars of 1 kg of cashews,
y be the cost in dollars of 1 kg of macadamias, and
z be the cost in dollars of 1 kg of Brazil nuts.

The cost of 1 kg of mix A is $0.5x + 0.3y + 0.2z = 12.5,$
the cost of 1 kg of mix B is $0.2x + 0.4y + 0.4z = 12.4,$
the cost of 1 kg of mix C is $0.6x + 0.1y + 0.3z = 11.7.$

Using technology, $x = 12, \quad y = 15$ and $z = 10$
i.e., the cost of 1 kg of cashews is \$12, the cost of 1 kg of macadamias is \$15 and the cost of 1 kg of Brazil nuts is \$10.

Cost per kg of 400 g cashews, 200 g macadamias and 400 g Brazil nuts
$= 0.4 \times 12 + 0.2 \times 15 + 0.4 \times 10$ dollars
$= \$11.80$

11 **a** Number of students who study Chemistry is $\frac{1}{3}p + \frac{1}{3}q + \frac{2}{5}r = 27$ ..... (1)

number of students who study Maths is $\frac{1}{2}p + \frac{2}{3}q + \frac{1}{5}r = 35$ ..... (2)

number of students who study Geography is $\frac{1}{4}p + \frac{1}{3}q + \frac{3}{5}r = 30$ ..... (3)

The required system of equations is

$$\begin{aligned} 5p + 5q + 6r &= 405 && \{(1) \times 15\} \\ 15p + 20q + 6r &= 1050 && \{(2) \times 30\} \\ 15p + 20q + 36r &= 1800 && \{(3) \times 60\} \end{aligned}$$

b Using technology, $p = 24$, $q = 27$, $r = 25$.

12 **a** As t is the number of years after 2000, then

profit in year 2000 is $P(0) = b + \dfrac{c}{4} = 160\,000$

profit in year 2001 is $P(1) = a + b + \dfrac{c}{5} = 198\,000$

profit in year 2002 is $P(2) = 2a + b + \dfrac{c}{6} = 240\,000$

Using technology, $a = 50\,000$, $b = 100\,000$ and $c = 240\,000$.

b Using the model given, the profit in 1999 would be

$$P(-1) = -a + b + \frac{c}{3} - 50\,000 + 100\,000 + 80\,000 = 130\,000$$

i.e., the profit would be \$130 000, which fits the model.

c Predicted profit in 2003 is $P(3) = 3a + b + \dfrac{c}{7} = 3(50\,000) + 100\,000 + \dfrac{240\,000}{7}$
$\doteqdot \$284\,000$

Predicted profit in 2005 is $P(5) = 5a + b + \dfrac{c}{9} = 5(50\,000) + 100\,000 + \dfrac{240\,000}{9}$
$\doteqdot \$377\,000$

REVIEW SET 14A

1 **a** $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

b $3\mathbf{A}$

$$= 3\begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix}$$

c $-2\mathbf{B}$

$$= -2\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 4 & -8 \end{bmatrix}$$

d $\mathbf{A} - \mathbf{B}$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

e $\mathbf{B} - 2\mathbf{A}$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 0 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} -5 & -4 \\ -2 & 6 \end{bmatrix}$$

f $3\mathbf{A} - 2\mathbf{B}$

$$= 3\begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -4 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 6 \\ 4 & -11 \end{bmatrix}$$

g $\mathbf{AB}$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}$$

h $\mathbf{BA}$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -6 & -8 \end{bmatrix}$$

i $\mathbf{A}^{-1}$

$$= \frac{1}{-3} \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{bmatrix}$$

j $\mathbf{A}^2$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 \\ 0 & 1 \end{bmatrix}$$

k $\mathbf{ABA}$

$$= (\mathbf{AB})\mathbf{A}$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -10 \\ 6 & 8 \end{bmatrix}$$

l $(\mathbf{AB})^{-1}$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}^{-1}$$

$$= \frac{1}{4-16} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \tfrac{1}{-12} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

2 **a** Equating corresponding elements,

$$\begin{aligned} a &= -a \\ b-2 &= 3 \\ c &= 2-c \\ d &= -4 \end{aligned} \qquad \therefore \quad a=0, \quad b=5, \quad c=1, \quad d=-4$$

b Equating corresponding elements,

$$\begin{aligned} 3+b &= a \\ 2a-a &= 2 \\ b+c &= 2 \\ -2+d &= 6 \end{aligned} \qquad \therefore \quad a=2, \quad b=-1, \quad c=3, \quad d=8$$

3 **a**

$$\begin{aligned} \mathbf{B}-\mathbf{Y} &= \mathbf{A} \\ \therefore\ -\mathbf{Y} &= \mathbf{A}-\mathbf{B} \\ \therefore\ \mathbf{Y} &= -(\mathbf{A}-\mathbf{B}) \\ \therefore\ \mathbf{Y} &= \mathbf{B}-\mathbf{A} \end{aligned}$$

b

$$\begin{aligned} 2\mathbf{Y}+\mathbf{C} &= \mathbf{D} \\ \therefore\ 2\mathbf{Y} &= \mathbf{D}-\mathbf{C} \\ \therefore\ \mathbf{Y} &= \tfrac{1}{2}(\mathbf{D}-\mathbf{C}) \end{aligned}$$

c

$$\begin{aligned} \mathbf{AY} &= \mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{IY} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{Y} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

d

$$\begin{aligned} \mathbf{YB} &= \mathbf{C} \\ \therefore\ \mathbf{YBB}^{-1} &= \mathbf{CB}^{-1} \\ \therefore\ \mathbf{YI} &= \mathbf{CB}^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{CB}^{-1} \end{aligned}$$

e

$$\begin{aligned} \mathbf{C}-\mathbf{AY} &= \mathbf{B} \\ \therefore\ -\mathbf{AY} &= \mathbf{B}-\mathbf{C} \\ \therefore\ \mathbf{AY} &= \mathbf{C}-\mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY} &= \mathbf{A}^{-1}(\mathbf{C}-\mathbf{B}) \\ \therefore\ \mathbf{Y} &= \mathbf{A}^{-1}(\mathbf{C}-\mathbf{B}) \end{aligned}$$

f

$$\begin{aligned} \mathbf{AY}^{-1} &= \mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{Y}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ (\mathbf{Y}^{-1})^{-1} &= (\mathbf{A}^{-1}\mathbf{B})^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{B}^{-1}\mathbf{A} \end{aligned}$$

4 **a**

$$\begin{aligned} 3x-4y &= 2 \\ 5x+2y &= -1 \end{aligned}$$

$$\therefore \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \tfrac{1}{26} \begin{bmatrix} 2 & 4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \tfrac{1}{26} \begin{bmatrix} 0 \\ -13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$$

$$\therefore \quad x = 0, \quad y = -\tfrac{1}{2}$$

b

$$\begin{aligned} 4x-y &= 5 \\ 2x+3y &= 9 \end{aligned}$$

$$\therefore \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$= \tfrac{1}{14} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$= \tfrac{1}{14} \begin{bmatrix} 24 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{7} \\ \frac{13}{7} \end{bmatrix}$$

$$\therefore \quad x = \tfrac{12}{7}, \quad y = \tfrac{13}{7}$$

c $\mathbf{X}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\tfrac{1}{-1}\begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \begin{bmatrix} -1 & 8 \\ -2 & 6 \end{bmatrix}$$

d $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \tfrac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

e $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \tfrac{1}{-3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \tfrac{1}{-3}\begin{bmatrix} -14 \\ -1 \end{bmatrix}$$

$$\therefore \ \mathbf{X} = \begin{bmatrix} \frac{14}{3} \\ \frac{1}{3} \end{bmatrix}$$

f $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{X}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

$$\therefore \ \mathbf{X} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$\therefore \ \mathbf{X} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}\tfrac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\tfrac{1}{6}\begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\tfrac{1}{6}\begin{bmatrix} -3 & -9 \\ -9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

5 **a** $2\mathbf{B}$

$$= 2\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{bmatrix}$$

b $\frac{1}{2}\mathbf{B}$

$$= \tfrac{1}{2}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

c $\mathbf{AB}$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

1×3 $\quad$ 3×2

$$= \begin{bmatrix} 11 & 12 \end{bmatrix}$$

d **B** is 3×2
and **A** is 1×3.

not equal

$\therefore$ **BA** does not exist.

6 **a** $\mathbf{P}+\mathbf{Q}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}$$

b $\mathbf{Q}-\mathbf{P}$

$$= \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{bmatrix}$$

c $\frac{3}{2}\mathbf{P}-\mathbf{Q}$

$$= \begin{bmatrix} \frac{3}{2} & 3 \\ \frac{3}{2} & 0 \\ 3 & \frac{9}{2} \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{bmatrix}$$

REVIEW SET 14B

1 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\therefore a=1, \quad b=0$

$a+c=1,$
$b+d=1$

i.e., $a=1, \ b=0, \ c=0, \ d=1$

$\therefore$ matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

2 **a** $\mathbf{AB}$

$$= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= [10]$$

b $\mathbf{BA}$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

c $\mathbf{AC}$

$$= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 & 21 \end{bmatrix}$$

d $\mathbf{CA}$ does not exist as $\mathbf{C}$ is 3×3 and $\mathbf{A}$ is 1×3
not equal

e $\mathbf{CB} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$

3 **a** $\mathbf{A}^{-1} = \dfrac{1}{42-40} \begin{bmatrix} 7 & -8 \\ -5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{bmatrix}$$

b $\mathbf{A}^{-1}$ does not exist as $|\mathbf{A}|$
$= -24 - -24$
$= 0$

c $\mathbf{A}^{-1} = \frac{1}{-3} \begin{bmatrix} -3 & -5 \\ 6 & 11 \end{bmatrix}$

$$= \begin{bmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{bmatrix}$$

4 Sales matrix is $\begin{bmatrix} 42-27 & 54-31 \\ 36-28 & 27-15 \\ 34-28 & 30-22 \end{bmatrix} = \begin{bmatrix} 15 & 23 \\ 8 & 12 \\ 6 & 8 \end{bmatrix}$ Totals matrix is $\begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix}$

$\therefore$ total profit $= \begin{bmatrix} 0.75 & 0.55 & 1.20 \end{bmatrix} \begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix} = [56.3]$ dollars i.e., \$56.30

5 $\mathbf{A} = 2\mathbf{A}^{-1}$

a $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$
$= \mathbf{A}(2\mathbf{A}^{-1})$
$= 2\mathbf{A}\mathbf{A}^{-1}$
$= 2\mathbf{I}$

b $(\mathbf{A}-\mathbf{I})(\mathbf{A}+3\mathbf{I}) = (\mathbf{A}-\mathbf{I})\mathbf{A} + (\mathbf{A}-\mathbf{I})3\mathbf{I}$
$= \mathbf{A}^2 - \mathbf{IA} + 3\mathbf{AI} - 3\mathbf{I}^2$
$= 2\mathbf{I} - \mathbf{A} + 3\mathbf{A} - 3\mathbf{I}$
$= 2\mathbf{A} - \mathbf{I}$

6

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+-8+6 & -2+2+0 & -1-2+3 \\ 6-20+14 & -4+5+0 & -2-5+7 \\ -6+16-10 & 4-4+0 & 2+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 3-4+2 & 6-10+4 & 9-14+5 \\ -4+2+2 & -8+5+4 & -12+7+5 \\ 2+0-2 & 4+0-4 & 6+0-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $\mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \therefore \quad \mathbf{A}^{-1} = \mathbf{B}$

REVIEW SET 14C

1 In matrix form $\begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ has a unique solution if $\begin{vmatrix} k & 3 \\ 1 & k+2 \end{vmatrix} \neq 0$

i.e., $k^2 + 2k - 3 \neq 0$

$(k-1)(k+3) \neq 0$

$k \neq 1$ or -3

and if $k \neq 1$ or -3, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}^{-1}\begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$$= \frac{1}{(k-1)(k+3)}\begin{bmatrix} k+2 & -3 \\ -1 & k \end{bmatrix}\begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(k-1)(k+3)}\begin{bmatrix} -6k-18 \\ 6+2k \end{bmatrix}$$

$$= \frac{1}{(k-1)(k+3)}(k+3)\begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{k-1} \\ \frac{2}{k-1} \end{bmatrix}$$

So the unique solution is $x = \dfrac{-6}{k-1}$, $y = \dfrac{2}{k-1}$

If $k = 1$, the equations are: $\left.\begin{array}{l} x + 3y = -6 \\ x + 3y = 2 \end{array}\right\}$ parallel lines $\therefore$ no solutions exist

If $k = -3$, the equations are: $\left.\begin{array}{r} -3x + 3y = -6 \\ x - y = 2 \end{array}\right\}$ coincident lines $\therefore$ infinitely many solutions

2 $\begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} = x\begin{vmatrix} x+1 & -2 \\ -2 & x+2 \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ x+2 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & x+1 \\ 0 & -2 \end{vmatrix}$

$$\begin{aligned} &= x[(x+1)(x+2)-4] + 2[0-2(x+2)] \\ &= x(x^2+3x+2-4) - 4(x+2) \\ &= x(x^2+3x-2) - 4x - 8 \\ &= x^3 + 3x^2 - 2x - 4x - 8 \\ &= x^3 + 3x^2 - 6x - 8 \\ &= (x+4)(x+1)(x-2) \qquad \{\text{using technology}\} \end{aligned}$$

But $(x+4)(x+1)(x-2) = 0$ {given}

$\therefore$ $x = -4, -1$ or 2

3 $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$

a $2\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -4 & 6 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} -14 & 18 \\ 18 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ -10 & 4 \end{bmatrix}$

b $\mathbf{A}$ is 2×2 and $\mathbf{C}$ is 2×3 $\therefore$ $\mathbf{AC}$ is 2×3

$$\mathbf{AC} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+6 & -6+3 \\ -4+0 & 0-2 & 12-1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{bmatrix}$$

c $\mathbf{C}$ is 2×3 and $\mathbf{B}$ is 2×2 $\therefore$ $\mathbf{CB}$ is not possible.

d

$$\begin{aligned} \mathbf{DA} &= \mathbf{B} \\ \therefore\ \mathbf{DAA}^{-1} &= \mathbf{BA}^{-1} \quad \{\text{post multiplying by } \mathbf{A}^{-1}\} \\ \therefore\ \mathbf{D} &= \mathbf{BA}^{-1} \\ \therefore\ \mathbf{D} &= \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix} \frac{1}{-2(-1)-3(4)}\begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = -\tfrac{1}{10}\begin{bmatrix} 7-36 & 21-18 \\ -9+12 & -27+6 \end{bmatrix} \\ \therefore\ \mathbf{D} &= \begin{bmatrix} \frac{29}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{21}{10} \end{bmatrix} \end{aligned}$$

4

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\mathbf{X} &= \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ \therefore\ \mathbf{X} &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ &= \tfrac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ &= \tfrac{1}{11}\begin{bmatrix} 11 & 33 & 22 \\ -11 & 11 & 33 \end{bmatrix} \\ \therefore\ \mathbf{X} &= \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{bmatrix} \end{aligned}$$

5

$$\begin{aligned} 5\mathbf{A}^2 - 6\mathbf{B} &= 3\mathbf{I} \\ \therefore\ \mathbf{A}(5\mathbf{A} - 6\mathbf{I}) &= 3\mathbf{I} \\ \therefore\ \mathbf{A}\times \tfrac{1}{3}(5\mathbf{A} - 6\mathbf{I}) &= \mathbf{I} \\ \therefore\ \mathbf{A}^{-1} &= \tfrac{5}{3}\mathbf{A} - 2\mathbf{I} \end{aligned}$$

6 **a** **i** If $\mathbf{AB}=\mathbf{B}$ then $\mathbf{ABB}^{-1}=\mathbf{BB}^{-1}$ provided $\mathbf{B}^{-1}$ exists.

$\therefore$ $\mathbf{A}=\mathbf{I}$ provided $\mathbf{B}^{-1}$ exists, i.e., provided that $|\mathbf{B}|\neq 0$.

ii

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^2 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2 \\ &= \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2 \quad \text{provided that } \mathbf{AB}=\mathbf{BA}. \end{aligned}$$

b $\mathbf{M} = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}\begin{bmatrix} k-1 & -2 \\ -3 & -k \end{bmatrix}$ and $\mathbf{M}^{-1}$ exists provided that $|\mathbf{M}|\neq 0$.

$$\begin{aligned} \text{Now}\quad |\mathbf{M}| &= \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix}\begin{vmatrix} k-1 & -2 \\ -3 & k \end{vmatrix} \\ &= (k^2-4)(k^2-k-6) \\ &= (k+2)(k-2)(k+2)(k-3) \end{aligned}$$

$\therefore$ $\mathbf{M}^{-1}$ exists provided that $k\neq 3$ or ± 2

REVIEW SET 14D

1 This system in matrix form is $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$

$$\therefore\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \text{So,} \quad x = 1, \quad y = -1, \quad z = 2. \quad \{\text{using technology}\}$$

2 $$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = (a+b)\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} + c\begin{vmatrix} a & a \\ c+a & b \end{vmatrix} + c\begin{vmatrix} a & b+c \\ b & b \end{vmatrix}$$

$$\begin{aligned} &= (a+b)\,[(b+c)(c+a) - ab] + c\,[ab - a(c+a)] + c\,[ab - (b+c)b] \\ &= (a+b)(bc + ab + c^2 + ac - ab) + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= abc + b^2c + ac^2 + bc^2 + a^2c + abc + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= 4abc \end{aligned}$$

3 If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$,

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}(5\mathbf{A} + 2\mathbf{I}) \\ &= 5\mathbf{A}^2 + 2\mathbf{A} \\ &= 5(5\mathbf{A} + 2\mathbf{I}) + 2\mathbf{A} \\ &= 25\mathbf{A} + 10\mathbf{I} + 2\mathbf{A} \\ &= 27\mathbf{A} + 10\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A}(27\mathbf{A} + 10\mathbf{I}) \\ &= 27\mathbf{A}^2 + 10\mathbf{AI} \\ &= 27(5\mathbf{A} + 2\mathbf{I}) + 10\mathbf{A} \\ &= 135\mathbf{A} + 54\mathbf{I} + 10\mathbf{A} \\ &= 145\mathbf{A} + 54\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^5 &= \mathbf{A}(145\mathbf{A} + 54\mathbf{I}) \\ &= 145\mathbf{A}^2 + 54\mathbf{AI} \\ &= 145(5\mathbf{A} + 2\mathbf{I}) + 54\mathbf{A} \\ &= 725\mathbf{A} + 290\mathbf{I} + 54\mathbf{A} \\ &= 779\mathbf{A} + 290\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{A}(779\mathbf{A} + 290\mathbf{I}) \\ &= 779\mathbf{A}^2 + 290\mathbf{AI} \\ &= 779(5\mathbf{A} + 2\mathbf{I}) + 290\mathbf{AI} \\ &= 4185\mathbf{A} + 1558\mathbf{I} \end{aligned}$$

4 **a** $C(0) = 80 \quad \therefore \quad a(0) + b(0) + c(0) + d = 80 \quad \therefore \quad d = 80$

b
$$\begin{aligned} C(1) &= 100 & \therefore \quad a + b + c + 80 &= 100 \\ C(2) &= 148 & 8a + 4b + 2c + 80 &= 148 \\ C(4) &= 376 & 64a + 16b + 4c + 80 &= 376 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

{technology}

$\therefore \quad a = 2, \quad b = 8, \quad c = 10$

5
$$\begin{aligned} \mathbf{AXB} &= \mathbf{C} \\ \therefore \quad \mathbf{A}^{-1}\mathbf{AXBB}^{-1} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \\ \therefore \quad \mathbf{IXI} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \\ \therefore \quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \end{aligned}$$

So $$\mathbf{X} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -12 & -11 \\ -10 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}^{-1}$$

$$\therefore \quad \mathbf{X} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \qquad \{\text{using technology}\}$$

6 **a** $$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 6, \quad y = -2, \quad z = 1$

b $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{6} \\ -\frac{7}{6} \end{bmatrix}$$

$$\therefore \quad x = \tfrac{3}{2}, \quad y = -\tfrac{7}{6}, \quad z = -\tfrac{7}{6}$$

REVIEW SET 14E

1 **a** Let $\$x$ be the cost of an opera ticket
$\$y$ be the cost of a play ticket
$\$z$ be the cost of a concert ticket

$$3x + 2y + 5z = 267$$
$$2x + 3y + z = 145$$
$$x + 5y + 4z = 230$$

b So, $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$ $\quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18 \\ 27 \end{bmatrix}$ using technology

$\therefore$ cost of each ticket is \$32 for opera, \$18 for play, \$27 for concert.

c Total cost $= 4 \times \$32 + 1 \times \$18 + 2 \times \$27$
$= \$128 + \$18 + \$54 = \$200$

2 $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix}$ $\quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$\therefore \quad x = 2, \quad y = 1, \quad z = 3$ {using technology}

3 **a** $3\mathbf{A} = 3\begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ 5 & 3 \end{bmatrix}$

c $\mathbf{BA}$
$= \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$
$= \begin{bmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{bmatrix}$

d **A** is 2×3 and **C** is 2×2
$\therefore$ **AC** does not exist.

e $\mathbf{BC} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{bmatrix}$

4 $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 & 2-2+4 & -3+8+1 \\ -2+8+18 & 4-4+24 & -6+16+6 \\ -3+2+6 & 6-1+8 & -9+4+2 \end{bmatrix}$$

$$\therefore \mathbf{AB} = \begin{bmatrix} 6 & 4 & 6 \\ 24 & 24 & 16 \\ 5 & 13 & -3 \end{bmatrix}$$

and $\det(\mathbf{AB}) = 6\begin{vmatrix} 24 & 16 \\ 13 & -3 \end{vmatrix} + 4\begin{vmatrix} 16 & 24 \\ -3 & 5 \end{vmatrix} + 6\begin{vmatrix} 24 & 24 \\ 5 & 13 \end{vmatrix}$

$= 6(-72-208)+4(80+72)+6(312-120)$

$= 80$

$\det\mathbf{A} = 1\begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

$= 1(8-6)+2(18-4)+1(2-12)$

$= 2+28-10$

$= 20$ and $\det\mathbf{B} = -1\begin{vmatrix} -1 & 4 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$

$= -1(-1-16)+2(12-2)-3(8--3)$

$= 4$ $\therefore$ $\det\mathbf{A}\times\det\mathbf{B} = 20\times 4 = 80 = \det(\mathbf{AB})$

5 **a** $s(t) = at^2+bt+c$

But $s(1) = 63$ $\therefore$ $a+b+c = 63$
$s(2) = 72$ $\quad 4a+2b+c = 72$
$s(7) = 27$ $\quad 49a+7b+c = 27$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 63 \\ 72 \\ 27 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 63 \\ 72 \\ 27 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \\ 48 \end{bmatrix}$$

b $s(t) = -3t^2+18t+48$ metres
$s(0) = 48$ m, so, the cliff is 48 m high

c $s(t) = 0$ when $-3t^2+18t+48 = 0$

$\therefore\ t^2-6t-16 = 0$

$\therefore\ (t+2)(t-8) = 0$

$\therefore\ t = -2$ or 8

$\therefore\ t = 8$ $\{$as $t>0\}$

$\therefore$ it will take 8 seconds to reach the water.

6 If $\mathbf{A}^2 = \mathbf{A}-\mathbf{I}$ then $\mathbf{A}^3 = \mathbf{A}(\mathbf{A}-\mathbf{I}) = \mathbf{A}^2-\mathbf{AI} = \mathbf{A}-\mathbf{I}-\mathbf{A} = -\mathbf{I}$

$\mathbf{A}^4 = \mathbf{AA}^3 = \mathbf{A}(-\mathbf{I}) = -\mathbf{A}$

$\mathbf{A}^5 = \mathbf{AA}^4 = \mathbf{A}(-\mathbf{A}) = -\mathbf{A}^2 = -(\mathbf{A}-\mathbf{I}) = \mathbf{I}-\mathbf{A}$

$\mathbf{A}^6 = \mathbf{AA}^5 = \mathbf{A}(\mathbf{I}-\mathbf{A}) = \mathbf{AI}-\mathbf{A}^2 = \mathbf{A}-(\mathbf{A}-\mathbf{I}) = \mathbf{I}$

$\mathbf{A}^7 = \mathbf{AA}^6 = \mathbf{AI} = \mathbf{A}$

$\mathbf{A}^8 = \mathbf{AA}^7 = \mathbf{AA} = \mathbf{A}^2 = \mathbf{A}-\mathbf{I}$

a $\mathbf{A}^{6n+3} = -\mathbf{I}$
$\mathbf{A}^{6n+5} = \mathbf{I}-\mathbf{A}$

b Now $\mathbf{A}^2 = \mathbf{A}-\mathbf{I}$

$\therefore\ \mathbf{A}^{-1}\mathbf{AA} = \mathbf{A}^{-1}\mathbf{A}-\mathbf{A}^{-1}\mathbf{I}$ $\{$premultiplying by $\mathbf{A}^{-1}\}$

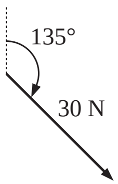
$\therefore\ \mathbf{IA} = \mathbf{I}-\mathbf{A}^{-1}$

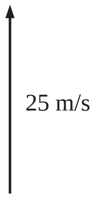
$\therefore\ \mathbf{A}^{-1} = \mathbf{I}-\mathbf{A}$

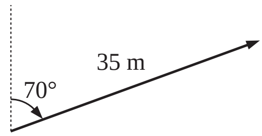
Chapter 15

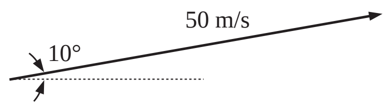
VECTORS IN 2-DIMENSIONS

EXERCISE 15A.1

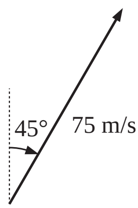
1 **a**


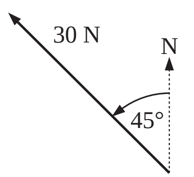
b


c


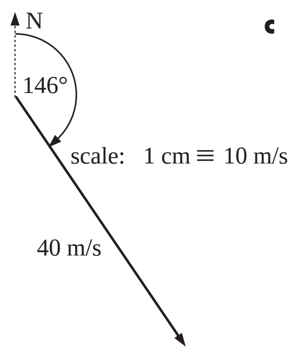
d


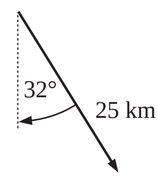
2 **a**


b


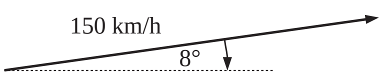
3 **a**


scale: 1 cm ≡ 10 N

b


c


Scale: 1 cm ≡ 10 km

d


Scale: 1 cm ≡ 30 km/h

EXERCISE 15A.2

1 **a** If they are equal in magnitude, they have the same length. These are **p**, **q**, **s** and **t**.

b Those parallel are **p**, **q**, **r** and **t**.

c Those in the same direction are: **p** and **r**, **q** and **t**.

d To be equal they must have the same direction and be equal in length $\therefore$ $\mathbf{q} = \mathbf{t}$.

e **p** and **q** are negatives (equal length, but opposite direction). Likewise **p** and **t** are negatives. We write $\mathbf{p} = -\mathbf{q}$ and $\mathbf{p} = -\mathbf{t}$.

2 **a** True, as they have the same length and are parallel.

b True, as they are sides of an equilateral triangle.

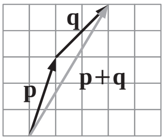
c False, as they do not have the same direction.

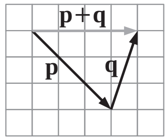
d False, as they have opposite directions.

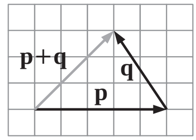
e True, as they have the same length and direction.

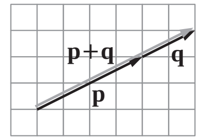
f False, as they do not have the same direction.

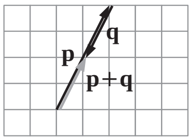
EXERCISE 15B.1

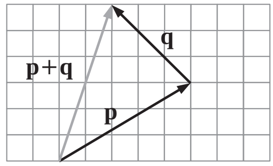
1 **a** 

b 

c 

d 

e 

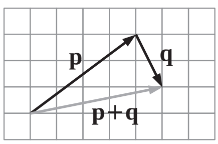
f 

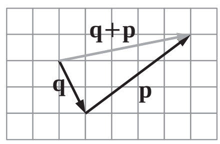
2 **a** $\overrightarrow{AB}+\overrightarrow{BC}$
$=\overrightarrow{AC}$

b $\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{BD}$

c $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

d $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$
$=\overrightarrow{AB}+\overrightarrow{BD}$
$=\overrightarrow{AD}$

3 **a** **i** 

ii 

b yes

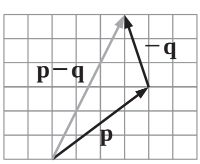
4 $\overrightarrow{PS}=\overrightarrow{PR}+\overrightarrow{RS}$
$=(\mathbf{a}+\mathbf{b})+\mathbf{c}$

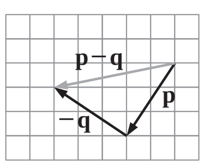
But $\overrightarrow{PS}=\overrightarrow{PQ}+\overrightarrow{QS}$
$=\mathbf{a}+(\mathbf{b}+\mathbf{c})$

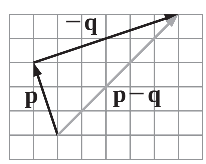
$\therefore\ (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ {as both are equal to $\overrightarrow{PS}$}

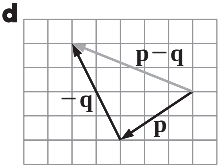
Note: $\overrightarrow{PS}=\mathbf{a}+\mathbf{b}+\mathbf{c}$ also

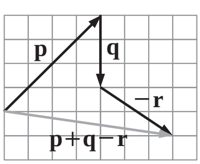
EXERCISE 15B.2

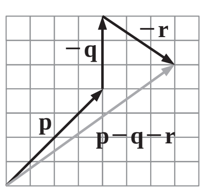
1 **a** 

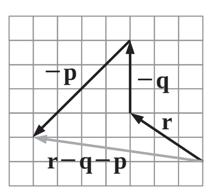
b 

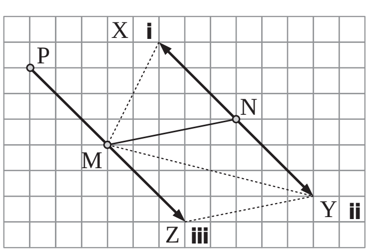
c 

d 

2 **a** 

b 

c 

3 **a** 

b a parallelogram

4 **a** $\overrightarrow{AC}+\overrightarrow{CB}$
$=\overrightarrow{AB}$

b $\overrightarrow{AD}-\overrightarrow{BD}$
$=\overrightarrow{AD}+\overrightarrow{DB}$
$=\overrightarrow{AB}$

c $\overrightarrow{AC}+\overrightarrow{CA}$
$=\overrightarrow{AA}$
$=\mathbf{0}$

d $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

e $\overrightarrow{BA}-\overrightarrow{CA}+\overrightarrow{CB}$
$=\overrightarrow{BA}+\overrightarrow{AC}+\overrightarrow{CB}$
$=\overrightarrow{BC}+\overrightarrow{CB}$
$=\overrightarrow{BB}$
$=\mathbf{0}$

f $\overrightarrow{AB}-\overrightarrow{CB}-\overrightarrow{DC}$
$=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

5 **a** $\mathbf{t}=\mathbf{r}+\mathbf{s}$ **b** $\mathbf{r}=-\mathbf{s}-\mathbf{t}$ **c** $\mathbf{r}=-\mathbf{p}-\mathbf{q}-\mathbf{s}$ **d** $\mathbf{r}=\mathbf{q}-\mathbf{p}+\mathbf{s}$
e $\mathbf{p}=\mathbf{t}+\mathbf{s}+\mathbf{r}-\mathbf{q}$ **f** $\mathbf{p}=-\mathbf{u}+\mathbf{t}+\mathbf{s}-\mathbf{r}-\mathbf{q}$

6 **a** **i** $\overrightarrow{OB}$
$=\overrightarrow{OA}+\overrightarrow{AB}$
$=\mathbf{r}+\mathbf{s}$

ii $\overrightarrow{CA}$
$=\overrightarrow{CB}+\overrightarrow{BA}$
$=-\overrightarrow{BC}-\overrightarrow{AB}$
$=-\mathbf{t}-\mathbf{s}$

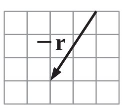
iii $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}$
$=\mathbf{r}+\mathbf{s}+\mathbf{t}$

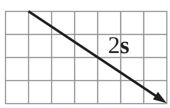
b **i** $\overrightarrow{AD}$
$=\overrightarrow{AB}+\overrightarrow{BD}$
$=\mathbf{p}+\mathbf{q}$

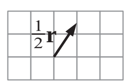
ii $\overrightarrow{BC}$
$=\overrightarrow{BD}+\overrightarrow{DC}$
$=\mathbf{q}+\mathbf{r}$

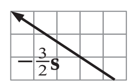
iii $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$
$=\mathbf{p}+\mathbf{q}+\mathbf{r}$

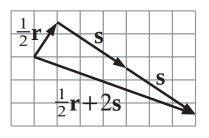
EXERCISE 15B.3

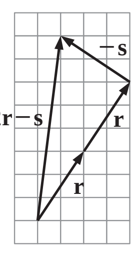
1 **a**


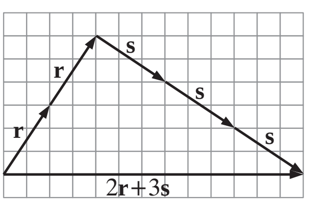
b


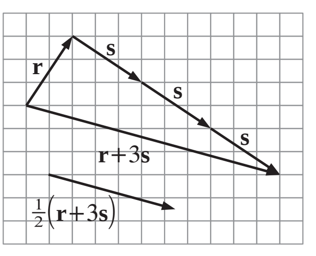
c


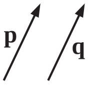
d


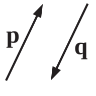
g


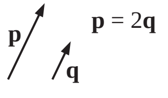
e


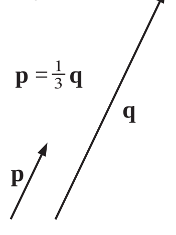
f


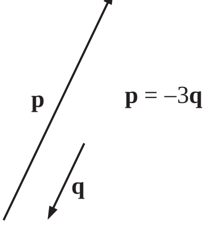
h


2 **a**


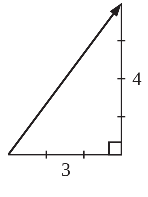
b


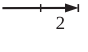
c


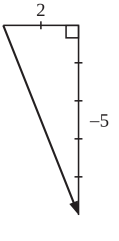
d


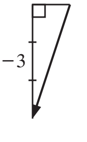
e


EXERCISE 15C.1

1 **a**


b


c


d


2 **a** $\begin{bmatrix}7\\3\end{bmatrix}$ **b** $\begin{bmatrix}-6\\0\end{bmatrix}$ **c** $\begin{bmatrix}2\\-5\end{bmatrix}$ **d** $\begin{bmatrix}0\\6\end{bmatrix}$ **e** $\begin{bmatrix}-6\\3\end{bmatrix}$ **f** $\begin{bmatrix}-5\\-5\end{bmatrix}$

EXERCISE 15C.2

1 **a** $\mathbf{a}+\mathbf{b} = \begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$

b $\mathbf{b}+\mathbf{a} = \begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$

c $\mathbf{b}+\mathbf{c} = \begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

d $\mathbf{c}+\mathbf{b} = \begin{bmatrix}-2\\-5\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

e $\mathbf{a}+\mathbf{c} = \begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix} = \begin{bmatrix}-5\\-3\end{bmatrix}$

f $\mathbf{c}+\mathbf{a} = \begin{bmatrix}-2\\-5\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}-5\\-3\end{bmatrix}$

g $\mathbf{a}+\mathbf{a} = \begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}-6\\4\end{bmatrix}$

h $\mathbf{b}+\mathbf{a}+\mathbf{c} = \begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$

2 **a** $\mathbf{p}-\mathbf{q} = \begin{bmatrix}-4\\2\end{bmatrix}-\begin{bmatrix}-1\\-5\end{bmatrix} = \begin{bmatrix}-3\\7\end{bmatrix}$

b $\mathbf{q}-\mathbf{r} = \begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}-4\\-3\end{bmatrix}$

c $\mathbf{p}+\mathbf{q}-\mathbf{r} = \begin{bmatrix}-4\\2\end{bmatrix}+\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}-8\\-1\end{bmatrix}$

d $\mathbf{p}-\mathbf{q}-\mathbf{r} = \begin{bmatrix}-4\\2\end{bmatrix}-\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}-6\\9\end{bmatrix}$

e $\mathbf{q}-\mathbf{r}-\mathbf{p} = \begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}0\\-5\end{bmatrix}$

f $\mathbf{r}+\mathbf{q}-\mathbf{p} = \begin{bmatrix}3\\-2\end{bmatrix}+\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}6\\-9\end{bmatrix}$

3 **a** $\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC} = -\overrightarrow{BA}+\overrightarrow{BC} = -\begin{bmatrix}2\\-3\end{bmatrix}+\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-5\\4\end{bmatrix}$

b $\overrightarrow{CB} = \overrightarrow{CA}+\overrightarrow{AB} = \begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

c $\overrightarrow{SP} = \overrightarrow{SR}+\overrightarrow{RQ}+\overrightarrow{QP} = -\overrightarrow{RS}+\overrightarrow{RQ}-\overrightarrow{PQ} = -\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}-1\\4\end{bmatrix} = \begin{bmatrix}6\\-5\end{bmatrix}$

EXERCISE 15C.3

1 **a** $-3\mathbf{p} = -3\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}-3\\-15\end{bmatrix}$

b $\frac{1}{2}\mathbf{q} = \frac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$

c $2\mathbf{p}+\mathbf{q} = 2\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}2\\10\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}0\\14\end{bmatrix}$

d $\mathbf{p}-2\mathbf{q} = \begin{bmatrix}1\\5\end{bmatrix}-2\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}-\begin{bmatrix}-4\\8\end{bmatrix} = \begin{bmatrix}5\\-3\end{bmatrix}$

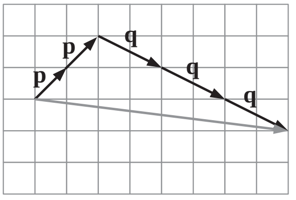
e $\mathbf{p}-\frac{1}{2}\mathbf{r} = \begin{bmatrix}1\\5\end{bmatrix}-\begin{bmatrix}-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}2\frac{1}{2}\\5\frac{1}{2}\end{bmatrix}$

f $2\mathbf{p}+3\mathbf{r} = 2\begin{bmatrix}1\\5\end{bmatrix}+3\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}2\\10\end{bmatrix}+\begin{bmatrix}-9\\-3\end{bmatrix} = \begin{bmatrix}-7\\7\end{bmatrix}$

g $2\mathbf{q}-3\mathbf{r} = 2\begin{bmatrix}-2\\4\end{bmatrix}-3\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}-4\\8\end{bmatrix}-\begin{bmatrix}-9\\-3\end{bmatrix} = \begin{bmatrix}5\\11\end{bmatrix}$

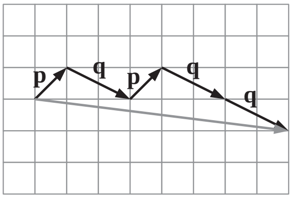
h $2\mathbf{p}-\mathbf{q}+\frac{1}{3}\mathbf{r} = \begin{bmatrix}2\\10\end{bmatrix}-\begin{bmatrix}-2\\4\end{bmatrix}+\begin{bmatrix}-1\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}3\\5\frac{2}{3}\end{bmatrix}$

2 **a**



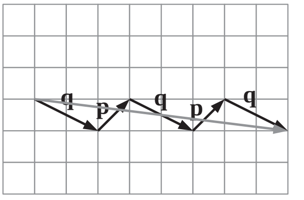
$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$

b



$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$

c



$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$

All vector expressions are equal {each consists of 2**p**s and 3**q**s}.
Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

EXERCISE 15C.4

1 **a** $|\mathbf{r}| = \sqrt{2^2+3^2} = \sqrt{13}$ units

b $|\mathbf{s}| = \sqrt{(-1)^2+4^2} = \sqrt{17}$ units

c $\mathbf{r}+\mathbf{s} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$\therefore\ |\mathbf{r}+\mathbf{s}| = \sqrt{1^2+7^2} = \sqrt{50}$ units

d $\mathbf{r}-\mathbf{s} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\therefore\ |\mathbf{r}-\mathbf{s}| = \sqrt{3^2+(-1)^2} = \sqrt{10}$ units

e $\mathbf{s}-2\mathbf{r} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$

$\therefore\ |\mathbf{s}-2\mathbf{r}| = \sqrt{(-5)^2+(-2)^2} = \sqrt{29}$ units

2 **a** $|\mathbf{p}| = \sqrt{1^2+3^2} = \sqrt{10}$ units

b $2\mathbf{p} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$\therefore\ |2\mathbf{p}| = \sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$ units

c $-2\mathbf{p} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

$\therefore\ |-2\mathbf{p}| = \sqrt{(-2)^2+(-6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$ units

d $3\mathbf{p} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

$\therefore\ |3\mathbf{p}| = \sqrt{3^2+9^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$ units

e $-3\mathbf{p} = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

$\therefore\ |-3\mathbf{p}| = \sqrt{(-3)^2+(-9)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$ units

f $|\mathbf{q}| = \sqrt{(-2)^2+4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ units

g $4\mathbf{q} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

$\therefore\ |4\mathbf{q}| = \sqrt{(-8)^2+16^2} = \sqrt{64+256} = \sqrt{320} = 8\sqrt{5}$ units

h $-4\mathbf{q} = \begin{bmatrix} 8 \\ -16 \end{bmatrix}$

$\therefore\ |-4\mathbf{q}| = \sqrt{8^2+(-16)^2} = \sqrt{64+256} = \sqrt{320} = 8\sqrt{5}$ units

i $\frac{1}{2}\mathbf{q} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore\ \left|\frac{1}{2}\mathbf{q}\right| = \sqrt{(-1)^2+2^2} = \sqrt{5}$ units

j $-\frac{1}{2}\mathbf{q} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\therefore\ \left|-\frac{1}{2}\mathbf{q}\right| = \sqrt{1^2+(-2)^2} = \sqrt{5}$ units

3 $k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix}$ $\therefore\ |k\mathbf{a}|$

$$\begin{aligned} |k\mathbf{a}| &= \sqrt{(ka_1)^2+(ka_2)^2} \\ &= \sqrt{k^2a_1^2+k^2a_2^2} \\ &= \sqrt{k^2(a_1^2+a_2^2)} \\ &= \sqrt{k^2}\sqrt{a_1^2+a_2^2} \\ &= |k|\,\sqrt{a_1^2+a_2^2} \\ &= |k|\,|\mathbf{a}| \end{aligned}$$

EXERCISE 15D

1 **a**
$$\begin{aligned} 2\mathbf{x} &= \mathbf{q} \\ \therefore\ \tfrac{1}{2}(2\mathbf{x}) &= \tfrac{1}{2}\mathbf{q} \\ \therefore\ \mathbf{x} &= \tfrac{1}{2}\mathbf{q} \end{aligned}$$

b
$$\begin{aligned} \tfrac{1}{2}\mathbf{x} &= \mathbf{n} \\ \therefore\ 2(\tfrac{1}{2}\mathbf{x}) &= 2\mathbf{n} \\ \therefore\ \mathbf{x} &= 2\mathbf{n} \end{aligned}$$

c
$$\begin{aligned} -3\mathbf{x} &= \mathbf{p} \\ \therefore\ 3\mathbf{x} &= -\mathbf{p} \\ \therefore\ \tfrac{1}{3}(3\mathbf{x}) &= -\tfrac{1}{3}\mathbf{p} \\ \therefore\ \mathbf{x} &= -\tfrac{1}{3}\mathbf{p} \end{aligned}$$

d
$$\begin{aligned} \mathbf{q} + 2\mathbf{x} &= \mathbf{r} \\ \therefore\ 2\mathbf{x} &= \mathbf{r} - \mathbf{q} \\ \therefore\ \mathbf{x} &= \tfrac{1}{2}(\mathbf{r} - \mathbf{q}) \end{aligned}$$

e
$$\begin{aligned} 4\mathbf{s} - 5\mathbf{x} &= \mathbf{t} \\ \therefore\ -5\mathbf{x} &= \mathbf{t} - 4\mathbf{s} \\ \therefore\ 5\mathbf{x} &= 4\mathbf{s} - \mathbf{t} \\ \therefore\ \mathbf{x} &= \tfrac{4}{5}\mathbf{s} - \tfrac{1}{5}\mathbf{t} \end{aligned}$$

f
$$\begin{aligned} 4\mathbf{m} - \tfrac{1}{3}\mathbf{x} &= \mathbf{n} \\ \therefore\ 4\mathbf{m} - \mathbf{n} &= \tfrac{1}{3}\mathbf{x} \\ \therefore\ \mathbf{x} &= 12\mathbf{m} - 3\mathbf{n} \end{aligned}$$

2 **a**
$$\begin{aligned} 2\mathbf{y} &= \mathbf{r} \\ \therefore\ \mathbf{y} &= \tfrac{1}{2}\mathbf{r} \\ &= \tfrac{1}{2}\begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} \end{aligned}$$

b
$$\begin{aligned} \tfrac{1}{2}\mathbf{y} &= \mathbf{s} \\ \therefore\ \mathbf{y} &= 2\mathbf{s} \\ &= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{aligned}$$

c
$$\begin{aligned} \mathbf{r} + 2\mathbf{y} &= \mathbf{s} \\ \therefore\ 2\mathbf{y} &= \mathbf{s} - \mathbf{r} \\ \therefore\ \mathbf{y} &= \tfrac{1}{2}\mathbf{s} - \tfrac{1}{2}\mathbf{r} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \end{aligned}$$

d
$$\begin{aligned} 3\mathbf{s} - 4\mathbf{y} &= \mathbf{r} \\ \therefore\ 3\mathbf{s} - \mathbf{r} &= 4\mathbf{y} \\ \therefore\ \mathbf{y} &= \tfrac{3}{4}\mathbf{s} - \tfrac{1}{4}\mathbf{r} \\ &= \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -\frac{2}{4} \\ \frac{3}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix} \end{aligned}$$

3 $k\mathbf{x} = \mathbf{a}$ $\quad\therefore\ k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\therefore\ kx_1 = a_1$ and $kx_2 = a_2$

$\therefore\ x_1 = \frac{1}{k}a_1$ and $x_2 = \frac{1}{k}a_2$ $\quad\therefore\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

EXERCISE 15E

1 **a** $\overrightarrow{AB} = \begin{bmatrix} 2-3 \\ 6--2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$

and $AB = \left|\overrightarrow{AB}\right| = \sqrt{1+64}$

$\therefore\ AB = \sqrt{65}$ units

b $\overrightarrow{BC} = \begin{bmatrix} -1-2 \\ -4-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix}$

and $BC = \left|\overrightarrow{BC}\right|$
$= \sqrt{9+100}$
$= \sqrt{109}$ units

c $\overrightarrow{CA} = \begin{bmatrix} 3--1 \\ -2--4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

and $CA = \left|\overrightarrow{CA}\right|$
$= \sqrt{16+4}$
$= \sqrt{20}$ units

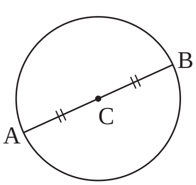
2 **a** M is $\left(\frac{3+-1}{2}, \frac{6+2}{2}\right)$

i.e., M is (1, 4)

b
$$\begin{aligned} \overrightarrow{CA} &= \begin{bmatrix} 3--4 \\ 6-1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \\ \overrightarrow{CM} &= \begin{bmatrix} 1--4 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ \overrightarrow{CB} &= \begin{bmatrix} -1--4 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

c
$$\begin{aligned} &\tfrac{1}{2}\overrightarrow{CA} + \tfrac{1}{2}\overrightarrow{CB} \\ &= \tfrac{1}{2}\begin{bmatrix} 7 \\ 5 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \text{which is } \overrightarrow{CM} \end{aligned}$$

3



a $\overrightarrow{AC} = \begin{bmatrix} 1-3 \\ 4--2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ $\quad\therefore$ B is $(1-2, 4+6)$, i.e., $(-1, 10)$

b $\overrightarrow{AC} = \begin{bmatrix} -1-0 \\ -2-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ $\quad\therefore$ B is $(-1-1, -2-7)$, i.e., $(-2, -9)$

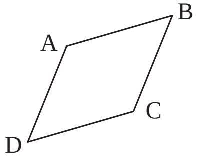
c $\overrightarrow{AC} = \begin{bmatrix} 3--1 \\ 0--4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\quad\therefore$ B is $(3+4, 0+4)$, i.e., $(7, 4)$

4 $\overrightarrow{AB} = \begin{bmatrix} 2--1 \\ 3-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\quad\therefore$ C is $(2+3, 3-2)$ i.e., $(5, 1)$

D is $(5+3, 1-2)$ i.e., $(8, -1)$

E is $(8+3, -1-2)$ i.e., $(11, -3)$

5



a $\overrightarrow{AB} = \begin{bmatrix} 4-3 \\ 2--1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} -1--2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ sides AB and DC are equal in length and parallel. This is sufficient to deduce that ABCD is a parallelogram.

b $\overrightarrow{AB} = \begin{bmatrix} -1-5 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} 4-10 \\ -3--6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

So, $\overrightarrow{AB} \neq \overrightarrow{DC}$

$\therefore$ ABCD cannot be a parallelogram.

c $\overrightarrow{AB} = \begin{bmatrix} 1-2 \\ 4--3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} -2--1 \\ 6--1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$.

$\therefore$ sides AB and DC are equal in length and parallel. This is sufficient to deduce that ABCD is a parallelogram.

6 **a** Let D be (a, b)

Now $\overrightarrow{CD} = \overrightarrow{BA}$

$\therefore \begin{bmatrix} a-8 \\ b--2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0--1 \end{bmatrix}$

$\therefore \begin{bmatrix} a-8 \\ b+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore a = 9, \quad b = -1$

So, D is $(9, -1)$

b Let R be (a, b)

Now $\overrightarrow{SR} = \overrightarrow{PQ}$

$\therefore \begin{bmatrix} a-4 \\ b-0 \end{bmatrix} = \begin{bmatrix} -2--1 \\ 5-4 \end{bmatrix}$

$\therefore \begin{bmatrix} a-4 \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore a = 3, \quad b = 1$

So, R is $(3, 1)$

c Let X be (a, b)

Now $\overrightarrow{WX} = \overrightarrow{ZY}$

$\therefore \begin{bmatrix} a--1 \\ b-5 \end{bmatrix} = \begin{bmatrix} 3-0 \\ -2-4 \end{bmatrix}$

$\therefore \begin{bmatrix} a+1 \\ b-5 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

$\therefore a = 2, \quad b = -1$

So, X is $(2, -1)$

7 **a**

$$r\begin{bmatrix} 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\therefore \begin{bmatrix} r+2s \\ -r+5s \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad r + 2s &= -8 \quad \text{...... (1)} \\ -r + 5s &= -27 \\ \hline \text{adding} \quad 7s &= -35 \\ \therefore \quad s &= -5 \end{aligned}$$

and in (1) $r + 2(-5) = -8$

$\therefore r - 10 = -8$

$\therefore r = 2$

So, $r = 2, \quad s = -5$

b

$$r\begin{bmatrix} 2 \\ -3 \end{bmatrix} + s\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2r+s \\ -3r+7s \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad 2r + s &= 7 \quad \times 3 \quad \text{...... (1)} \\ -3r + 7s &= -19 \quad \times 2 \end{aligned}$$

$$\begin{aligned} \therefore \quad 6r + 3s &= 21 \\ -6r + 14s &= -38 \\ \hline \text{adding} \quad 17s &= -17 \\ \therefore \quad s &= -1 \end{aligned}$$

and in (1) $2r - 1 = 7$

$\therefore 2r = 8$

$\therefore r = 4$

So, $r = 4, \quad s = -1$

EXERCISE 15F

1 $\mathbf{d} = \begin{bmatrix} -20 \\ 50 \end{bmatrix} = 10\begin{bmatrix} -2 \\ 5 \end{bmatrix} = 10\mathbf{c}, \quad \therefore$ **d** is parallel to **c**

$\mathbf{e} = \begin{bmatrix} -12 \\ 4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ -1 \end{bmatrix} = -4\mathbf{b} \quad \therefore$ **e** is parallel to **b**

$\mathbf{f} = \begin{bmatrix} 10 \\ 25 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5\mathbf{a} \quad \therefore$ **f** is parallel to **a**

2 **a** $\begin{bmatrix} 9 \\ t \end{bmatrix} = k\begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \therefore \begin{cases} 3k = 9 \\ t = 4k \end{cases}$ so, $k = 3$ and $t = 12$

b $\begin{bmatrix} -10 \\ t \end{bmatrix} = k\begin{bmatrix} 5 \\ -4 \end{bmatrix} \quad \therefore \begin{cases} 5k = -10 \\ -4k = t \end{cases}$ so, $k = -2$ and $t = 8$

c $\begin{bmatrix} 16 \\ t \end{bmatrix} = k\begin{bmatrix} 12 \\ 9 \end{bmatrix} \quad \therefore \begin{cases} 12k = 16 \\ 9k = t \end{cases}$ so, $k = \frac{4}{3}$ and $t = 9 \times \frac{4}{3} = 12$

EXERCISE 15G

1 **a** length $= \sqrt{0^2 + (-1)^2}$ $= \sqrt{0+1}$ $= 1$ unit

$\therefore$ is a unit vector

b length $= \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$ $= \sqrt{\frac{1}{2} + \frac{1}{2}}$ $= 1$ unit

$\therefore$ is a unit vector

c length $= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$ $= \sqrt{\frac{4}{9} + \frac{1}{9}}$ $= \sqrt{\frac{5}{9}} \neq 1$

$\therefore$ is not a unit vector

d length $= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$ $= \sqrt{\frac{9}{25} + \frac{16}{25}}$ $= \sqrt{\frac{25}{25}}$ $= 1$ unit

$\therefore$ is a unit vector

e length $= \sqrt{\left(\frac{2}{7}\right)^2 + \left(-\frac{5}{7}\right)^2}$ $= \sqrt{\frac{4}{49} + \frac{25}{49}}$ $= \sqrt{\frac{29}{49}} \neq 1$

$\therefore$ is not a unit vector

2 **a** $\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\mathbf{i} - \mathbf{j}$ **b** $\begin{bmatrix} -3 \\ -4 \end{bmatrix} = -3\mathbf{i} - 4\mathbf{j}$ **c** $\begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3\mathbf{i}$ **d** $\begin{bmatrix} 0 \\ 7 \end{bmatrix} = 7\mathbf{j}$

e $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

3 **a** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ **c** $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

4 **a** length $= 1$

$\therefore \sqrt{0^2 + k^2} = 1$

$\therefore k^2 = 1$

$\therefore k = \pm 1$

b length $= 1$

$\therefore \sqrt{k^2 + 0} = 1$

$\therefore k^2 = 1$

$\therefore k = \pm 1$

c length $= 1$

$\therefore \sqrt{k^2 + 1} = 1$

$\therefore k^2 + 1 = 1$

$\therefore k^2 = 0$

$\therefore k = 0$

d length $= 1$

$\therefore \sqrt{\left(-\frac{1}{2}\right)^2 + k^2} = 1$

$\therefore \frac{1}{4} + k^2 = 1$

$\therefore k^2 = \frac{3}{4}$

$\therefore k = \pm\frac{\sqrt{3}}{2}$

e length $= 1$

$\therefore \sqrt{k^2 + \left(\frac{2}{3}\right)^2} = 1$

$\therefore k^2 + \frac{4}{9} = 1$

$\therefore k^2 = \frac{5}{9}$ and so $k = \pm\frac{\sqrt{5}}{3}$

5 **a** length $= \sqrt{3^2 + 4^2}$ $= 5$ units

b length $= \sqrt{3^2 + (-4)^2}$ $= \sqrt{9+16}$ $= \sqrt{25}$ $= 5$ units

c length $= \sqrt{(-2)^2 + (-7)^2}$ $= \sqrt{4+49}$ $= \sqrt{53}$ units

d length $= \sqrt{(-2.36)^2 + (5.65)^2} \doteqdot 6.12$ units

6 **a** $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ units $\therefore$ unit vector $= \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

b $2\mathbf{i} - 2\mathbf{j}$ has length $\sqrt{2^2 + (-2)^2} = \sqrt{8}$ or $2\sqrt{2}$ units $\therefore$ unit vector $= \frac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{j})$ $= \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

c $-2\mathbf{i} - 5\mathbf{j}$ has length $\sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$ units $\therefore$ unit vector is $\frac{1}{\sqrt{29}}(-2\mathbf{i} - 5\mathbf{j})$

EXERCISE 15H

1 **a** $\mathbf{q} \bullet \mathbf{p} = \begin{bmatrix}-1\\5\end{bmatrix} \bullet \begin{bmatrix}3\\2\end{bmatrix} = -3 + 10 = 7$

b $\mathbf{q} \bullet \mathbf{r} = \begin{bmatrix}-1\\5\end{bmatrix} \bullet \begin{bmatrix}-2\\4\end{bmatrix} = 2 + 20 = 22$

c $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) = \begin{bmatrix}-1\\5\end{bmatrix} \bullet \begin{bmatrix}1\\6\end{bmatrix} = -1 + 30 = 29$

d $3\mathbf{r} \bullet \mathbf{q} = \begin{bmatrix}-6\\12\end{bmatrix} \bullet \begin{bmatrix}-1\\5\end{bmatrix} = 6 + 60 = 66$

e $2\mathbf{p} \bullet 2\mathbf{p} = \begin{bmatrix}6\\4\end{bmatrix} \bullet \begin{bmatrix}6\\4\end{bmatrix} = 36 + 16 = 52$

f $\mathbf{i} \bullet \mathbf{p} = \begin{bmatrix}1\\0\end{bmatrix} \bullet \begin{bmatrix}3\\2\end{bmatrix} = 3 + 0 = 3$

g $\mathbf{q} \bullet \mathbf{j} = \begin{bmatrix}-1\\5\end{bmatrix} \bullet \begin{bmatrix}0\\1\end{bmatrix} = 0 + 5 = 5$

h $\mathbf{i} \bullet \mathbf{i} = \begin{bmatrix}1\\0\end{bmatrix} \bullet \begin{bmatrix}1\\0\end{bmatrix} = 1 + 0 = 1$

2 **a**
$$\begin{aligned}\begin{bmatrix}3\\t\end{bmatrix} \bullet \begin{bmatrix}-2\\1\end{bmatrix} &= 0\\ \therefore\ -6 + t &= 0\\ \therefore\ t &= 6\end{aligned}$$

b
$$\begin{aligned}\begin{bmatrix}3\\5\end{bmatrix} \bullet \begin{bmatrix}10\\t\end{bmatrix} &= 0\\ \therefore\ 30 + 5t &= 0\\ \therefore\ t &= -6\end{aligned}$$

c
$$\begin{aligned}\begin{bmatrix}t\\-2\end{bmatrix} \bullet \begin{bmatrix}t\\6\end{bmatrix} &= 0\\ \therefore\ t^2 - 12 &= 0\\ \therefore\ t^2 &= 12\\ \therefore\ t &= \pm\sqrt{12}\end{aligned}$$

d
$$\begin{aligned}\begin{bmatrix}t\\t+2\end{bmatrix} \bullet \begin{bmatrix}3\\-4\end{bmatrix} &= 0\\ \therefore\ 3t - 4(t+2) &= 0\\ \therefore\ 3t - 4t - 8 &= 0\\ \therefore\ -t &= 8\\ \therefore\ t &= -8\end{aligned}$$

e
$$\begin{aligned}\begin{bmatrix}t\\t+2\end{bmatrix} \bullet \begin{bmatrix}2-3t\\t\end{bmatrix} &= 0\\ \therefore\ 2t - 3t^2 + t^2 + 2t &= 0\\ \therefore\ -2t^2 + 4t &= 0\\ \therefore\ t^2 - 2t &= 0\\ \therefore\ t(t-2) &= 0\\ \therefore\ t &= 0 \text{ or } 2\end{aligned}$$

3 **a**
$$\begin{aligned}\overrightarrow{AB} &= \begin{bmatrix}-2--2\\5-1\end{bmatrix} = \begin{bmatrix}0\\4\end{bmatrix}\\ \overrightarrow{BC} &= \begin{bmatrix}3--2\\1-5\end{bmatrix} = \begin{bmatrix}5\\-4\end{bmatrix}\\ \overrightarrow{AC} &= \begin{bmatrix}3--2\\1-1\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}\\ \text{and}\quad \overrightarrow{AB} \bullet \overrightarrow{AC} &= \begin{bmatrix}0\\4\end{bmatrix} \bullet \begin{bmatrix}5\\0\end{bmatrix}\\ &= 0 + 0\\ &= 0\end{aligned}$$

$\therefore$ $\measuredangle$BAC is a right angle
i.e., ΔABC is right-angled at B.

b
$$\begin{aligned}\overrightarrow{AB} &= \begin{bmatrix}1-4\\2-7\end{bmatrix} = \begin{bmatrix}-3\\-5\end{bmatrix}\\ \overrightarrow{BC} &= \begin{bmatrix}-1-1\\6-2\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix}\\ \overrightarrow{AC} &= \begin{bmatrix}-1-4\\6-7\end{bmatrix} = \begin{bmatrix}-5\\-1\end{bmatrix}\end{aligned}$$

None of the possible dot products result in 0.
$\therefore$ ΔABC is not right-angled.

c
$$\begin{aligned}\overrightarrow{AB} &= \begin{bmatrix}5-2\\7--2\end{bmatrix} = \begin{bmatrix}3\\9\end{bmatrix}\\ \overrightarrow{BC} &= \begin{bmatrix}-1-5\\-1-7\end{bmatrix} = \begin{bmatrix}-6\\-8\end{bmatrix}\\ \overrightarrow{AC} &= \begin{bmatrix}-1-2\\-1--2\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}\\ \text{and}\quad \overrightarrow{AB} \bullet \overrightarrow{AC} &= \begin{bmatrix}3\\9\end{bmatrix} \bullet \begin{bmatrix}-3\\1\end{bmatrix}\\ &= -9 + 9\\ &= 0\end{aligned}$$

$\therefore$ $\measuredangle$BAC is a right angle
i.e., ΔABC is right-angled at A.

d
$$\begin{aligned}\overrightarrow{AB} &= \begin{bmatrix}5-10\\2-1\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix}\\ \overrightarrow{BC} &= \begin{bmatrix}7-5\\4-2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}\\ \overrightarrow{AC} &= \begin{bmatrix}7-10\\4-1\end{bmatrix} = \begin{bmatrix}-3\\3\end{bmatrix}\\ \text{and}\quad \overrightarrow{BC} \bullet \overrightarrow{AC} &= \begin{bmatrix}2\\2\end{bmatrix} \bullet \begin{bmatrix}-3\\3\end{bmatrix}\\ &= -6 + 6\\ &= 0\end{aligned}$$

$\therefore$ $\measuredangle$BCA is a right angle
i.e., ΔABC is right-angled at C.

4 **a** $\begin{bmatrix}5\\2\end{bmatrix} \bullet \begin{bmatrix}-2\\5\end{bmatrix} = -10 + 10 = 0$, so $\begin{bmatrix}-2\\5\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}-2\\5\end{bmatrix}$ where $k \neq 0$.

Note: $k\begin{bmatrix}2\\-5\end{bmatrix}$, $k \neq 0$ is also ok.

b $\begin{bmatrix}-1\\-2\end{bmatrix} \bullet \begin{bmatrix}-2\\1\end{bmatrix} = 2 - 2 = 0$, so $\begin{bmatrix}-2\\1\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}-2\\1\end{bmatrix}$, $k \neq 0$.

c $\begin{bmatrix}3\\-1\end{bmatrix} \bullet \begin{bmatrix}1\\3\end{bmatrix} = 3 - 3 = 0$, so $\begin{bmatrix}1\\3\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}1\\3\end{bmatrix}$, $k \neq 0$.

d $\begin{bmatrix}-4\\3\end{bmatrix} \bullet \begin{bmatrix}3\\4\end{bmatrix} = -12 + 12 = 0$, so $\begin{bmatrix}3\\4\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}3\\4\end{bmatrix}$, $k \neq 0$

e $\begin{bmatrix}2\\0\end{bmatrix} \bullet \begin{bmatrix}0\\1\end{bmatrix} = 0 + 0 = 0$, so $\begin{bmatrix}0\\1\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}0\\1\end{bmatrix}$, $k \neq 0$.

5 **a**
$$\begin{aligned}\mathbf{p} \bullet \mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}| \cos\theta \\ &= 2 \times 5 \times \cos 60^o \\ &= 5\end{aligned}$$

b
$$\begin{aligned}\mathbf{p} \bullet \mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}| \cos\theta \\ &= 6 \times 3 \times \cos 120^o \\ &= -9\end{aligned}$$

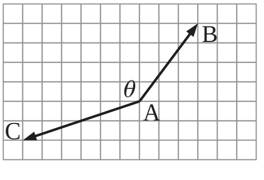
6 **a** **i** $\mathbf{a} = \begin{bmatrix}4\\4\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}7\\-1\end{bmatrix}$

ii
$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \\ \therefore \begin{bmatrix}4\\4\end{bmatrix} \bullet \begin{bmatrix}7\\-1\end{bmatrix} &= \sqrt{4^2+4^2}\sqrt{7^2+(-1)^2}\cos\theta \\ \therefore 28 - 4 &= \sqrt{32}\sqrt{50}\cos\theta \\ \therefore \frac{24}{\sqrt{32}\sqrt{50}} &= \cos\theta \\ \therefore \theta &= \cos^{-1}\left(\frac{24}{\sqrt{32}\sqrt{50}}\right) \\ \therefore \theta &\doteqdot 53.1^o\end{aligned}$$

b **i** $\mathbf{a} = \begin{bmatrix}-2\\6\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}9\\-1\end{bmatrix}$

ii
$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta \\ \therefore \begin{bmatrix}-2\\6\end{bmatrix} \bullet \begin{bmatrix}9\\-1\end{bmatrix} &= \sqrt{(-2)^2+6^2}\sqrt{9^2+(-1)^2}\cos\theta \\ \therefore -18 - 6 &= \sqrt{40}\sqrt{82}\cos\theta \\ \therefore \cos\theta &= \frac{-24}{\sqrt{40}\sqrt{82}} \\ \therefore \theta &= \cos^{-1}\left(\frac{-24}{\sqrt{40}\sqrt{82}}\right) \\ \therefore \theta &\doteqdot 114.8^o\end{aligned}$$

7 **a**



b $\overrightarrow{AC} = \begin{bmatrix}-6\\-2\end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix}3\\4\end{bmatrix}$

$$\begin{aligned}\overrightarrow{AC} \bullet \overrightarrow{AB} &= |\overrightarrow{AC}|\,|\overrightarrow{AB}| \cos\theta \\ \therefore \begin{bmatrix}-6\\-2\end{bmatrix} \bullet \begin{bmatrix}3\\4\end{bmatrix} &= \sqrt{(-6)^2+(-2)^2}\sqrt{3^2+4^2}\cos\theta \\ \therefore -18 - 8 &= \sqrt{40}(5)\cos\theta \\ \therefore \frac{-26}{5\sqrt{40}} &= \cos\theta \\ \therefore \theta &= \cos^{-1}\left(\frac{-26}{5\sqrt{40}}\right) \\ \therefore \theta &\doteqdot 145.3^o\end{aligned}$$

8 **a**

$$\begin{aligned} \mathbf{r} \bullet \mathbf{s} &= |\mathbf{r}|\,|\mathbf{s}|\cos\theta \\ \therefore\ \begin{bmatrix}3\\3\end{bmatrix} \bullet \begin{bmatrix}-1\\2\end{bmatrix} &= \sqrt{3^2+3^2}\sqrt{(-1)^2+2^2}\cos\theta \\ \therefore\ -3+6 &= \sqrt{18}\sqrt{5}\cos\theta \\ \therefore\ \tfrac{3}{\sqrt{90}} &= \cos\theta \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{3}{\sqrt{90}}\right) \\ \therefore\ \theta &\doteqdot 71.6^o \end{aligned}$$

b

$$\begin{aligned} \mathbf{r} \bullet \mathbf{s} &= |\mathbf{r}|\,|\mathbf{s}|\cos\theta \\ \therefore\ \begin{bmatrix}-1\\-3\end{bmatrix} \bullet \begin{bmatrix}2\\5\end{bmatrix} &= \sqrt{(-1)^2+(-3)^2}\sqrt{2^2+5^2}\cos\theta \\ \therefore\ -2-15 &= \sqrt{10}\sqrt{29}\cos\theta \\ \therefore\ \frac{-17}{\sqrt{290}} &= \cos\theta \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{-17}{\sqrt{290}}\right) \\ \therefore\ \theta &\doteqdot 176.6^o \end{aligned}$$

c

$$\begin{aligned} \mathbf{r} \bullet \mathbf{s} &= |\mathbf{r}|\,|\mathbf{s}|\cos\theta \\ \therefore\ \begin{bmatrix}1\\-1\end{bmatrix} \bullet \begin{bmatrix}2\\-1\end{bmatrix} &= \sqrt{1+1}\sqrt{4+1}\cos\theta \\ \therefore\ 2+1 &= \sqrt{2}\sqrt{5}\cos\theta \\ \therefore\ \tfrac{3}{\sqrt{10}} &= \cos\theta \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{3}{\sqrt{10}}\right) \\ \therefore\ \theta &\doteqdot 18.4^o \end{aligned}$$

d

$$\begin{aligned} \mathbf{r} \bullet \mathbf{s} &= |\mathbf{r}|\,|\mathbf{s}|\cos\theta \\ \therefore\ \begin{bmatrix}1\\0\end{bmatrix} \bullet \begin{bmatrix}1\\1\end{bmatrix} &= \sqrt{1+0}\sqrt{1+1}\cos\theta \\ \therefore\ 1+0 &= \sqrt{1}\sqrt{2}\cos\theta \\ \therefore\ \tfrac{1}{\sqrt{2}} &= \cos\theta \\ \therefore\ \theta &= \cos^{-1}(\tfrac{1}{\sqrt{2}}) \\ \therefore\ \theta &= 45^o \end{aligned}$$

9 **a** Let Y be (a, b)

$$\begin{aligned} \text{Now}\quad \overrightarrow{XY} &= \overrightarrow{WZ} \\ \therefore\ \begin{bmatrix}a-7\\b-12\end{bmatrix} &= \begin{bmatrix}20-4\\5-3\end{bmatrix} \\ \therefore\ \begin{bmatrix}a-7\\b-12\end{bmatrix} &= \begin{bmatrix}16\\2\end{bmatrix} \\ \therefore\ a &= 23, \quad b = 14 \end{aligned}$$

$\therefore$ Y is (23, 14).

b

$$\begin{aligned} \overrightarrow{WX} &= \begin{bmatrix}7-4\\12-3\end{bmatrix} = \begin{bmatrix}3\\9\end{bmatrix} \\ \overrightarrow{WZ} &= \begin{bmatrix}20-4\\5-3\end{bmatrix} = \begin{bmatrix}16\\2\end{bmatrix} \\ \overrightarrow{WX} \bullet \overrightarrow{WZ} &= |\,\overrightarrow{WX}\,||\,\overrightarrow{WZ}\,|\cos\theta \\ \therefore\ \begin{bmatrix}3\\9\end{bmatrix}\begin{bmatrix}16\\2\end{bmatrix} &= \sqrt{9+81}\sqrt{256+4}\cos\theta \\ \therefore\ 48+18 &= \sqrt{90}\sqrt{260}\cos\theta \\ \therefore\ \frac{66}{\sqrt{90}\sqrt{260}} &= \cos\theta \\ \therefore\ \theta &= \cos^{-1}\left(\frac{66}{\sqrt{90}\sqrt{260}}\right) \\ \therefore\ \theta &\doteqdot 64.4^o \end{aligned}$$

c

$$\begin{aligned} \text{WX} &= \sqrt{90}\text{ units,} \quad \text{WZ} = \sqrt{260}\text{ units} \\ \text{and area} &= 2\left(\tfrac{1}{2}(\text{WX})(\text{WZ})\sin\theta\right) \\ &= \sqrt{90}\sqrt{260}\sin 64.44^o \\ &= 138\text{ units}^2 \end{aligned}$$

10 **a**

$$\begin{aligned} \overrightarrow{BA} &= \begin{bmatrix}3-1\\1-2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix} \\ \overrightarrow{BC} &= \begin{bmatrix}4-1\\-1-2\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix} \\ \overrightarrow{BA} \bullet \overrightarrow{BC} &= |\,\overrightarrow{BA}\,||\,\overrightarrow{BC}\,|\cos B \\ \therefore\ \begin{bmatrix}2\\-1\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix} &= \sqrt{4+1}\sqrt{9+9}\cos B \\ \therefore\ 6+3 &= \sqrt{5}\sqrt{18}\cos B \\ \therefore\ \tfrac{9}{\sqrt{90}} &= \cos B \\ \therefore\ B &= \cos^{-1}\left(\tfrac{9}{\sqrt{90}}\right) \doteqdot 18.4^o \end{aligned}$$

b

$$\begin{aligned} \overrightarrow{BA} &= \begin{bmatrix}5--1\\0-3\end{bmatrix} = \begin{bmatrix}6\\-3\end{bmatrix} \\ \overrightarrow{BC} &= \begin{bmatrix}2--1\\8-3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \\ \overrightarrow{BA} \bullet \overrightarrow{BC} &= |\,\overrightarrow{BA}\,||\,\overrightarrow{BC}\,|\cos B \\ \therefore\ \begin{bmatrix}6\\-3\end{bmatrix} \bullet \begin{bmatrix}3\\5\end{bmatrix} &= \sqrt{36+9}\sqrt{9+25}\cos B \\ \therefore\ 18-15 &= \sqrt{45}\sqrt{34}\cos B \\ \therefore\ \frac{3}{\sqrt{45\times 34}} &= \cos B \\ \therefore\ B &\doteqdot 85.6^o \end{aligned}$$

11 **a**

$$\begin{aligned} \overrightarrow{AB} &= \begin{bmatrix}-2-3\\4--1\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix} \\ \overrightarrow{AC} &= \begin{bmatrix}1-3\\0--1\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix} \\ \therefore\ \begin{bmatrix}-5\\5\end{bmatrix} \bullet \begin{bmatrix}-2\\1\end{bmatrix} &= \sqrt{25+25}\sqrt{4+1}\cos A \\ \therefore\ 10+5 &= \sqrt{50}\sqrt{5}\cos A \\ \therefore\ \tfrac{15}{\sqrt{250}} &= \cos A \\ \therefore\ \measuredangle\text{A} &\doteqdot 18.4^o \end{aligned}$$

$$\begin{aligned} \overrightarrow{BC} &= \begin{bmatrix}1--2\\0-4\end{bmatrix} = \begin{bmatrix}3\\-4\end{bmatrix} \\ \overrightarrow{BA} &= -\begin{bmatrix}-5\\5\end{bmatrix} = \begin{bmatrix}5\\-5\end{bmatrix} \\ \therefore\ \begin{bmatrix}3\\-4\end{bmatrix} \bullet \begin{bmatrix}5\\-5\end{bmatrix} &= \sqrt{9+16}\sqrt{25+25}\cos B \\ \therefore\ 15+20 &= \sqrt{25}\sqrt{50}\cos B \\ \therefore\ 35 &= 5\sqrt{50}\cos B \\ \therefore\ \tfrac{7}{\sqrt{50}} &= \cos B \\ \therefore\ \measuredangle\text{B} &\doteqdot 8.1^o \end{aligned}$$

$$\therefore\ \measuredangle\text{C} \doteqdot 180^o - 18.43^o - 8.13^o \doteqdot 153.4^o$$

b $\overrightarrow{AB} = \begin{bmatrix} 3-1 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ $\quad$ $\overrightarrow{BC} = \begin{bmatrix} -1-3 \\ 2--1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$\overrightarrow{AC} = \begin{bmatrix} -1-1 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $\quad$ $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$$\begin{aligned} \therefore \quad \begin{bmatrix} 2 \\ -5 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -2 \end{bmatrix} &= \sqrt{4+25}\sqrt{4+4}\cos A \\ \therefore \quad -4+10 &= \sqrt{29}\sqrt{8}\cos A \\ \therefore \quad \tfrac{6}{\sqrt{29\times 8}} &= \cos A \\ \therefore \quad \measuredangle A &\doteq 66.8^o \end{aligned}$$

$$\begin{aligned} \therefore \quad \begin{bmatrix} -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} &= \sqrt{16+9}\sqrt{4+25}\cos B \\ \therefore \quad 8+15 &= \sqrt{25}\sqrt{29}\cos B \\ \therefore \quad \tfrac{23}{5\sqrt{29}} &= \cos B \\ \therefore \quad \measuredangle B &\doteq 31.3^o \end{aligned}$$

$\therefore \quad \measuredangle C \doteq 180^o - 66.80^o - 31.33^o \doteq 81.9^o$

12 **a** $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$$\begin{aligned} \therefore \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -3 \end{bmatrix} &= \sqrt{1+1}\sqrt{4+9}\cos\theta \\ \therefore \quad 2-3 &= \sqrt{2}\sqrt{13}\cos\theta \\ \therefore \quad \tfrac{-1}{\sqrt{26}} &= \cos\theta \\ \therefore \quad \theta &\doteq 101.3 \end{aligned}$$

$\therefore$ the angle is 101.3^o or 78.7^o

b $y = x + 2$ has slope $1 = \frac{1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$y = 1 - 3x$ has slope $-3 = \frac{-3}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

$$\begin{aligned} \therefore \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix} &= \sqrt{1+1}\sqrt{1+9}\cos\theta \\ \therefore \quad 1-3 &= \sqrt{2}\sqrt{10}\cos\theta \\ \therefore \quad \tfrac{-2}{\sqrt{20}} &= \cos\theta \\ \therefore \quad \theta &\doteq 116.6 \end{aligned}$$

$\therefore$ the angle is 116.6^o or 63.4^o

c $y + x = 7$ has slope $-1 = \frac{-1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$x - 3y + 2 = 0$ has slope $\frac{1}{3}$ $\therefore$ direction vector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \therefore \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{9+1}\cos\theta \\ \therefore \quad 3-1 &= \sqrt{2}\sqrt{10}\cos\theta \\ \therefore \quad \tfrac{2}{\sqrt{20}} &= \cos\theta \\ \therefore \quad \theta &\doteq 63.4 \end{aligned}$$

$\therefore$ the angle is 63.4^o or 116.6^o

d $y = 2 - x$ has slope $-1 = \frac{-1}{1}$ $\therefore$ has direction vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

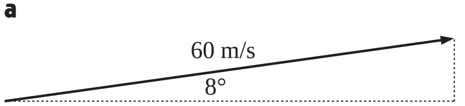
$x - 2y = 7$ has slope $\frac{1}{2}$ $\therefore$ has direction vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{aligned} \therefore \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{4+1}\cos\theta \\ \therefore \quad 2-1 &= \sqrt{2}\sqrt{5}\cos\theta \\ \therefore \quad \cos\theta &= \tfrac{1}{\sqrt{10}} \\ \therefore \quad \theta &= \cos^{-1}\left(\tfrac{1}{\sqrt{10}}\right) \doteq 71.6 \end{aligned}$$

$\therefore$ the angle is 71.6^o or 108.4^o

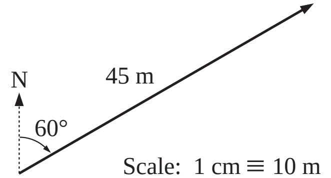
REVIEW SET 15A

1 **a**

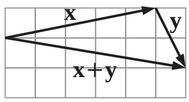


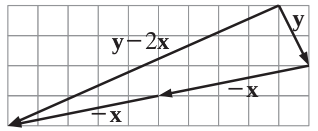
Scale: 1 cm $\equiv$ 10 m/s

b



Scale: 1 cm $\equiv$ 10 m

2 **a** 

b 

3 **a** $\overrightarrow{PR}+\overrightarrow{RQ}$
$= \overrightarrow{PQ}$

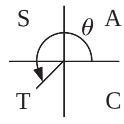
b $\overrightarrow{PS}+\overrightarrow{SQ}+\overrightarrow{QR}$
$= \overrightarrow{PQ}+\overrightarrow{QR}$
$= \overrightarrow{PR}$

4 Dino's first displacement vector is $9\begin{bmatrix}\cos 246^o\\ \sin 246^o\end{bmatrix}$, Dino's second displacement vector is $6\begin{bmatrix}\cos 96^o\\ \sin 96^o\end{bmatrix}$

$\therefore$ Dino's resultant displacement vector is $\begin{bmatrix}9\cos 246^o\\ 9\sin 246^o\end{bmatrix}+\begin{bmatrix}6\cos 96^o\\ 6\sin 96^o\end{bmatrix} \doteqdot \begin{bmatrix}-4.288\\ -2.255\end{bmatrix}$

which has length $\sqrt{(-4.288)^2+(-2.255)^2} \doteqdot 4.845$

$\therefore$ the resultant displacement vector $= 4.845\begin{bmatrix}-0.8850\\ -0.4654\end{bmatrix} \begin{array}{l}\leftarrow\cos\theta\\ \leftarrow\sin\theta\end{array}$



If $\cos\theta = -0.8850$ and $\sin\theta = -0.4654$
θ is in Quadrant 3 $\therefore$ $\theta = 180^o + \cos^{-1}(0.8850) \doteqdot 207.7^o$
$\therefore$ Dino is 4.845 km from the start at bearing 208^o.

5 **a** $\overrightarrow{AB}-\overrightarrow{CB}$
$= \overrightarrow{AB}+\overrightarrow{BC}$
$= \overrightarrow{AC}$

b $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{DC}$
$= \overrightarrow{AC}+\overrightarrow{CD}$
$= \overrightarrow{AD}$

6 **a** If $\overrightarrow{AB}=\frac{1}{2}\overrightarrow{CD}$ then
AB $\parallel$ CD and AB $= \frac{1}{2}$(CD)

b If $\overrightarrow{AB}=2\overrightarrow{AC}$ then
AB $\parallel$ AC and AB $=$ 2(AC)
i.e., A, B and C are collinear and AB $=$ 2(AC).
So, C is the midpoint of AB.

7 **a** $\mathbf{p}+\mathbf{r}-\mathbf{q}=0$
$\therefore$ $\mathbf{p}+\mathbf{r}=\mathbf{q}$

b $\mathbf{l}+\mathbf{m}-\mathbf{n}+\mathbf{j}-\mathbf{k}=0$
$\therefore$ $\mathbf{l}+\mathbf{m}+\mathbf{j}=\mathbf{n}+\mathbf{k}$

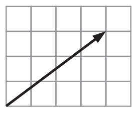
8 **a** $\overrightarrow{OQ}=\overrightarrow{OR}+\overrightarrow{RQ}=\mathbf{r}+\mathbf{q}$

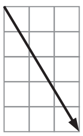
b $\overrightarrow{PQ}=\overrightarrow{PO}+\overrightarrow{OR}+\overrightarrow{RQ}=-\mathbf{p}+\mathbf{r}+\mathbf{q}$

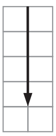
c $\overrightarrow{ON}=\overrightarrow{OR}+\overrightarrow{RN}=\mathbf{r}+\frac{1}{2}\mathbf{q}$

d $\overrightarrow{MN} = \overrightarrow{MQ}+\overrightarrow{QN}$
$= \frac{1}{2}\overrightarrow{PQ}+\frac{1}{2}\overrightarrow{QR}$
$= \frac{1}{2}(-\mathbf{p}+\mathbf{r}+\mathbf{q})+\frac{1}{2}(-\mathbf{q})$
$= -\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{r}+\frac{1}{2}\mathbf{q}-\frac{1}{2}\mathbf{q}$
$= \frac{1}{2}\mathbf{r}-\frac{1}{2}\mathbf{p}$

REVIEW SET 15B

1 **a** $\begin{bmatrix}4\\3\end{bmatrix}$ 

b $\begin{bmatrix}3\\-5\end{bmatrix}$ 

c $\begin{bmatrix}0\\-4\end{bmatrix}$ 

2 **a** $\begin{bmatrix}5\\2\end{bmatrix}$ **b** $\begin{bmatrix}2\\-4\end{bmatrix}$

3 **a** $2\mathbf{p}+\mathbf{q}$
$= 2\begin{bmatrix}-3\\1\end{bmatrix}+\begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-6\\2\end{bmatrix}+\begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-4\\-2\end{bmatrix}$

b $\mathbf{q}-3\mathbf{r}$
$= \begin{bmatrix}2\\-4\end{bmatrix}-3\begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}2\\-4\end{bmatrix}-\begin{bmatrix}3\\9\end{bmatrix}$
$= \begin{bmatrix}-1\\-13\end{bmatrix}$

c $\mathbf{p}-\mathbf{q}+\mathbf{r}$
$= \begin{bmatrix}-3\\1\end{bmatrix}-\begin{bmatrix}2\\-4\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}-5\\5\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}-4\\8\end{bmatrix}$

4 **a** $\mathbf{p}+\mathbf{q}-\mathbf{r}$
$= \begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}1\\-3\end{bmatrix}-\begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-2\\-1\end{bmatrix}-\begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-4\\3\end{bmatrix}$

b $2\mathbf{q}-3\mathbf{r}$
$= 2\begin{bmatrix}1\\-3\end{bmatrix}-3\begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}2\\-6\end{bmatrix}-\begin{bmatrix}6\\-12\end{bmatrix}$
$= \begin{bmatrix}-4\\6\end{bmatrix}$

c $\mathbf{r}+2\mathbf{p}-\mathbf{q}$
$= \begin{bmatrix}2\\-4\end{bmatrix}+2\begin{bmatrix}-3\\2\end{bmatrix}-\begin{bmatrix}1\\-3\end{bmatrix}$
$= \begin{bmatrix}2\\-4\end{bmatrix}+\begin{bmatrix}-6\\4\end{bmatrix}-\begin{bmatrix}1\\-3\end{bmatrix}$
$= \begin{bmatrix}-4\\0\end{bmatrix}-\begin{bmatrix}1\\-3\end{bmatrix}$
$= \begin{bmatrix}-5\\3\end{bmatrix}$

5
$$\begin{aligned}&\overrightarrow{CB}\\ &= \overrightarrow{CA} + \overrightarrow{AB}\\ &= \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix}\\ &= \begin{bmatrix}2\\1\end{bmatrix}\end{aligned}$$

6
$$\begin{aligned}&\overrightarrow{SP}\\ &= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}\\ &= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}\\ &= -\begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}-4\\1\end{bmatrix}\\ &= \begin{bmatrix}1\\4\end{bmatrix}\end{aligned}$$

7 **a**
$$\begin{aligned}&|\mathbf{r}|\\ &= \sqrt{4^2+1^2}\\ &= \sqrt{17} \text{ units}\end{aligned}$$

b
$$\begin{aligned}&|\mathbf{s}|\\ &= \sqrt{(-3)^2+2^2}\\ &= \sqrt{13} \text{ units}\end{aligned}$$

c
$$\begin{aligned}&\mathbf{r}+\mathbf{s}\\ &= \begin{bmatrix}4\\1\end{bmatrix} + \begin{bmatrix}-3\\2\end{bmatrix}\\ &= \begin{bmatrix}1\\3\end{bmatrix}\\ \therefore\quad &|\mathbf{r}+\mathbf{s}|\\ &= \sqrt{1^2+3^2}\\ &= \sqrt{10} \text{ units}\end{aligned}$$

d
$$\begin{aligned}&2\mathbf{s}-\mathbf{r}\\ &= 2\begin{bmatrix}-3\\2\end{bmatrix} - \begin{bmatrix}4\\1\end{bmatrix}\\ &= \begin{bmatrix}-10\\3\end{bmatrix}\\ \therefore\quad &|2\mathbf{s}-\mathbf{r}|\\ &= \sqrt{100+9}\\ &= \sqrt{109} \text{ units}\end{aligned}$$

8 **a**
$$\begin{aligned}\overrightarrow{BC} &= 2\overrightarrow{OA} = 2\mathbf{p}\\ \text{Now}\quad \overrightarrow{AC} &= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BC}\\ &= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}\\ &= \mathbf{p} + \mathbf{q}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= \mathbf{p} + \tfrac{1}{2}\overrightarrow{AC}\\ &= \mathbf{p} + \tfrac{1}{2}(\mathbf{p}+\mathbf{q})\\ &= \tfrac{3}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}\end{aligned}$$

REVIEW SET 15C

1 **a**
$$\begin{aligned}\mathbf{p} - 3\mathbf{x} &= \mathbf{0}\\ \therefore\quad \mathbf{p} &= 3\mathbf{x}\\ \therefore\quad \tfrac{1}{3}\mathbf{p} &= \mathbf{x}\\ \therefore\quad \mathbf{x} &= \begin{bmatrix}-1\\ \frac{1}{3}\end{bmatrix}\end{aligned}$$

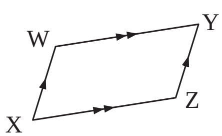
b
$$\begin{aligned}2\mathbf{q} - \mathbf{x} &= \mathbf{r}\\ \therefore\quad 2\mathbf{q} - \mathbf{r} &= \mathbf{x}\\ \therefore\quad \mathbf{x} &= 2\begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix}\\ \therefore\quad \mathbf{x} &= \begin{bmatrix}1\\-10\end{bmatrix}\end{aligned}$$

2 **a** $\overrightarrow{PQ} = \begin{bmatrix}2--3\\-1-5\end{bmatrix} = \begin{bmatrix}5\\-6\end{bmatrix}$

b $\overrightarrow{PR} = \begin{bmatrix}1--3\\3-5\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$

c
$$\begin{aligned}|\overrightarrow{PR}| &= \sqrt{4^2+(-2)^2}\\ &= \sqrt{20} \text{ units}\end{aligned}$$

3



$\overrightarrow{WY} = \begin{bmatrix}3--3\\4--1\end{bmatrix} = \begin{bmatrix}6\\5\end{bmatrix}$

$\overrightarrow{XZ} = \begin{bmatrix}4--2\\10-5\end{bmatrix} = \begin{bmatrix}6\\5\end{bmatrix}$

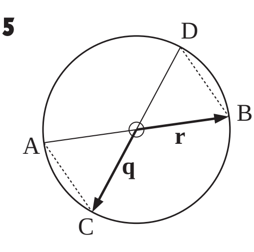
So, $\overrightarrow{WY} = \overrightarrow{XZ}$

$\therefore$ WY is parallel to XZ and they are equal in length. This is sufficient to deduce that WYZX is a parallelogram.

4
$$\begin{aligned}r\begin{bmatrix}-2\\1\end{bmatrix} + s\begin{bmatrix}3\\-4\end{bmatrix} &= \begin{bmatrix}13\\-24\end{bmatrix}\\ \therefore\quad \begin{bmatrix}-2r+3s\\r-4s\end{bmatrix} &= \begin{bmatrix}13\\-24\end{bmatrix}\\ \therefore\quad -2r+3s &= 13\\ r-4s &= -24 \quad \times 2 \text{ .... (1)}\\ \therefore\quad -2r+3s &= 13\\ 2r-8s &= -48\\ \hline -5s &= -35\\ \therefore\quad s &= 7\end{aligned}$$

$$\begin{aligned}\text{and in (1)}\quad r-4(7) &= -24\\ \therefore\quad r &= -24+28\\ \therefore\quad r &= 4\\ \text{i.e.,}\quad r = 4 \quad &\text{and} \quad s = 7\end{aligned}$$

5



a
$$\begin{aligned}\overrightarrow{DB} &= \overrightarrow{DO} + \overrightarrow{OB}\\ &= \overrightarrow{OC} + \overrightarrow{OB}\\ &= \mathbf{q} + \mathbf{r}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= \overrightarrow{OB} + \overrightarrow{OC}\\ &= \mathbf{r} + \mathbf{q}\end{aligned}$$

We see that $\overrightarrow{DB} = \overrightarrow{AC}$

$\therefore$ DB is parallel to AC and equal in length.

This is sufficient to deduce that ACBD is a parallelogram.

6 **a** $\mathbf{p} = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore \quad |\mathbf{p}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ units

$\therefore$ the required unit vector is

$\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ or $\frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$

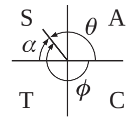
b The magnitude of $\mathbf{p}$ is $|\mathbf{p}| = \sqrt{5}$ units

$\cos\theta = \frac{-2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

$\therefore \quad \theta = 180 - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$\therefore \quad \theta \doteqdot 153^o$ and $\alpha = 27^o$

i.e., the direction of $\mathbf{p}$ is $\phi \doteqdot 270^o + 27^o = 297^o$



7 **a**
$$\begin{aligned} &3\mathbf{x} - \mathbf{y} \\ &= 3(-\mathbf{i} + 3\mathbf{j}) - (-\mathbf{i} - 2\mathbf{j}) \\ &= -3\mathbf{i} + 9\mathbf{j} + \mathbf{i} + 2\mathbf{j} \\ &= -2\mathbf{i} + 11\mathbf{j} \end{aligned}$$

b
$$\begin{aligned} &|\mathbf{x}| \\ &= \sqrt{(1)^2 + 3^2} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

c
$$\begin{aligned} &\mathbf{y} - \mathbf{x} \\ &= -\mathbf{i} - 2\mathbf{j} - (-\mathbf{i} + 3\mathbf{j}) \\ &= -\mathbf{i} - 2\mathbf{j} + \mathbf{i} - 3\mathbf{j} \\ &= -5\mathbf{j} \end{aligned}$$

$\therefore$ the unit vector is $-\mathbf{j}$

8 **a** Is a unit vector if

$$\begin{aligned} \sqrt{\left(\tfrac{4}{7}\right)^2 + \left(\tfrac{1}{k}\right)^2} &= 1 \\ \therefore \quad \tfrac{16}{49} + \frac{1}{k^2} &= 1 \\ \therefore \quad \frac{1}{k^2} &= \tfrac{33}{49} \\ \therefore \quad k &= \pm\tfrac{7}{\sqrt{33}} \end{aligned}$$

b Is a unit vector if

$$\begin{aligned} \sqrt{k^2 + k^2} &= 1 \\ \therefore \quad 2k^2 &= 1 \\ \therefore \quad k^2 &= \tfrac{1}{2} \\ \therefore \quad k &= \pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

REVIEW SET 15D

1 **a**
$$\begin{aligned} &\mathbf{p} \bullet \mathbf{q} \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix} \\ &= -3 + (-10) \\ &= -13 \end{aligned}$$

b
$$\begin{aligned} &\mathbf{p} - \mathbf{r} \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore \quad &\mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\ &= -6 - 30 \\ &= -36 \end{aligned}$$

2
$$\begin{aligned} \text{LHS} &= \mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \left(\begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right) \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\ &= -9 - 16 \\ &= -25 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r} \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= (-6 - 10) - (3 + 6) \\ &= -16 - 9 \\ &= -25 \end{aligned}$$

$\therefore$ LHS = RHS ✓

3 Since they are perpendicular

$$\begin{aligned} \begin{bmatrix} 3 \\ 3-2t \end{bmatrix} \bullet \begin{bmatrix} t^2+t \\ -2 \end{bmatrix} &= 0 \\ \therefore \quad 3(t^2 + t) - 2(3 - 2t) &= 0 \\ \therefore \quad 3t^2 + 3t - 6 + 4t &= 0 \\ \therefore \quad 3t^2 + 7t - 6 &= 0 \\ \therefore \quad (3t - 2)(t + 3) &= 0 \\ \therefore \quad t &= \tfrac{2}{3} \text{ or } -3 \end{aligned}$$

4
$$\begin{aligned} \overrightarrow{AB} &= \begin{bmatrix} -1-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ \overrightarrow{AC} &= \begin{bmatrix} 3-2 \\ k-3 \end{bmatrix} = \begin{bmatrix} 1 \\ k-3 \end{bmatrix} \end{aligned}$$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0 \quad \{\text{as } \measuredangle BAC = 90^o\}$

$$\begin{aligned} \therefore \quad \begin{bmatrix} -3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ k-3 \end{bmatrix} &= 0 \\ \therefore \quad -3 + k - 3 &= 0 \\ \therefore \quad k &= 6 \end{aligned}$$

5 One vector perpendicular to $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as dot product $= -20 + 20$
$= 0$

$\therefore$ all vectors have form $k\begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad k \neq 0$

6

$$\overrightarrow{KL} = \begin{bmatrix} 3--2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$
$$\overrightarrow{KM} = \begin{bmatrix} 1--2 \\ -3-1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$
Now $\overrightarrow{KL} \bullet \overrightarrow{KM} = |\overrightarrow{KL}||\overrightarrow{KM}|\cos K$
$$\therefore \begin{bmatrix} 5 \\ 1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \sqrt{25+1}\sqrt{9+16}\cos K$$
$$\therefore 15-4 = \sqrt{26}\sqrt{25}\cos K$$
$$\therefore \tfrac{11}{5\sqrt{26}} = \cos K$$
$$\therefore K = \cos^{-1}\left(\tfrac{11}{5\sqrt{26}}\right)$$
$$\therefore K = 64.4^o$$

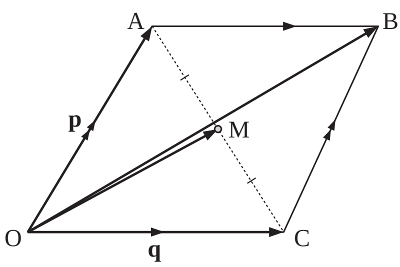
$$\overrightarrow{LK} = -\overrightarrow{KL} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$
$$\overrightarrow{LM} = \begin{bmatrix} 1-3 \\ -3-2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$
Now $\overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}||\overrightarrow{LM}|\cos L$
$$\therefore \begin{bmatrix} -5 \\ -1 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \sqrt{25+1}\sqrt{4+25}\cos L$$
$$\therefore 10+5 = \sqrt{26}\sqrt{29}\cos L$$
$$\therefore \cos L = \tfrac{15}{\sqrt{26\times 29}}$$
$$\therefore L \doteqdot 56.9^o$$
and $\measuredangle M \doteqdot 180^o - 56.89^o - 64.44^o \doteqdot 58.7^o$

7 $4x-5y=11$ has gradient $\frac{4}{5}$ $\therefore$ direction vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$2x+3y=7$ has gradient $-\frac{2}{3}$ $\therefore$ direction vector is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

If the angle is θ, $\begin{bmatrix} 5 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \sqrt{5^2+4^2}\sqrt{3^2+(-2)^2}\cos\theta$
$$\therefore 15-8 = \sqrt{41}\sqrt{13}\cos\theta$$
$$\therefore \tfrac{7}{\sqrt{41\times 13}} = \cos\theta$$
$\therefore \theta \doteqdot 72.3$ $\therefore$ the angle is 72.3^o (or 107.7^o)

8 **a**



i (1) $\overrightarrow{OB}$
$= \overrightarrow{OA} + \overrightarrow{AB}$
$= \overrightarrow{OA} + \overrightarrow{OC}$
$= \mathbf{p} + \mathbf{q}$

(2) $\overrightarrow{OM}$
$= \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
$= \mathbf{p} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OC})$
$= \mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q})$
$= \mathbf{p} - \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$
$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

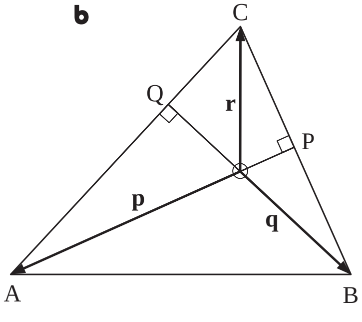
ii We notice that $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$

$\therefore$ $\overline{OM} \parallel \overline{OB}$ and $OM = \frac{1}{2}(OB)$

i.e., O, M and B are collinear (as O is common)

and $\therefore$ M is the midpoint of $\overline{OB}$.

b



i $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\mathbf{p} + \mathbf{r}$
$= \mathbf{r} - \mathbf{p}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$
$= -\mathbf{q} + \mathbf{r}$
$= \mathbf{r} - \mathbf{q}$

ii AP $\perp$ BC and BQ $\perp$ AC

$\therefore \mathbf{p} \perp \mathbf{r} - \mathbf{q}$
$\therefore \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) = 0$
$\therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0$
$\therefore \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$

$\therefore \mathbf{q} \perp (\mathbf{r} - \mathbf{p})$
$\therefore \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0$
$\therefore \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0$
$\therefore \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$

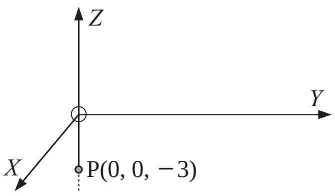
iii $\mathbf{r} \bullet \overrightarrow{AB} = \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q})$
$= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$
$= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q}$ {from **ii**}
$= 0$ $\therefore \mathbf{r} \perp \overrightarrow{AB}$ i.e., OC $\perp$ AB

Chapter 16

VECTORS IN 3-DIMENSIONS

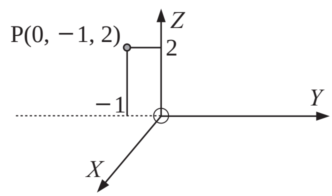
EXERCISE 16A

1 **a**



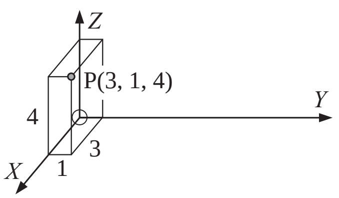
$\text{OP} = \sqrt{0^2 + 0^2 + (-3)^2} = 3$ units

b



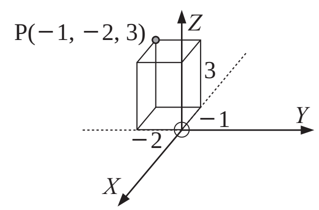
$\text{OP} = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$ units

c



$\text{OP} = \sqrt{3^2 + 1^2 + 4^2}$
$= \sqrt{26}$ units

d



$\text{OP} = \sqrt{(-1)^2 + (-2)^2 + 3^2}$
$= \sqrt{14}$ units

2 **a** **i** AB
$= \sqrt{(0 - -1)^2 + (-1 - 2)^2 + (1 - 3)^2}$
$= \sqrt{1 + 9 + 4}$
$= \sqrt{14}$ units

ii Midpoint is at $\left(\dfrac{-1+0}{2}, \dfrac{2-1}{2}, \dfrac{3+1}{2}\right)$

i.e., $\left(-\frac{1}{2}, \frac{1}{2}, 2\right)$

b **i** AB
$= \sqrt{(2 - 0)^2 + (-1 - 0)^2 + (3 - 0)^2}$
$= \sqrt{4 + 1 + 9}$
$= \sqrt{14}$ units

ii Midpoint is at $\left(\dfrac{0+2}{2}, \dfrac{0-1}{2}, \dfrac{0+3}{2}\right)$

i.e., $\left(1, -\frac{1}{2}, \frac{3}{2}\right)$

c **i** AB
$= \sqrt{(-1 - 3)^2 + (0 - -1)^2 + (1 - -1)^2}$
$= \sqrt{16 + 1 + 4}$
$= \sqrt{21}$ units

ii Midpoint is at $\left(\dfrac{3-1}{2}, \dfrac{-1+0}{2}, \dfrac{-1+1}{2}\right)$

i.e., $\left(1, -\frac{1}{2}, 0\right)$

d **i** AB
$= \sqrt{(0 - 2)^2 + (1 - 0)^2 + (0 - -3)^2}$
$= \sqrt{4 + 1 + 9}$
$= \sqrt{14}$ units

ii Midpoint is at $\left(\dfrac{2+0}{2}, \dfrac{0+1}{2}, \dfrac{-3+0}{2}\right)$

i.e., $\left(1, \frac{1}{2}, -\frac{3}{2}\right)$

3 P(0, 4, 4) Q(2, 6, 5) R(1, 4, 3)

$\text{PQ} = \sqrt{(2 - 0)^2 + (6 - 4)^2 + (5 - 4)^2}$
$= \sqrt{4 + 4 + 1}$
$= 3$

$\text{PR} = \sqrt{(1 - 0)^2 + (4 - 4)^2 + (3 - 4)^2}$
$= \sqrt{1 + 0 + 1}$
$= \sqrt{2}$

$\text{QR} = \sqrt{(1 - 2)^2 + (4 - 6)^2 + (3 - 5)^2}$
$= \sqrt{1 + 4 + 4}$
$= 3$

$\therefore$ PQ = QR and so Δ PQR is isosceles.

4 **a** A(2, −1, 7) B(3, 1, 4) C(5, 4, 5)

$$\begin{aligned} AB &= \sqrt{(3-2)^2+(1--1)^2+(4-7)^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(5-2)^2+(4--1)^2+(5-7)^2} \\ &= \sqrt{9+25+4} \\ &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5-3)^2+(4-1)^2+(5-4)^2} \\ &= \sqrt{4+9+1} \\ &= \sqrt{14} \end{aligned}$$

$\therefore$ as AB = BC, Δ ABC is isosceles.

b A(0, 0, 3) B(2, 8, 1) C(−9, 6, 18)

$$\begin{aligned} AB &= \sqrt{(2-0)^2+(8-0)^2+(1-3)^2} \\ &= \sqrt{4+64+4} \\ &= \sqrt{72} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(-9-0)^2+(6-0)^2+(18-3)^2} \\ &= \sqrt{81+36+225} \\ &= \sqrt{342} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-9-2)^2+(6-8)^2+(18-1)^2} \\ &= \sqrt{121+4+289} \\ &= \sqrt{414} \end{aligned}$$

Since $BC^2 = AB^2 + AC^2$, Δ ABC is right-angled.

c A(5, 6, −2) B(6, 12, 9) C(2, 4, 2)

$$\begin{aligned} AB &= \sqrt{(6-5)^2+(12-6)^2+(9--2)^2} \\ &= \sqrt{1+36+121} \\ &= \sqrt{158} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(2-5)^2+(4-6)^2+(2--2)^2} \\ &= \sqrt{9+4+16} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-6)^2+(4-12)^2+(2-9)^2} \\ &= \sqrt{16+64+49} \\ &= \sqrt{129} \end{aligned}$$

Since $AB^2 = AC^2 + BC^2$, Δ ABC is right-angled.

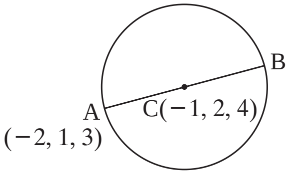
d A(1, 0, −3) B(2, 2, 0) C(4, 6, 6)

$$\begin{aligned} AB &= \sqrt{(2-1)^2+(2-0)^2+(0--3)^2} \\ &= \sqrt{1^2+2^2+3^2} \\ &= \sqrt{14} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(4-1)^2+(6-0)^2+(6--3)^2} \\ &= \sqrt{3^2+6^2+9^2} \\ &= \sqrt{126} \\ &= 3\sqrt{14} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2+(6-2)^2+(6-0)^2} \\ &= \sqrt{2^2+4^2+6^2} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

Since AB + BC = AC, the points A, B and C lie on a straight line, i.e., do not form a triangle.

5



If B is (a, b, c) then $\dfrac{a-2}{2} = -1$, $\dfrac{b+1}{2} = 2$, $\dfrac{c+3}{2} = 4$

$\therefore$ $a = 0$, $b = 3$, $c = 5$

$\therefore$ B is (0, 3, 5)

$$\begin{aligned} r = AC &= \sqrt{(-1--2)^2+(2-1)^2+(4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

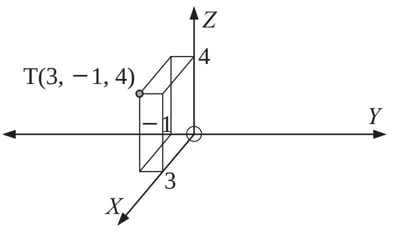
6 **a** $(0, y, 0)$ for any y

b Distance between $(0, y, 0)$ and $B(-1, -1, 2)$ is $\sqrt{(-1)^2+(-1-y)^2+2^2}$

$$\therefore \sqrt{1+(y+1)^2+4} = \sqrt{14}$$
$$\therefore (y+1)^2 = 9$$
$$\therefore y+1 = \pm 3$$
$$\therefore y = -1 \pm 3$$
$$\therefore y = -4 \text{ or } 2 \quad \therefore \text{ the two points are } (0, -4, 0) \text{ and } (0, 2, 0)$$

EXERCISE 16B.1

1 **a**



b $\overrightarrow{OT} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

c $OT = \sqrt{(3-0)^2+(-1-0)^2+(4-0)^2}$
$= \sqrt{9+1+16}$
$= \sqrt{26}$ units

2 **a** $\overrightarrow{AB} = \begin{bmatrix} 1-(-3) \\ 0-1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$

$\overrightarrow{BA} = \begin{bmatrix} -3-1 \\ 1-0 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$

b $|\overrightarrow{AB}| = \sqrt{4^2+(-1)^2+(-3)^2}$
$= \sqrt{26}$ units

$|\overrightarrow{BA}| = \sqrt{(-4)^2+1^2+3^2}$
$= \sqrt{26}$ units

3 $\overrightarrow{OA} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $\quad \overrightarrow{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ $\quad \overrightarrow{AB} = \begin{bmatrix} -1-3 \\ 1-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

4 **a** The position vector of M from N
$= \overrightarrow{NM}$
$= \begin{bmatrix} 4-(-1) \\ -2-2 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$

b The position vector of N from M
$= \overrightarrow{MN}$
$= \begin{bmatrix} -1-4 \\ 2-(-2) \\ 0-(-1) \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$

c $|\overrightarrow{MN}| = \sqrt{(-5)^2+4^2+1^2} = \sqrt{25+16+1} = \sqrt{42}$ units

5 **a** The position vector of A from O
$= \overrightarrow{OA} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

$|\overrightarrow{OA}| = \sqrt{(-1)^2+2^2+5^2}$
$= \sqrt{1+4+25}$
$= \sqrt{30}$ units

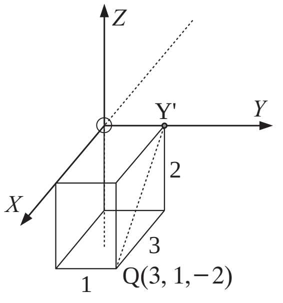
b The position vector of C from A
$= \overrightarrow{AC} = \begin{bmatrix} -3-(-1) \\ 1-2 \\ 0-5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix}$

$|\overrightarrow{AC}| = \sqrt{(-2)^2+(-1)^2+(-5)^2}$
$= \sqrt{4+1+25}$
$= \sqrt{30}$ units

c The position vector of B from C
$= \overrightarrow{CB} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

and $|\overrightarrow{CB}| = \sqrt{5^2+(-1)^2+3^2}$
$= \sqrt{25+1+9}$
$= \sqrt{35}$ units

6



a The distance from Q to the Y-axis is the distance from Q to $Y'(0, 1, 0)$

$$\therefore \quad QY' = \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2} = \sqrt{9+4} = \sqrt{13} \text{ units}$$

b The distance from Q to the origin is

$$QO = \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{9+1+4} = \sqrt{14} \text{ units}$$

c The distance from Q to the ZOY plane is the distance from Q to (0, 1, −2), i.e., 3 units.

EXERCISE 16B.2

1 **a** $\begin{bmatrix} a-4 \\ b-3 \\ c+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$

$\therefore \quad a-4 = 1, \quad b-3 = 3, \quad c+2 = -4$

$\therefore \quad a = 5, \quad b = 6, \quad c = -6$

b $\begin{bmatrix} a-5 \\ b-2 \\ c+3 \end{bmatrix} = \begin{bmatrix} 3-a \\ 2-b \\ 5-c \end{bmatrix}$

$\therefore \quad a-5 = 3-a, \quad b-2 = 2-b, \quad c+3 = 5-c,$

$\therefore \quad 2a = 8, \quad 2b = 4, \quad 2c = 2$

$\therefore \quad a = 4, \quad b = 2, \quad c = 1$

2 **a** $2\begin{bmatrix} 1 \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 6a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$

$\therefore \quad 6a = 2, \; b = 2, \; c-1 = 0 \quad \therefore \quad a = \frac{1}{3}, \; b = 2, \; c = 1$

b $\begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} = \begin{bmatrix} b \\ a^2 \\ a+b \end{bmatrix}$

$\therefore \quad b = 2, \quad a^2 = a, \quad a+b = 3$

i.e., $a = 1, \quad b = 2$

c $a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

$\therefore \quad a + 2b = -1$ ...... (1), $\quad a + c = 3$ ...... (2) and $-b + c = 3$ ...... (3)

(1) − (2) gives: $2b - c = -4$ ...... (4)

Adding (3) and (4) gives $b = -1$

$\therefore$ using (3), $c = 2$

and using (2), $a = 1$

$\therefore \quad a = 1, \quad b = -1, \quad c = 2$

3 A(−1, 3, 4) B(2, 5, −1) C(−1, 2, −2) D(r, s, t)

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then $\begin{bmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{bmatrix} = \begin{bmatrix} r-2 \\ s-5 \\ t+1 \end{bmatrix}$

$\therefore \quad r-2 = 0, \quad s-5 = -1$ and $t+1 = -6 \qquad \therefore \quad r = 2, \quad s = 4$ and $t = -7$

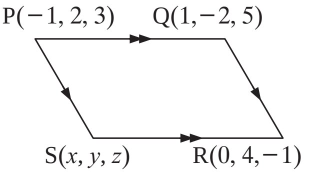
b If $\overrightarrow{AB} = \overrightarrow{DC}$ then $\begin{bmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{bmatrix} = \begin{bmatrix} -1-r \\ 2-s \\ -2-t \end{bmatrix}$

$\therefore \quad -1-r = 3, \quad 2-s = 2$ and $-2-t = -5 \qquad \therefore \quad r = -4, \quad s = 0$ and $t = 3$

4 **a** $\overrightarrow{AB} = \begin{bmatrix} 3-1 \\ -3-2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\overrightarrow{DC} = \begin{bmatrix} 7-5 \\ -4-1 \\ 5-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

5 **a**



Suppose S is at (x, y, z)

$\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}

$\therefore \begin{bmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 0-x \\ 4-y \\ -1-z \end{bmatrix}$

$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -x \\ 4-y \\ -1-z \end{bmatrix}$

$\therefore\ -x = 2 \qquad 4-y = -4 \qquad -1-z = 2$

$\therefore\ x = -2 \qquad y = 8 \qquad z = -3$

$\therefore$ S is at $(-2, 8, -3)$

b The midpoint of PR is $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

The midpoint of QS is $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

i.e., PR and QS have the same midpoint.

EXERCISE 16C

1 **a** $\mathbf{a}+\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ **b** $\mathbf{a}-\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

c $\mathbf{b}+2\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}$

d $\mathbf{a}-3\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix}$

e $\mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

f $\mathbf{c}-\frac{1}{2}\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{bmatrix}$

g $\mathbf{a}-\mathbf{b}-\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$

h $2\mathbf{b}-\mathbf{c}+\mathbf{a} = 2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

2 **a** $|\mathbf{a}| = \sqrt{(-1)^2+1^2+3^2} = \sqrt{11}$ units

b $|\mathbf{b}| = \sqrt{1^2+(-3)^2+2^2} = \sqrt{14}$ units

c $|\mathbf{b}+\mathbf{c}| = \left|\begin{bmatrix}1\\-3\\2\end{bmatrix}+\begin{bmatrix}-2\\2\\4\end{bmatrix}\right| = \left|\begin{bmatrix}-1\\-1\\6\end{bmatrix}\right|$
$= \sqrt{(-1)^2+(-1)^2+6^2}$
$= \sqrt{1+1+36}$
$= \sqrt{38}$ units

d $|\mathbf{a}-\mathbf{c}| = \left|\begin{bmatrix}-1\\1\\3\end{bmatrix}-\begin{bmatrix}-2\\2\\4\end{bmatrix}\right| = \left|\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right|$
$= \sqrt{1^2+(-1)^2+(-1)^2}$
$= \sqrt{3}$ units

e $|\mathbf{a}|\mathbf{b} = \sqrt{11}\begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{bmatrix}$

f $\dfrac{1}{|\mathbf{a}|}\times\mathbf{a} = \frac{1}{\sqrt{11}}\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{bmatrix}$

3 **a** $2\mathbf{x}+\mathbf{a} = \mathbf{b}$
$\therefore\ 2\mathbf{x} = \mathbf{b}-\mathbf{a}$
$\therefore\ \mathbf{x} = \frac{1}{2}(\mathbf{b}-\mathbf{a})$

b $\mathbf{b}-3\mathbf{x} = 2\mathbf{a}$
$\therefore\ \mathbf{b}-2\mathbf{a} = 3\mathbf{x}$
$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{b}-2\mathbf{a})$

c $\mathbf{a}+2\mathbf{x} = \mathbf{b}-\mathbf{x}$
$\therefore\ 3\mathbf{x} = \mathbf{b}-\mathbf{a}$
$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{b}-\mathbf{a})$

4 **a** $2\mathbf{a}+\mathbf{x} = \mathbf{b}$
$\therefore\ \mathbf{x} = \mathbf{b}-2\mathbf{a} = \begin{bmatrix}2\\-2\\1\end{bmatrix}-2\begin{bmatrix}-1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}-2\\4\\6\end{bmatrix} = \begin{bmatrix}4\\-6\\-5\end{bmatrix}$

b $3\mathbf{x}-\mathbf{a} = 2\mathbf{b}$
$\therefore\ 3\mathbf{x} = \mathbf{a}+2\mathbf{b}$
$\therefore\ \mathbf{x} = \frac{1}{3}(\mathbf{a}+2\mathbf{b}) = \frac{1}{3}\left(\begin{bmatrix}-1\\2\\3\end{bmatrix}+2\begin{bmatrix}2\\-2\\1\end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix}-1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\-4\\2\end{bmatrix}\right)$
$\therefore\ \mathbf{x} = \frac{1}{3}\begin{bmatrix}3\\-2\\5\end{bmatrix} = \begin{bmatrix}1\\-\frac{2}{3}\\\frac{5}{3}\end{bmatrix}$

c $2\mathbf{b}-2\mathbf{x} = -\mathbf{a}$
$\therefore\ \mathbf{a}+2\mathbf{b} = 2\mathbf{x}$
$\therefore\ \mathbf{x} = \frac{1}{2}(\mathbf{a}+2\mathbf{b}) = \frac{1}{2}\begin{bmatrix}3\\-2\\5\end{bmatrix}$ {using **b**} $= \begin{bmatrix}\frac{3}{2}\\-1\\\frac{5}{2}\end{bmatrix}$

5 $\overrightarrow{AB} = \overrightarrow{AO}+\overrightarrow{OB} = -\overrightarrow{OA}+\overrightarrow{OB} = -\begin{bmatrix}-2\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix}+\begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}3\\4\\-2\end{bmatrix}$

$\therefore\ |\overrightarrow{AB}| = \sqrt{3^2+4^2+(-2)^2} = \sqrt{9+16+4} = \sqrt{29}$ units

6 $\overrightarrow{OA} = \begin{bmatrix}2\\1\\-2\end{bmatrix},\quad \overrightarrow{OB} = \begin{bmatrix}0\\3\\-4\end{bmatrix},\quad \overrightarrow{OC} = \begin{bmatrix}1\\-2\\1\end{bmatrix},\quad \overrightarrow{OD} = \begin{bmatrix}-2\\-3\\2\end{bmatrix}$

$\therefore\ \overrightarrow{BD} = \overrightarrow{BO}+\overrightarrow{OD} = -\overrightarrow{OB}+\overrightarrow{OD} = -\begin{bmatrix}0\\3\\-4\end{bmatrix}+\begin{bmatrix}-2\\-3\\2\end{bmatrix} = \begin{bmatrix}-2\\-6\\6\end{bmatrix}$

and $\overrightarrow{AC} = \overrightarrow{AO}+\overrightarrow{OC} = -\begin{bmatrix}2\\1\\-2\end{bmatrix}+\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}-1\\-3\\3\end{bmatrix}$ $\therefore\ \overrightarrow{BD} = \begin{bmatrix}-2\\-6\\6\end{bmatrix} = 2\begin{bmatrix}-1\\-3\\3\end{bmatrix} = 2\overrightarrow{AC}$

7 **a** $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

c $\overrightarrow{BA} = -\overrightarrow{AB} = -(\mathbf{b} - \mathbf{a}) = -\mathbf{b} + \mathbf{a}$ or $\mathbf{a} - \mathbf{b}$

d $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \mathbf{b} + \frac{1}{2}\mathbf{a}$

e $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = -\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{a} = -\frac{1}{2}\mathbf{a} + \mathbf{b}$

f $\overrightarrow{DA} = -\overrightarrow{AD} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

8 **a** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$

b $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$

c $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ {using **b**} $= \begin{bmatrix} -3 \\ 6 \\ -5 \end{bmatrix}$

EXERCISE 16D

1 Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{b} = k\mathbf{a}$ $\quad\therefore \begin{bmatrix} -6 \\ r \\ s \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ 3k \end{bmatrix}$

$\therefore\ 2k = -6,\quad r = -k,\quad s = 3k \qquad \therefore\ k = -3,\quad r = 3,\quad s = -9$

2 If $\begin{bmatrix} a \\ 2 \\ b \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ are parallel, then $\begin{bmatrix} a \\ 2 \\ b \end{bmatrix} = k\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

i.e., $2 = -k,\quad a = 3k,\quad b = 2k \qquad \therefore\ k = -2,\quad a = -6$ and $b = -4$

3 **a** Let the vector parallel to $\mathbf{a}$ be $k\mathbf{a}$

i.e., $k\mathbf{a} = k\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ -2k \end{bmatrix}$

$k\mathbf{a}$ has length $= 1$,

so $\sqrt{(2k)^2 + (-k)^2 + (-2k)^2} = 1$

$\therefore\ 4k^2 + k^2 + 4k^2 = 1$

$\therefore\ 9k^2 = 1$

$\therefore\ k^2 = \frac{1}{9}$

$\therefore\ k = \pm\frac{1}{3}$

Choosing $k = \frac{1}{3}$,

the vector is $\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$

b Let the vector parallel to $\mathbf{b}$ be $k\mathbf{b}$

i.e., $k\mathbf{b} = k\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix}$

$k\mathbf{b}$ has length $= 2$,

so $\sqrt{(-2k)^2 + (-k)^2 + (2k)^2} = 2$

$\therefore\ 4k^2 + k^2 + 4k^2 = 4$

$\therefore\ 9k^2 = 4$

$\therefore\ k^2 = \frac{4}{9}$

$\therefore\ k = \pm\frac{2}{3}$

Choosing $k = \frac{2}{3}$,

the vector is $\begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$

4 **a** $\overrightarrow{AB} = 3\overrightarrow{CD}$ means that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and 3 times its length.

b $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ means that $\overrightarrow{RS}$ is parallel to $\overrightarrow{KL}$, half its length and in the opposite direction.

c 

$\overrightarrow{AB} = 2\overrightarrow{BC}$ means that A, B and C are collinear and the length of $\overrightarrow{AB}$ is twice the length of $\overrightarrow{BC}$ (i.e., B cuts $\overrightarrow{AC}$ internally in the ratio 2 : 1).

d 

$\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AC}$ means that A, B and C are collinear and the length of $\overrightarrow{BC}$ is one third the length of $\overrightarrow{AC}$ (i.e., B cuts $\overrightarrow{AC}$ internally in the ratio 2 : 1).

5 $\overrightarrow{OP} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\overrightarrow{OQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\overrightarrow{OR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\overrightarrow{OS} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

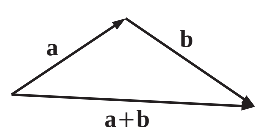
a $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ and

$\overrightarrow{QS} = \overrightarrow{QO} + \overrightarrow{OS} = -\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$

$\therefore$ $\overrightarrow{QS} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\overrightarrow{PR}$ and so QS $\parallel$ PR

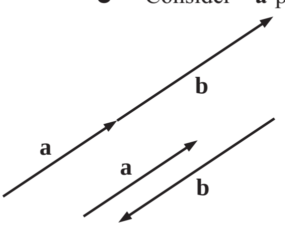
b As $\overrightarrow{QS} = 2\overrightarrow{PR}$ then $\left|\overrightarrow{QS}\right| = 2\left|\overrightarrow{PR}\right|$, i.e., QS is twice as long as PR.

6 **a** Consider **a** not parallel to **b**:



Clearly, $|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a}+\mathbf{b}|$

b Consider **a** parallel to **b**:



$|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$

or

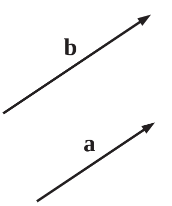
$|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a}+\mathbf{b}|$

c If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}+\mathbf{b} = \mathbf{b}$

$\therefore$ $|\mathbf{a}| + |\mathbf{b}| = \mathbf{0} + |\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$

Similarly if $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, then $\mathbf{a}+\mathbf{b} = \mathbf{a}$

$\therefore$ $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}| + \mathbf{0} = |\mathbf{a}| = |\mathbf{a}+\mathbf{b}|$



If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, then $\mathbf{a}+\mathbf{b} = \mathbf{0}$

$\therefore$ $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$

Combining **all** possibilities, $|\mathbf{a}| + |\mathbf{b}| \geqslant |\mathbf{a}+\mathbf{b}|$ i.e., $|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$

EXERCISE 16E

1 **a** $\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $|\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{3}$ units

b $3\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $|3\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{9+1+1} = \sqrt{11}$ units

c $\mathbf{i} - 5\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

$\therefore$ $|\mathbf{i} - 5\mathbf{k}| = \sqrt{1+25}$
$= \sqrt{26}$ units

d $\frac{1}{2}(\mathbf{j} + \mathbf{k}) = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ $\therefore$ $\left|\frac{1}{2}(\mathbf{j} + \mathbf{k})\right|$
$= \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$
$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ units

2 **a** $\begin{bmatrix}2\\1\\3\end{bmatrix} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

b $\begin{bmatrix}2\\0\\-5\end{bmatrix} = 2\mathbf{i} - 5\mathbf{k}$

c $\begin{bmatrix}0\\0\\-3\end{bmatrix} = -3\mathbf{k}$

d $\begin{bmatrix}-3\\0\\1\end{bmatrix} = -3\mathbf{i} + \mathbf{k}$

3 **a** $\left|\begin{bmatrix}1\\0\\-1\end{bmatrix}\right| = \sqrt{2}$ units

b $\left|\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}$
$= \sqrt{1}$
$= 1$ unit

c $|\mathbf{i} - \mathbf{j} + \mathbf{k}| = \left|\begin{bmatrix}1\\-1\\1\end{bmatrix}\right| = \sqrt{3}$ units

d $|2\mathbf{i} - \mathbf{j}| = \left|\begin{bmatrix}2\\-1\\0\end{bmatrix}\right| = \sqrt{5}$ units

4 **a** $\mathbf{a} + \mathbf{b}$
$= (-\mathbf{i} + \mathbf{j} + \mathbf{k}) + (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
$= 2\mathbf{i} + 3\mathbf{k}$

b $\mathbf{b} - \mathbf{a}$
$= (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (-\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mathbf{i} - \mathbf{j} - \mathbf{k}$
$= 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

c $2\mathbf{a} + 5\mathbf{b}$
$= 2(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + 5(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
$= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + 15\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$
$= 13\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$

d $3\mathbf{a} - 2\mathbf{b}$
$= 3(-\mathbf{i} + \mathbf{j} + \mathbf{k}) - 2(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
$= -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} - 6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
$= -9\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

5 **a** $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ has length
$\sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$ units
$\therefore$ the unit vector is $\frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

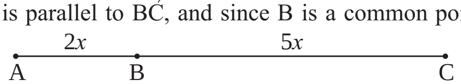
b $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ has length
$\sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14}$ units
$\therefore$ the unit vector is $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

EXERCISE 16F

1 **a** $\overrightarrow{AB} = \begin{bmatrix}4-(-2)\\3-1\\0-4\end{bmatrix} = \begin{bmatrix}6\\2\\-4\end{bmatrix} = 2\begin{bmatrix}3\\1\\-2\end{bmatrix}$ and

$\overrightarrow{BC} = \begin{bmatrix}19-4\\8-3\\-10-0\end{bmatrix} = \begin{bmatrix}15\\5\\-10\end{bmatrix} = 5\begin{bmatrix}3\\1\\-2\end{bmatrix}$

$\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, and since B is a common point, A, B and C are collinear.

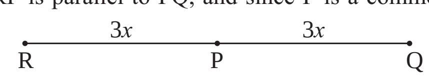


A divides CB in the ratio 7 : 2 externally.

b $\overrightarrow{RP} = \begin{bmatrix}2-(-1)\\1-7\\1-4\end{bmatrix} = \begin{bmatrix}3\\-6\\-3\end{bmatrix} = 3\begin{bmatrix}1\\-2\\-1\end{bmatrix}$ and

$\overrightarrow{PQ} = \begin{bmatrix}5-2\\-5-1\\-2-1\end{bmatrix} = \begin{bmatrix}3\\-6\\-3\end{bmatrix} = 3\begin{bmatrix}1\\-2\\-1\end{bmatrix}$

$\therefore$ $\overrightarrow{RP}$ is parallel to $\overrightarrow{PQ}$, and since P is a common point, P, Q and R are collinear.



$\therefore$ Q divides PR in the ratio 1 : 2 externally.

2 **a**

C$(-13, a, b)$ A$(2, -3, 4)$ B$(11, -9, 7)$

Since A, B and C are collinear, $\overrightarrow{CA}$ is parallel to $\overrightarrow{AB}$.

$$\therefore \quad \begin{bmatrix} 15 \\ -3-a \\ 4-b \end{bmatrix} = k \begin{bmatrix} 9 \\ -6 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad 9k &= 15, \\ -3-a &= -6k \\ \text{and} \quad 4-b &= 3k \\ \therefore \quad k &= \tfrac{5}{3}, \\ \text{and so} \quad a &= -3+6k \\ \therefore \quad a &= -3+10 \\ \text{i.e.,} \quad a &= 7 \\ \text{and} \quad b &= 4-3k = 4-5 = -1 \end{aligned}$$

b

L$(4, -3, 7)$ K$(1, -1, 0)$ M$(a, 2, b)$

Since K, L and M are collinear, $\overrightarrow{LK}$ is parallel to $\overrightarrow{KM}$.

$$\therefore \quad \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix} = k \begin{bmatrix} a-1 \\ 3 \\ b \end{bmatrix}$$

$$\begin{aligned} \therefore \quad 3k &= 2, \\ k(a-1) &= -3 \\ \text{and} \quad kb &= -7 \\ \therefore \quad k &= \tfrac{2}{3}, \\ a-1 &= -\tfrac{3}{k} = -\tfrac{9}{2} \\ \text{i.e.,} \quad a &= -\tfrac{7}{2} \\ \text{and} \quad b &= -\tfrac{7}{k} = -\tfrac{21}{2} \end{aligned}$$

EXERCISE 16G

1 **a**
$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2(-1) + 1(1) + 3(1) \\ &= -2+1+3 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} \mathbf{b} \bullet \mathbf{a} &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= (-1)(2) + 1(1) + 1(3) \\ &= -2+1+3 \\ &= 2 \end{aligned}$$

c
$$\begin{aligned} |\mathbf{a}|^2 &= (\sqrt{2^2+1^2+3^2})^2 \\ &= 14 \end{aligned}$$

d
$$\begin{aligned} \mathbf{a} \bullet \mathbf{a} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= 2(2) + 1(1) + 3(3) \\ &= 14 \end{aligned}$$

e
$$\begin{aligned} &\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= 2(-1) + 1(0) + 3(2) \\ &= 4 \end{aligned}$$

f
$$\begin{aligned} &\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \\ &= 2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \{\text{using } \mathbf{a}\} \\ &= 2 + 2(0) + 1(-1) + 3(1) \\ &= 4 \end{aligned}$$

2 **a**
$$\begin{aligned} &(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k}) \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ &= 1(0) + 1(2) - 1(1) \\ &= 0+2-1 \\ &= 1 \end{aligned}$$

b
$$\begin{aligned} \mathbf{i} \bullet \mathbf{i} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} \mathbf{i} \bullet \mathbf{j} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= 0 \end{aligned}$$

3

$$\begin{aligned}
\mathbf{a} \bullet (\mathbf{b}+\mathbf{c}) &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{bmatrix} \\
&= a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3) \\
&= a_1b_1+a_1c_1+a_2b_2+a_2c_2+a_3b_3+a_3c_3 \\
&= (a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3) \\
&= \mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c} \\
\therefore \quad \mathbf{p}\bullet(\mathbf{c}+\mathbf{d}) &= \mathbf{p}\bullet\mathbf{c}+\mathbf{p}\bullet\mathbf{d} \qquad \text{and if we let} \quad \mathbf{p}=\mathbf{a}+\mathbf{b}, \\
\text{then} \quad (\mathbf{a}+\mathbf{b})\bullet(\mathbf{c}+\mathbf{d}) &= \mathbf{p}\bullet(\mathbf{c}+\mathbf{d}) \\
&= \mathbf{p}\bullet\mathbf{c}+\mathbf{p}\bullet\mathbf{d} \\
&= (\mathbf{a}+\mathbf{b})\bullet\mathbf{c}+(\mathbf{a}+\mathbf{b})\bullet\mathbf{d} \\
&= \mathbf{c}\bullet(\mathbf{a}+\mathbf{b})+\mathbf{d}\bullet(\mathbf{a}+\mathbf{b}) \\
&= \mathbf{c}\bullet\mathbf{a}+\mathbf{c}\bullet\mathbf{b}+\mathbf{d}\bullet\mathbf{a}+\mathbf{d}\bullet\mathbf{b} \\
&= \mathbf{a}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{c}+\mathbf{a}\bullet\mathbf{d}+\mathbf{b}\bullet\mathbf{d}
\end{aligned}$$

4 Since the vectors are perpendicular, $\begin{bmatrix} 3 \\ -1 \\ t \end{bmatrix} \bullet \begin{bmatrix} 2t \\ -3 \\ -4 \end{bmatrix} = 0$

$$\begin{aligned}
\therefore \quad 3(2t)+(-1)(-3)+t(-4) &= 0 \\
\therefore \quad 6t+3-4t &= 0 \\
\therefore \quad 2t+3 &= 0 \\
\therefore \quad t &= -\tfrac{3}{2}
\end{aligned}$$

5

$\mathbf{a}\bullet\mathbf{b}$

$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$= 3(-1)+1(1)+2(1)$

$= 0$

$\mathbf{b}\bullet\mathbf{c}$

$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$

$= (-1)(1)+1(5)+1(-4)$

$= 0$

$\mathbf{a}\bullet\mathbf{c}$

$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$

$= (3)(1)+1(5)+2(-4)$

$= 0$

$\therefore$ **a**, **b** and **c** are mutually perpendicular.

6 **a**

$\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$= 1(2)+1(3)+5(-1)$

$= 0$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ are perpendicular

b

$$\begin{aligned}
\begin{bmatrix} 3 \\ t \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1-t \\ -3 \\ 4 \end{bmatrix} &= 0 \\
\therefore \quad 3(1-t)+t(-3)+(-2)4 &= 0 \\
\therefore \quad 3-3t-3t-8 &= 0 \\
\therefore \quad -6t &= 5 \\
\therefore \quad t &= -\tfrac{5}{6}
\end{aligned}$$

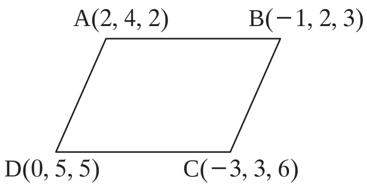
7 We have three points: A(5, 1, 2) B(6, −1, 0) C(3, 2, 0)

Then $\overrightarrow{AB} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ and $\overrightarrow{BC} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = (-2)+(-2)+4 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so ΔABC is right-angled.

8



a $\overrightarrow{AB} = \begin{bmatrix}-3\\-2\\1\end{bmatrix}$ $\quad \overrightarrow{BC} = \begin{bmatrix}-2\\1\\3\end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix}-3\\-2\\1\end{bmatrix}$ $\quad \overrightarrow{AD} = \begin{bmatrix}-2\\1\\3\end{bmatrix}$

$\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ and $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$.
$\therefore$ ABCD is a parallelogram.

b $|\overrightarrow{AB}| = \sqrt{14}$ units and $|\overrightarrow{BC}| = \sqrt{14}$ units $\quad\therefore$ ABCD is a rhombus.

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{bmatrix}-5\\-1\\4\end{bmatrix} \bullet \begin{bmatrix}1\\3\\2\end{bmatrix} = (-5) \times 1 + (-1) \times 3 + 4(2) = 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

9 **a** $\mathbf{a} = -\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$

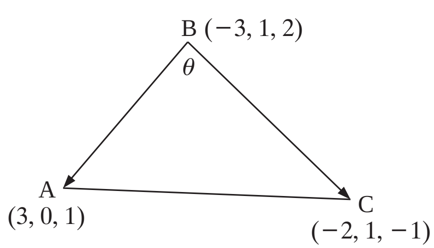
$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix}-1\\-1\\1\end{bmatrix} \bullet \begin{bmatrix}1\\1\\1\end{bmatrix}\\ &= (-1) + (-1) + 1\\ &= -1\end{aligned}$$

b
$$\begin{aligned}\cos\theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{-1}{\sqrt{3} \times \sqrt{3}}\\ &= -\tfrac{1}{3}\end{aligned}$$
$\therefore\ \theta \doteqdot 109.5^o$

10 Given A(3, 0, 1), B(−3, 1, 2) and C(−2, 1, −1),

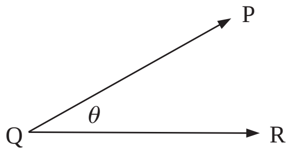
$\overrightarrow{BC} = \begin{bmatrix}1\\0\\-3\end{bmatrix}$ and $\overrightarrow{BA} = \begin{bmatrix}6\\-1\\-1\end{bmatrix}$



$$\begin{aligned}\therefore\ \cos\theta &= \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}|\ |\overrightarrow{BA}|}\\ &= \frac{\begin{bmatrix}1\\0\\-3\end{bmatrix} \bullet \begin{bmatrix}6\\-1\\-1\end{bmatrix}}{\sqrt{1+9} \times \sqrt{36+1+1}}\\ &= \frac{6+0+3}{\sqrt{10} \times \sqrt{38}}\\ &= \frac{9}{\sqrt{380}}\end{aligned}$$
and so $\theta \doteqdot 62.5^o$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, i.e., 117.5^o.

11 **a**



Now $\overrightarrow{QP} = \begin{bmatrix}1\\0\\4\end{bmatrix}$ and $\overrightarrow{QR} = \begin{bmatrix}2\\-3\\3\end{bmatrix}$

$$\cos\theta = \frac{\begin{bmatrix}1\\0\\4\end{bmatrix} \bullet \begin{bmatrix}2\\-3\\3\end{bmatrix}}{\sqrt{1+16}\times\sqrt{4+9+9}}$$
$$= \frac{2+0+12}{\sqrt{17}\times\sqrt{22}}$$
$$= \frac{14}{\sqrt{374}}$$
$$\therefore \quad \theta = 43.6^o$$

b $\overrightarrow{QP} = \begin{bmatrix}1\\0\\-3\end{bmatrix}$ $\overrightarrow{QR} = \begin{bmatrix}2\\-1\\-5\end{bmatrix}$

$$\therefore \quad \cos\theta = \frac{\begin{bmatrix}1\\0\\-3\end{bmatrix} \bullet \begin{bmatrix}2\\-1\\-5\end{bmatrix}}{\sqrt{1+9}\times\sqrt{4+1+25}}$$
$$= \frac{2+15}{\sqrt{10}\times\sqrt{30}}$$
$$= \frac{17}{\sqrt{300}}$$
$$\therefore \quad \theta = 11.0^o$$

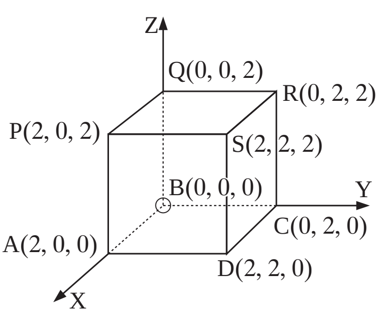
c $\overrightarrow{QP} = \begin{bmatrix}4\\0\\2\end{bmatrix}$ $\overrightarrow{QR} = \begin{bmatrix}-3\\-3\\3\end{bmatrix}$

$$\cos\theta = \frac{\begin{bmatrix}4\\0\\2\end{bmatrix} \bullet \begin{bmatrix}-3\\-3\\3\end{bmatrix}}{\sqrt{16+4}\times\sqrt{9+9+9}}$$
$$= \frac{-12+0+6}{\sqrt{20}\times\sqrt{27}}$$
$$= \frac{-6}{\sqrt{540}}$$
$$\therefore \quad \theta = 105.0^o$$

d $\overrightarrow{QP} = \begin{bmatrix}2\\-1\\0\end{bmatrix}$ $\overrightarrow{QR} = \begin{bmatrix}-1\\-2\\-4\end{bmatrix}$

$$\cos\theta = \frac{\begin{bmatrix}2\\-1\\0\end{bmatrix} \bullet \begin{bmatrix}-1\\-2\\-4\end{bmatrix}}{\sqrt{4+1}\times\sqrt{1+4+16}}$$
$$= \frac{-2+2}{\sqrt{5}\times\sqrt{21}}$$
$$= 0$$
$$\therefore \quad \theta = 90^o$$

12



a Suppose the origin is at B.

Now $\overrightarrow{BA} = \begin{bmatrix}2\\0\\0\end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$$\therefore \quad \overrightarrow{BA} \bullet \overrightarrow{BS} = \begin{bmatrix}2\\0\\0\end{bmatrix} \bullet \begin{bmatrix}2\\2\\2\end{bmatrix}$$
$$= 4+0+0$$
$$= 4$$
$$\therefore \quad \cos\measuredangle ABS = \frac{4}{\sqrt{4+0+0}\times\sqrt{4+4+4}}$$
$$= \frac{4}{2\times 2\sqrt{3}}$$
$$= \tfrac{1}{\sqrt{3}}$$
$$\therefore \quad \measuredangle ABS \doteqdot 54.7^o$$

b Consider vectors away from B.

$$\overrightarrow{BR} = \begin{bmatrix}0\\2\\2\end{bmatrix} \text{ and } \overrightarrow{BP} = \begin{bmatrix}2\\0\\2\end{bmatrix}$$

$$\overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{bmatrix}0\\2\\2\end{bmatrix} \bullet \begin{bmatrix}2\\0\\2\end{bmatrix}$$

$$= 0+0+4 = 4$$

$$\therefore \quad \cos\measuredangle RBP = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}}$$

$$= \frac{4}{\sqrt{8}\times\sqrt{8}}$$

$$= \tfrac{1}{2} \text{ and so } \measuredangle RBP = 60^o$$

c $\overrightarrow{BP} = \begin{bmatrix}2\\0\\2\end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$$\therefore \quad \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{bmatrix}2\\0\\2\end{bmatrix} \bullet \begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$= 4+0+4$$

$$= 8$$

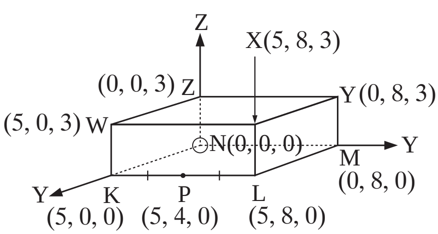
$$\therefore \quad \cos\measuredangle PBS = \frac{8}{\sqrt{4+4}\sqrt{4+4+4}}$$

$$= \frac{8}{\sqrt{96}}$$

$$\therefore \quad \measuredangle PBS \doteqdot 35.3^o$$

13 Suppose the origin is at N.

a



$$\overrightarrow{NY} = \begin{bmatrix}0\\8\\3\end{bmatrix} \text{ and } \overrightarrow{NX} = \begin{bmatrix}5\\8\\3\end{bmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{bmatrix}0\\8\\3\end{bmatrix} \bullet \begin{bmatrix}5\\8\\3\end{bmatrix}$$

$$= 0+64+9$$

$$= 73$$

$$\therefore \quad \cos\measuredangle YNX = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\tfrac{73}{98}}$$

$$\therefore \quad \measuredangle YNX \doteqdot 30.3^o$$

b $\overrightarrow{NY} = \begin{bmatrix}0\\8\\3\end{bmatrix}$ and $\overrightarrow{NP} = \begin{bmatrix}5\\4\\0\end{bmatrix}$

$$\overrightarrow{NY} \bullet \overrightarrow{NP} = \begin{bmatrix}0\\8\\3\end{bmatrix} \bullet \begin{bmatrix}5\\4\\0\end{bmatrix}$$

$$= 0+32+0$$

$$= 32$$

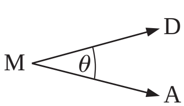
$$\therefore \quad \cos\measuredangle YNP = \frac{32}{\sqrt{64+9}\sqrt{25+16}}$$

$$\frac{32}{\sqrt{73}\sqrt{41}}$$

$$\therefore \quad \measuredangle YNP \doteqdot 54.2^o$$

14 **a** M is the midpoint of BC $\therefore$ M is at $\left(\dfrac{2+1}{2}, \dfrac{2+3}{2}, \dfrac{2+1}{2}\right)$ i.e., $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$

b Now $\overrightarrow{MD} = \begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}$ and $\overrightarrow{MA} = \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}$



$$\therefore \quad \cos\theta = \frac{\overrightarrow{MD} \bullet \overrightarrow{MA}}{|\overrightarrow{MD}|\ |\overrightarrow{MA}|} = \frac{\begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix} \bullet \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}}{\sqrt{\frac{9}{4}+\frac{1}{4}+\frac{9}{4}}\sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}}}$$

$$\therefore \quad \cos\theta = \frac{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}{\sqrt{\frac{19}{4}}\sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{201}}{4}} = \frac{9}{\sqrt{201}} \quad \text{and so} \quad \theta \doteqdot 51.5^o$$

15 **a** $\begin{bmatrix}2\\t\\t-2\end{bmatrix} \bullet \begin{bmatrix}t\\3\\t\end{bmatrix} = 0 \quad \therefore \quad 2t+3t+t(t-2) = 0$

$\therefore \quad 5t+t^2-2t = 0$

$\therefore \quad t^2+3t = 0$

$\therefore \quad t(t+3) = 0$ and so $t=0$ or $t=-3$

b Given that $\mathbf{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}2\\2\\r\end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix}s\\t\\1\end{bmatrix}$ are mutually perpendicular

$\therefore \quad \mathbf{a} \bullet \mathbf{b} = 0, \quad \mathbf{b} \bullet \mathbf{c} = 0$ and $\mathbf{a} \bullet \mathbf{c} = 0$

$\therefore \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}2\\2\\r\end{bmatrix} = 0 \qquad \therefore \quad 2+4+3r = 0$

$\therefore \quad 3r = -6$

$\therefore \quad r = -2$

and $\begin{bmatrix}2\\2\\-2\end{bmatrix} \bullet \begin{bmatrix}s\\t\\1\end{bmatrix} = 0 \qquad \therefore \quad 2s+2t-2 = 0$

$\therefore \quad s+t = 1$ ......(1)

and $\begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}s\\t\\1\end{bmatrix} = 0 \qquad \therefore \quad s+2t+3 = 0$

$\therefore \quad s+2t = -3$ ......(2)

(2) $-$ (1) gives $t = -4$ and so $s = 5$ i.e., $r=-2$, $s = 5$ and $t=-4$

16 **a** Choose any vector in the direction of the X-axis, e.g., $\mathbf{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

Then $\cos\theta = \dfrac{\begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}1\\2\\3\end{bmatrix}}{\left|\begin{bmatrix}1\\0\\0\end{bmatrix}\right|\left|\begin{bmatrix}1\\2\\3\end{bmatrix}\right|}$

$= \dfrac{1}{\sqrt{1}\sqrt{1+4+9}}$

$= \dfrac{1}{\sqrt{14}}$ and so $\theta = 74.5^o$

b A line parallel to the Y-axis is $\mathbf{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Then $\cos\theta = \dfrac{\begin{bmatrix}0\\1\\0\end{bmatrix} \bullet \begin{bmatrix}-1\\1\\3\end{bmatrix}}{\sqrt{1}\sqrt{1+1+9}}$

$= \dfrac{1}{\sqrt{11}}$

$\therefore \quad \theta \doteqdot 72.45^o$

17 $\mathbf{a} \bullet \mathbf{b}$ is a scalar, i.e., a number, so $\mathbf{a} \bullet \mathbf{b}$ dotted with $\mathbf{c}$ is not a vector dotted with a vector.

$\therefore \quad \mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ does not exist.

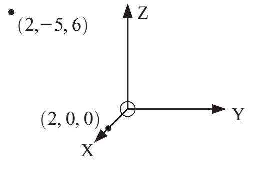
REVIEW SET 16A

1 **a** $\overrightarrow{PQ} = \begin{bmatrix}-1-2\\7--5\\9-6\end{bmatrix} = \begin{bmatrix}-3\\12\\3\end{bmatrix}$

b $PQ = \sqrt{(-3)^2+12^2+3^2}$

$= \sqrt{162}$ units

c



$\therefore$ distance

$= \sqrt{(2-2)^2+(0--5)^2+(0-6)^2}$

$= \sqrt{0+25+36}$

$= \sqrt{61}$ units

2 **a** $\mathbf{m}-\mathbf{n}+\mathbf{p}=\begin{bmatrix}6\\-3\\1\end{bmatrix}-\begin{bmatrix}2\\3\\-4\end{bmatrix}+\begin{bmatrix}-1\\3\\6\end{bmatrix}=\begin{bmatrix}3\\-3\\11\end{bmatrix}$

b $2\mathbf{n}-3\mathbf{p}=2\begin{bmatrix}2\\3\\-4\end{bmatrix}-3\begin{bmatrix}-1\\3\\6\end{bmatrix}=\begin{bmatrix}4\\6\\-8\end{bmatrix}-\begin{bmatrix}-3\\9\\18\end{bmatrix}=\begin{bmatrix}7\\-3\\-26\end{bmatrix}$

c $\mathbf{m}+\mathbf{p}=\begin{bmatrix}6\\-3\\1\end{bmatrix}+\begin{bmatrix}-1\\3\\6\end{bmatrix}=\begin{bmatrix}5\\0\\7\end{bmatrix}$ $\therefore$ $|\mathbf{m}+\mathbf{p}|=\sqrt{25+0+49}$ $=\sqrt{74}$ units

3 $\overrightarrow{CB}=\overrightarrow{CA}+\overrightarrow{AB}=-\overrightarrow{AC}+\overrightarrow{AB}=\begin{bmatrix}6\\-1\\3\end{bmatrix}+\begin{bmatrix}2\\-7\\4\end{bmatrix}=\begin{bmatrix}8\\-8\\7\end{bmatrix}$

4 The vectors are parallel $\therefore$ $\begin{bmatrix}-12\\-20\\2\end{bmatrix}=k\begin{bmatrix}3\\m\\n\end{bmatrix}$ $\therefore$ $3k=-12,\quad km=-20,\quad kn=2$

$\therefore$ $k=-4,\quad m=5,\quad n=-\frac{1}{2}$

5 $\overrightarrow{PQ}=\begin{bmatrix}4--6\\6-8\\8-2\end{bmatrix}=\begin{bmatrix}10\\-2\\6\end{bmatrix}=2\begin{bmatrix}5\\-1\\3\end{bmatrix}$ $\therefore$ Both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel to $\begin{bmatrix}5\\-1\\3\end{bmatrix}$

$\overrightarrow{QR}=\begin{bmatrix}19-4\\3-6\\17-8\end{bmatrix}=\begin{bmatrix}15\\-3\\9\end{bmatrix}=3\begin{bmatrix}5\\-1\\3\end{bmatrix}$ $\therefore$ PQ $\parallel$ QR with Q common to both.

$\therefore$ P, Q, R are collinear.

$$\overrightarrow{PQ}:\overrightarrow{QR}=2\begin{bmatrix}5\\-1\\3\end{bmatrix}:3\begin{bmatrix}5\\-1\\3\end{bmatrix}=2:3$$

$\therefore$ Q divides PR internally in the ratio 2 : 3.

6 As the vectors are perpendiculars,

$$\begin{bmatrix}-4\\t+2\\t\end{bmatrix}\bullet\begin{bmatrix}t\\1+t\\-3\end{bmatrix}=0$$

$$\therefore\quad 4t+(t+2)(1+t)-3t=0$$
$$\therefore\quad -4t+t+t^2+2+2t-3t=0$$
$$\therefore\quad t^2-4t+2=0$$

$$\therefore\quad t=\frac{4\pm\sqrt{16-4(1)(2)}}{2}$$
$$\therefore\quad t=\frac{4\pm\sqrt{8}}{2}=2\pm\sqrt{2}$$

7 If θ is the angle then $\begin{bmatrix}2\\-4\\3\end{bmatrix}\bullet\begin{bmatrix}-1\\1\\3\end{bmatrix}=\sqrt{4+16+9}\sqrt{1+1+9}\cos\theta$

$$\therefore\quad -2-4+9=\sqrt{29}\sqrt{11}\cos\theta$$
$$\therefore\quad \frac{3}{\sqrt{29\times 11}}=\cos\theta$$
$$\therefore\quad \theta\doteqdot 80.3^o$$

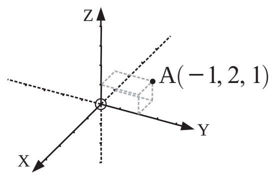
8 If D is the origin, DA the x-axis, DC the y-axis and DE the z-axis, then A is (4, 0, 0), C is (0, 8, 0) and G is (4, 8, 5)

$\overrightarrow{AG}=\begin{bmatrix}4-4\\8-0\\5-0\end{bmatrix}=\begin{bmatrix}0\\8\\5\end{bmatrix}$ $\overrightarrow{AC}=\begin{bmatrix}0-4\\8-0\\0-0\end{bmatrix}=\begin{bmatrix}-4\\8\\0\end{bmatrix}$

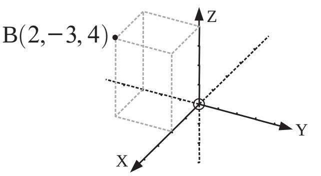
If the required angle is θ then
$$\begin{aligned}\begin{bmatrix}0\\8\\5\end{bmatrix}\bullet\begin{bmatrix}-4\\8\\0\end{bmatrix} &= \sqrt{0+64+25}\sqrt{16+64+0}\cos\theta\\
\therefore\quad 0+64+0 &= \sqrt{89}\sqrt{80}\cos\theta\\
\therefore\quad \cos\theta &= \tfrac{64}{\sqrt{89\times 80}}\\
\therefore\quad \theta &\doteqdot 40.7^o\end{aligned}$$

REVIEW SET 16B

1 **a**



b



2 **a** $\overrightarrow{PQ} = \begin{bmatrix}-4-2\\4-3\\2--1\end{bmatrix} = \begin{bmatrix}-6\\1\\3\end{bmatrix}$

b $\text{PQ} = \left|\overrightarrow{PQ}\right| = \sqrt{36+1+9} = \sqrt{46}$ units

c M is $\left(\frac{2+-4}{2}, \frac{3+4}{2}, \frac{-1+2}{2}\right)$

i.e., $(-1, \frac{7}{2}, \frac{1}{2})$

3 **a** $\mathbf{p}\bullet\mathbf{q} = \begin{bmatrix}-1\\2\\1\end{bmatrix}\bullet\begin{bmatrix}3\\-1\\4\end{bmatrix} = -3-2+4 = -1$

b $\mathbf{p}+2\mathbf{q}-\mathbf{r} = \begin{bmatrix}-1\\2\\1\end{bmatrix}+\begin{bmatrix}6\\-2\\8\end{bmatrix}-\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\-1\\7\end{bmatrix}$

c $\mathbf{p}\bullet\mathbf{r} = |\mathbf{p}|\,|\mathbf{r}|\cos\theta \quad \therefore$
$$\begin{aligned}\begin{bmatrix}-1\\2\\1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\2\end{bmatrix} &= \sqrt{1+4+1}\sqrt{1+1+4}\cos\theta\\
\therefore\quad -1+2+2 &= \sqrt{6}\sqrt{6}\cos\theta\\
\therefore\quad 3 &= 6\cos\theta\\
\therefore\quad \cos\theta &= \tfrac{1}{2}\\
\therefore\quad \theta &= 60^o\end{aligned}$$

4
$$\overrightarrow{KL} = \begin{bmatrix}-2-3\\1-1\\3-4\end{bmatrix} = \begin{bmatrix}-5\\0\\-1\end{bmatrix} \qquad \overrightarrow{LK} = \begin{bmatrix}3--2\\1-1\\4-3\end{bmatrix} = \begin{bmatrix}5\\0\\1\end{bmatrix}$$
$$\overrightarrow{KM} = \begin{bmatrix}4-3\\1-1\\3-4\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} \qquad \overrightarrow{LM} = \begin{bmatrix}4--2\\1-1\\3-3\end{bmatrix} = \begin{bmatrix}6\\0\\0\end{bmatrix}$$

$$\begin{aligned}\overrightarrow{KL}\bullet\overrightarrow{KM} &= \left|\overrightarrow{KL}\right|\left|\overrightarrow{KM}\right|\cos K\\
\therefore\quad -5+0+1 &= \sqrt{25+0+1}\sqrt{1+0+1}\cos K\\
\therefore\quad -4 &= \sqrt{26}\sqrt{2}\cos K\\
\therefore\quad \cos K &= -\tfrac{4}{\sqrt{52}}\\
\therefore\quad K &\doteqdot 123.7^o\end{aligned}$$

$$\begin{aligned}\therefore\quad \overrightarrow{LK}\bullet\overrightarrow{LM} &= \left|\overrightarrow{KL}\right|\left|\overrightarrow{KM}\right|\cos L\\
\therefore\quad 30+0+0 &= \sqrt{25+0+1}\sqrt{36+0+0}\cos L\\
\therefore\quad 30 &= \sqrt{26}\times 6\cos L\\
\therefore\quad \tfrac{5}{\sqrt{26}} &= \cos L\\
\therefore\quad L &\doteqdot 11.3^o\end{aligned}$$

and $M \doteqdot 180^o - 123.7^o - 11.3^o \doteqdot 45.0^o$

and $M \doteqdot 180^o - 123.7^o - 11.3^o \doteqdot 45.0^o$

5 As the vectors are perpendicular $\begin{bmatrix} -4 \\ t \\ 1-t \end{bmatrix} \bullet \begin{bmatrix} t \\ t \\ -6-t \end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -4t + t^2 - 6 - t + 6t + t^2 &= 0 \\ \therefore \quad 2t^2 + t - 6 &= 0 \\ \therefore \quad (2t-3)(t+2) &= 0 \\ \therefore \quad t &= \tfrac{3}{2} \text{ or } -2 \end{aligned}$$

6 **a** $\mathbf{x} \bullet \mathbf{y}$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$
$= 2 - 3 + 12$
$= 11$

$\mathbf{y} \bullet \mathbf{x}$

$$= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$
$= 2 - 3 + 12$
$= 11$

$\therefore \quad \mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$ is verified

b $\mathbf{x} \bullet (\mathbf{y} - \mathbf{z})$

$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$
$= 8 - 1 + 9$
$= 16$

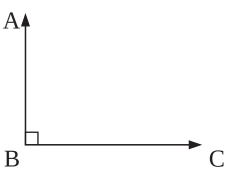
$\mathbf{x} \bullet \mathbf{y} - \mathbf{x} \bullet \mathbf{z}$

$$= 11 - \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$
$= 11 - (-6 - 2 + 3)$
$= 11 - (-5)$
$= 16$

7 If the angle is θ then, $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \sqrt{9+1+4}\,\sqrt{4+25+1}\,\cos\theta$

$$\begin{aligned} \therefore \quad 6 + 5 - 2 &= \sqrt{14}\sqrt{30}\cos\theta \\ \therefore \quad \frac{9}{\sqrt{14 \times 30}} &= \cos\theta \\ \therefore \quad \theta &\doteqdot 63.95^o \end{aligned}$$

8



$$\overrightarrow{BA} = \begin{bmatrix} 4 - -1 \\ 2 - 5 \\ -1 - 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 3 - -1 \\ -3 - 5 \\ c - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ c-2 \end{bmatrix}$$

But $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$$\begin{aligned} \therefore \quad 20 + 24 - 3(c-2) &= 0 \\ \therefore \quad 44 &= 3(c-2) \\ \therefore \quad 3c - 6 &= 44 \\ \therefore \quad 3c &= 50 \\ \therefore \quad c &= \tfrac{50}{3} \end{aligned}$$

Chapter 17

LINES IN THE PLANE AND IN SPACE

EXERCISE 17A

1 **a** **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ **ii** $x = 3 + t, \quad y = -4 + 4t, \quad t \in \mathcal{R}$

b **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + t\begin{bmatrix} -8 \\ 2 \end{bmatrix}$ **ii** $x = 5 - 8t, \quad y = 2 + 2t, \quad t \in \mathcal{R}$

c **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **ii** $x = -6 + 3t, \quad y = 7t, \quad t \in \mathcal{R}$

d **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **ii** $x = -1 - 2t, \quad y = 11 + t, \quad t \in \mathcal{R}$

2 $x = -1 + 2\lambda, \quad y = 4 - \lambda, \quad \lambda \in \mathcal{R}$

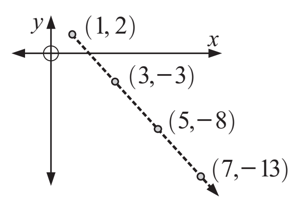
When $\lambda = 0$, $x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4$ $\therefore$ point is $(-1, 4)$
When $\lambda = 1$, $x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3$ $\therefore$ point is $(1, 3)$
When $\lambda = 3$, $x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1$ $\therefore$ point is $(5, 1)$
When $\lambda = -1$, $x = -1 + 2(-1) = -3$ and $y = 4 - -1 = 5$ $\therefore$ point is $(-3, 5)$
When $\lambda = -4$, $x = -1 + 2(-4) = -9$ and $y = 4 - -4 = 8$ $\therefore$ point is $(-9, 8)$

3 **a** If $t + 2 = 3$ and $1 - 3t = -2$, we get $t = 1$ and $-3t = -3$ i.e., $t = 1$
Since $t = 1$ in each case, $(3, -2)$ lies on the line.
If $t + 2 = 0$ and $1 - 3t + 6$, we get $t = -2$ and $-3t = 5$ i.e., $t = -\frac{5}{3}$
$\therefore$ $(0, 6)$ does not lie on the line.

b If $(k, 4)$ lies on $x = 1 - 2t, \ y = 1 + t$ then $k = 1 - 2t$ and $4 = 1 + t$
$\therefore$ $t = 3$ and $k = 1 - 6 = -5$, i.e., $k = -5$.

4 **a** $x(0) = 1$ and $y(0) = 2$,
$\therefore$ the initial position is $(1, 2)$

b



c In 1 second, the x-step is 2 and y-step is -5, which is a distance of $\sqrt{2^2 + (-5)^2} = \sqrt{29}$
$\therefore$ speed is $\sqrt{29}$ cm/sec.

EXERCISE 17B

1 **a** **i** when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$
$\therefore$ the object is at $(-4, 3)$

ii The velocity vector is $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$

iii The speed is $\sqrt{12^2 + 5^2}$
$= 13$ m/s

b **i** when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$
$\therefore$ the object is at $(0, -6)$

ii The velocity vector is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

iii The speed is $\sqrt{3^2 + (-4)^2}$
$= 5$ m/s

c **i** when $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$
$\therefore$ the object is at $(-2, -7)$

ii The velocity vector is $\begin{bmatrix} -6 \\ -4 \end{bmatrix}$

iii The speed is $\sqrt{(-6)^2 + (-4)^2}$
$= \sqrt{36 + 16}$
$= \sqrt{52}$ m/s

2 **a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} + t\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ **i** velocity vector $= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ **ii** speed $= \sqrt{8^2 + 4^2}$
$= \sqrt{80}$ km/h

b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **i** velocity vector $= \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **ii** speed $= \sqrt{6^2 + 2^2}$
$= \sqrt{40}$ km/h

c $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -12 \\ 15 \end{bmatrix} + t\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **i** velocity vector $= \begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **ii** speed $= \sqrt{7^2 + 24^2}$
$= 25$ km/h

3 **a** $\begin{bmatrix}4\\-3\end{bmatrix}$ has length $\sqrt{4^2+(-3)^2}=5$

$\therefore$ $30\begin{bmatrix}4\\-3\end{bmatrix}$ has length 150

$\therefore$ velocity vector is $\begin{bmatrix}120\\-90\end{bmatrix}$

b $\begin{bmatrix}24\\7\end{bmatrix}$ has length $\sqrt{24^2+7^2}=25$

$\therefore$ $\frac{1}{2}\begin{bmatrix}24\\7\end{bmatrix}$ has length 12.5

$\therefore$ velocity vector is $\begin{bmatrix}12\\3.5\end{bmatrix}$

c $2\mathbf{i}+\mathbf{j}=\begin{bmatrix}2\\1\end{bmatrix}$ has length $\sqrt{2^2+1^2}=\sqrt{5}$

$\therefore$ $10\sqrt{5}\begin{bmatrix}2\\1\end{bmatrix}$ has length 50

$\therefore$ velocity vector is $\begin{bmatrix}20\sqrt{5}\\10\sqrt{5}\end{bmatrix}$

d $-3\mathbf{i}+4\mathbf{j}=\begin{bmatrix}-3\\4\end{bmatrix}$ has length $\sqrt{(-3)^2+4^2}=5$

$\therefore$ $20\begin{bmatrix}-3\\4\end{bmatrix}$ has length 100

$\therefore$ velocity vector is $\begin{bmatrix}-60\\80\end{bmatrix}$.

EXERCISE 17C

1 **a** $x=-3+2t$, $y=-2+4t$, $t\geqslant 0$

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3+2t\\-2+4t\end{bmatrix}$

b when $t=2.5$,

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3+5\\-2+10\end{bmatrix}=\begin{bmatrix}2\\8\end{bmatrix}$

$\therefore$ position is (2, 8).

c **i** When the car is due north,

$x=0$ and so $-3+2t=0$

$\therefore$ $t=\frac{3}{2}$ sec.

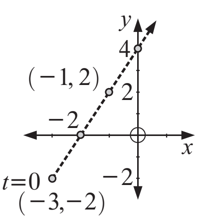
ii When the car is due west,

$y=0$ and so $-2+4t=0$

$\therefore$ $t=\frac{1}{2}$ sec

(O is the observation point)

d



2 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-10\end{bmatrix}+t\begin{bmatrix}4\\-3\end{bmatrix}$, $t\in\mathcal{R}$

b The direction vector is $\begin{bmatrix}18--2\\21-6\end{bmatrix}=\begin{bmatrix}20\\15\end{bmatrix}$ $\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\6\end{bmatrix}+\frac{t}{10}\begin{bmatrix}20\\15\end{bmatrix}$, $t\in\mathcal{R}$

c Since $2\mathbf{i}+\mathbf{j}=\begin{bmatrix}2\\1\end{bmatrix}$ has length $\sqrt{2^2+1^2}=\sqrt{5}$ then $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-5\\0\end{bmatrix}+t\begin{bmatrix}2\\1\end{bmatrix}$, $t\in\mathcal{R}$

d Since $3\mathbf{i}+4\mathbf{j}=\begin{bmatrix}3\\4\end{bmatrix}$ has length $\sqrt{3^2+4^2}=5$ then $3\begin{bmatrix}3\\4\end{bmatrix}$ has length 15

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}+(t-1)\begin{bmatrix}9\\12\end{bmatrix}$

or $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}-\begin{bmatrix}9\\12\end{bmatrix}+t\begin{bmatrix}9\\12\end{bmatrix}$ i.e., $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-8\\-16\end{bmatrix}+t\begin{bmatrix}9\\12\end{bmatrix}$, $t\in\mathcal{R}$

3 $\mathbf{r}=\begin{bmatrix}-20\\32\end{bmatrix}+t\begin{bmatrix}12\\-5\end{bmatrix}$, where t is the time in hours since 6 am.

a At 6 am, $t=0$ $\therefore$ $\mathbf{r}=\begin{bmatrix}-20\\32\end{bmatrix}$ and $|\mathbf{r}|=\sqrt{(-20)^2+(32)^2}$

$\doteqdot 37.7$ $\therefore$ the ship is 37.7 km away

b Speed $=\left|\begin{bmatrix}12\\-5\end{bmatrix}\right|=\sqrt{12^2+(-5)^2}=13$ km/h

c When due north, $x=0$. $\therefore$ $-20+12t=0$

$\therefore$ $t=\frac{5}{3}$ hours

$\therefore$ $t=1$ hour 40 min, so the time is 7:40 am

4 Yacht A: $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}+t\begin{bmatrix}1\\-2\end{bmatrix}$ Yacht B: $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-8\end{bmatrix}+t\begin{bmatrix}2\\1\end{bmatrix}$, $t\geqslant 0$

a when $t=0$, $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$ and $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-8\end{bmatrix}$

$\therefore$ A is at (4, 5) and B is at (1, −8)

b For A, the velocity vector is $\begin{bmatrix}1\\-2\end{bmatrix}$, and for B it is $\begin{bmatrix}2\\1\end{bmatrix}$

c Speed of A $= \sqrt{1^2+(-2)^2}$
$= \sqrt{5}$ km/h

Speed of B $= \sqrt{2^2+1^2}$
$= \sqrt{5}$ km/h

d The distance between them is
$$\begin{aligned} D &= \sqrt{[(1+2t)-(4+t)]^2+[(-8+t)-(5-2t)]^2} \\ &= \sqrt{(-3+t)^2+(-13+3t)^2} \\ &= \sqrt{9-6t+t^2+169-78t+9t^2} \\ &= \sqrt{10t^2-84t+178} \end{aligned}$$

This is a minimum when $10t^2-84t+178$ is a minimum. This occurs when

$t = \dfrac{-b}{2a} = \dfrac{84}{20} = 4.2$ hours. $\therefore$ the time is 4 h 12 min after 6 am i.e., 10:12 am.

e A has direction vector $\begin{bmatrix}1\\-2\end{bmatrix}$ and B has direction vector $\begin{bmatrix}2\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\-2\end{bmatrix} \bullet \begin{bmatrix}2\\1\end{bmatrix} = 2-2 = 0$, the paths of the yachts are at right angles to each other.

5 **a** P has position $\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}-5\\4\end{bmatrix} + t\begin{bmatrix}3\\-1\end{bmatrix}$ and at $t=0$, the time is 1.34 pm

$\therefore$ $x_1(t) = -5+3t$, $y_1(t) = 4-t$.

b Speed $= \sqrt{3^2+(-1)^2} = \sqrt{10}$ km/min

c Q fires its torpedo after a minutes.

$\therefore$ at time t, its torpedo has travelled for $(t-a)$ minutes.

$\therefore$ $\begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}15\\7\end{bmatrix} + (t-a)\begin{bmatrix}-4\\-3\end{bmatrix}$, $t > a$

i.e., $x_2(t) = 15-4(t-a)$ and $y_2(t) = 7-3(t-a)$

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$\therefore$ $-5+3t = 15-4(t-a)$ and $4-t = 7-3(t-a)$
i.e., $7t-4a = 20$ and $2t-3a = 3$

Solving simultaneously,
$$\begin{aligned} 7t-4a &= 20 \quad \times 3 \\ 2t-3a &= 3 \quad \times(-4) \end{aligned}$$

$$\begin{aligned} \therefore\quad 21t-12a &= 60 \\ -8t+12a &= -12 \\ \hline 13t \qquad\quad &= 48 \end{aligned}$$

$\therefore$ $t = \frac{48}{13}$ and $7(\frac{48}{13})-4a = 20$

i.e., $t \doteqdot 3.6923$ $\therefore$ $5.8462 = 4a$

i.e., $t \doteqdot 3$ min 41.53 $\therefore$ $a \doteqdot 1.4615 \doteqdot 1$ min 27.7 sec

So, as $a \doteqdot 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

EXERCISE 17D

1 **a** $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-3\\-2\end{bmatrix} + t\begin{bmatrix}2\\4\end{bmatrix}$, $t \geqslant 0$ $\therefore$ its position is $(-3+2t, -2+4t)$.

b **i** It is due east when $y = 0$
$\therefore$ $-2+4t = 0$
$\therefore$ $t = \frac{1}{2}$ sec

ii It is due north when $x = 0$
$\therefore$ $-3+2t = 0$
$\therefore$ $t = 1\frac{1}{2}$ sec

c When $y=0$, $t=\frac{1}{2}$ and $x = -3+2\left(\frac{1}{2}\right) = -2$

When $x=0$, $t = 1\frac{1}{2}$ and $y = -2+4\left(1\frac{1}{2}\right) = 4$

$\therefore$ axis intercepts are $(-2, 0)$ and $(0, 4)$

2 **a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $t \geqslant 0$ $\therefore$ its position is $(-2-t, 1-3t)$

b Crosses the x-axis when $y = 0$ $\therefore$ $1 - 3t = 0$ $\therefore$ $t = \frac{1}{3}$ sec

c When $t = \frac{1}{3}$, the point is at $(-2\frac{1}{3}, 0)$.

3 **a** $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = 5$ $\therefore$ unit direction vector is $\frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

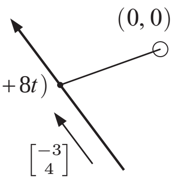
$\therefore$ the velocity vector is $\frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix} \times 10 = \begin{bmatrix} -6 \\ 8 \end{bmatrix} = -6\mathbf{i} + 8\mathbf{j}$

b $\therefore$ the liner's position is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} + t\begin{bmatrix} -6 \\ 8 \end{bmatrix}$, $t \geqslant 0$

c The liner is due east when $y = 0$, $\therefore$ $0 = -6 + 8t$

$\therefore$ $t = \frac{3}{4}$ hour

d The liner is nearest to the fishing boat when $\overrightarrow{OP} \perp \begin{bmatrix} -3 \\ 4 \end{bmatrix}$



$$\begin{aligned}
\therefore \quad \begin{bmatrix} 6-6t-0 \\ -6+8t-0 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 4 \end{bmatrix} &= 0 \\
\therefore \quad -3(6-6t) + 4(-6+8t) &= 0 \\
\therefore \quad -18 + 18t - 24 + 32t &= 0 \\
\therefore \quad 50t &= 42 \\
\therefore \quad t &= 0.84 \text{ hours}
\end{aligned}$$

i.e., after 50 min 24 sec.

At this time, the liner is at $(6 - 6(0.84), -6 + 8(0.84))$ i.e., at $(0.96, 0.72)$.

4 **a** **i** Initial position vector is $\begin{bmatrix} -8-0 \\ -5-0 \end{bmatrix} = -8\mathbf{i} - 5\mathbf{j}$

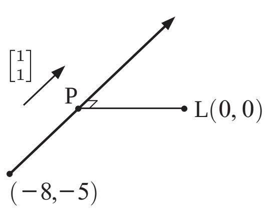
ii $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $\sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore$ $3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $3\sqrt{2}$

$\therefore$ the direction vector is $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{i} + 3\mathbf{j}$

iii position vector at time t is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 3 \end{bmatrix}$, or $(-8+3t)\mathbf{i} + (-5+3t)\mathbf{j}$

b



P is $(-8+3t, -5+3t)$

and $\overrightarrow{LP}$ is $\begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix}$

Now when closest, $\overrightarrow{LP} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$$\begin{aligned}
\therefore \quad \begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= 0 \\
\therefore \quad -8 + 3t - 5 + 3t &= 0 \\
\therefore \quad 6t &= 13 \\
\therefore \quad t &= \tfrac{13}{6} = 2\tfrac{1}{6}
\end{aligned}$$

i.e., at 2 hours 10 min

c When $t = \frac{13}{6}$, P is $\left(-8 + 3\left(\frac{13}{6}\right), -5 + 3\left(\frac{13}{6}\right)\right)$ i.e., $\left(-\frac{3}{2}, \frac{3}{2}\right)$

$$\begin{aligned}
\therefore \text{ the closest distance } &= \sqrt{\left(-\tfrac{3}{2} - 0\right)^2 + \left(\tfrac{3}{2} - 0\right)^2} \\
&= \sqrt{\tfrac{9}{4} + \tfrac{9}{4}} \\
&= \sqrt{\tfrac{9}{2}} \\
&\doteqdot 2.12 \text{ km}
\end{aligned}$$

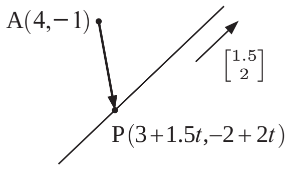
$\therefore$ the trawler would be breaking the law.

5 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$

a When $t = 4$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + 4\begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ $\quad\therefore$ the car is at (3, 6)

b Speed $= \sqrt{(1.5)^2 + 2^2} = 2.5$ m/s

c



$\overrightarrow{AP} = \begin{bmatrix} -3+1.5t-4 \\ -2+2t--1 \end{bmatrix} = \begin{bmatrix} -7+1.5t \\ -1+2t \end{bmatrix}$

and for shortest AP, $\overrightarrow{AP} \bullet \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = 0$

$$\begin{aligned} \therefore\quad (-7 + 1.5t)1.5 + (-1 + 2t)2 &= 0 \\ \therefore\quad -10.5 + 2.25t - 2 + 4t &= 0 \\ \therefore\quad 6.25t &= 12.5 \\ \therefore\quad t &= 2 \text{ sec} \end{aligned}$$

d $x = -3 + \frac{3}{2}t$ $\quad$ Equating t values, $\frac{2}{3}(x+3) = \frac{1}{2}(y+2)$

$y = -2 + 2t$

$$\begin{aligned} \therefore\quad 4(x+3) &= 3(y+2) \quad \{\times \text{ both sides by } 6\} \\ \therefore\quad 4x + 12 &= 3y + 6 \\ \therefore\quad 4x - 3y &= -6 \end{aligned}$$

6 **a** $|\mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km/h, the velocity vector is $40\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -120 \\ -40 \end{bmatrix}$.

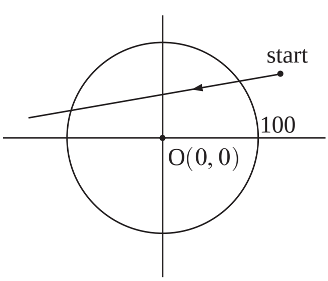
b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix} + t\begin{bmatrix} -120 \\ -40 \end{bmatrix}$, $t \geqslant 0$ $\quad\{t = 0$ at 12.00 noon$\}$

c At 1:00 pm, $t = 1$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200-120 \\ 100-40 \end{bmatrix} = \begin{bmatrix} 80 \\ 60 \end{bmatrix}$

d The distance from O(0, 0) to P_1 (80, 60) is $\left|\begin{bmatrix} 80 \\ 60 \end{bmatrix}\right| = \sqrt{80^2 + 60^2} = 100$ km,

which is when it becomes visible to radar. $\{$within 100 km of O(0, 0)$\}$

e



A general point on the path is P(200 − 120t, 100 − 40t).

Now $\overrightarrow{OP} = \begin{bmatrix} 200-120t \\ 100-40t \end{bmatrix}$,

and for the closest point $\overrightarrow{OP} \bullet \begin{bmatrix} -3 \\ -1 \end{bmatrix} = 0$

$$\begin{aligned} \therefore\quad -3(200 - 120t) - 1(100 - 40t) &= 0 \\ \therefore\quad -700 + 400t &= 0 \\ \therefore\quad t = \tfrac{7}{4} &= 1\tfrac{3}{4} \text{ hours} \end{aligned}$$

The time when the aircraft is closest is 1:45 pm, and

at this time $\overrightarrow{OP} = \begin{bmatrix} 200-120(\frac{7}{4}) \\ 100-40(\frac{7}{4}) \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix}$

$$\therefore\quad d_{\min} = \sqrt{(-10)^2 + 30^2} \doteqdot 31.6 \text{ km}$$

f It disappears from radar when $|\overrightarrow{OP}| = 100$ and $t > 1\frac{3}{4}$

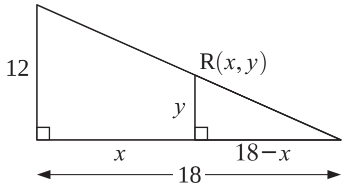
$$\begin{aligned} \therefore\quad \sqrt{(200 - 120t)^2 + (100 - 40t)^2} &= 100 \\ \therefore\quad 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1400t^2 &= 10\,000 \\ \therefore\quad 16\,000t^2 - 56\,000t + 40\,000 &= 0 \\ \therefore\quad 16t^2 - 56\,t + 40 &= 0 \quad \{\div 1000\} \\ \therefore\quad 2t^2 - 7t + 5 &= 0 \quad \{\div 8\} \\ \therefore\quad (2t - 5)(t - 1) &= 0 \\ \therefore\quad t &= \tfrac{5}{2} \quad \{\text{as } t > 1\tfrac{3}{4}\} \end{aligned}$$

i.e., at $2\frac{1}{2}$ hours i.e., at 2:30 pm

7 **a** when $x = 0$, $3y = 36$, $\therefore$ $y = 12$ $\therefore$ B is (0, 12)

when $y = 0$, $2x = 36$, $\therefore$ $x = 18$ $\therefore$ A is (18, 0)

b



Using the similar triangles,

$$\frac{y}{18-x} = \frac{12}{18} = \frac{2}{3}$$

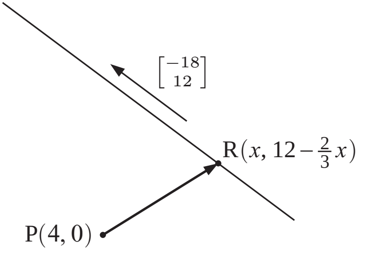
$$\therefore \quad y = \tfrac{2}{3}(18-x)$$

$\therefore$ R is $(x,\ 12 - \tfrac{2}{3}x)$

c $\overrightarrow{PR} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x-0 \end{bmatrix} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 0-18 \\ 12-0 \end{bmatrix} = \begin{bmatrix} -18 \\ 12 \end{bmatrix}$

d R is located so the shortest distance PR is when $\overrightarrow{RP} \bullet \begin{bmatrix} -18 \\ 12 \end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -18(x-4) + 12(12 - \tfrac{2}{3}x) &= 0 \\ \therefore \quad -18x + 72 + 144 - 8x &= 0 \\ \therefore \quad 26x &= 216 \\ \therefore \quad x &= \tfrac{108}{13} \\ \text{and} \quad y &= \tfrac{2}{3}\left(18 - \tfrac{108}{13}\right) \\ \therefore \quad y &= 12 - \tfrac{72}{13} = \tfrac{84}{13} \end{aligned}$$



i.e., R is at $\left(\frac{108}{13}, \frac{84}{13}\right)$

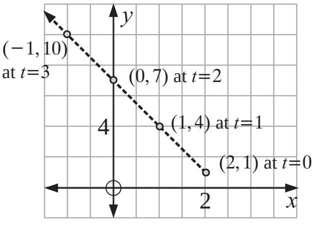
and $\overrightarrow{PR} = \begin{bmatrix} \frac{108}{13} - 4 \\ 12 - \frac{2}{3}\left(\frac{108}{13}\right) \end{bmatrix} \doteqdot \begin{bmatrix} 4.308 \\ 6.4615 \end{bmatrix}$

$\therefore$ $|\overrightarrow{PR}| \doteqdot \sqrt{(4.308)^2 + (6.4615)^2} \doteqdot 7.77$ km,

so the shortest distance is 7.77 km

8 **a** $x(0) = 2$, $y(0) = 1$ $\therefore$ the particle's initial position is (2, 1).

b



c If Q is a general point on the path, then Q is $(2-t,\ 1+3t)$

If R is (0, 10) then $\overrightarrow{RQ} = \begin{bmatrix} 2-t-0 \\ 1+3t-10 \end{bmatrix} = \begin{bmatrix} 2-t \\ 3t-9 \end{bmatrix}$

For the shortest distance, $\overrightarrow{RQ} \bullet \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -1(2-t) + 3(3t-9) &= 0 \\ \therefore \quad -2 + t + 9t - 27 &= 0 \\ \therefore \quad 10t &= 29 \\ \therefore \quad t &= 2.9 \text{ sec} \end{aligned}$$

When $t = 2.9$, $\overrightarrow{RQ} = \begin{bmatrix} -0.9 \\ -0.3 \end{bmatrix}$ and $|\overrightarrow{RQ}| = \sqrt{(-0.9)^2 + (-0.3)^2} \doteqdot 0.949$ cm

(or $\overrightarrow{RQ} = \begin{bmatrix} -\frac{9}{10} \\ -\frac{3}{10} \end{bmatrix}$ $\therefore$ $|\overrightarrow{RQ}| = \sqrt{\frac{81}{100} + \frac{9}{100}} = \sqrt{\frac{90}{100}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$ cm)

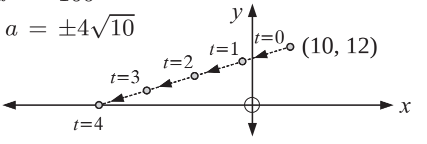
9 **a** $x(0) = 10$, $y(0) = 12$ $\therefore$ P is at (10, 12)

b The velocity vector is $\begin{bmatrix} a \\ -3 \end{bmatrix}$ and speed $= \sqrt{a^2 + 9} = 13$

$$\begin{aligned} \therefore \quad a^2 + 9 &= 169 \\ \therefore \quad a^2 &= 160 \\ \therefore \quad a &= \pm 4\sqrt{10} \end{aligned}$$

c For $a < 0$, i.e., $a = -4\sqrt{10}$,

$x(t) = 10 - 4\sqrt{10}t$, $y(t) = 12 - 3t$.



10 For A, $x(t) = 3 - t$, $y(t) = 2t - 4$ For B, $x(t) = 4 - 3t$, $y(t) = 3 - 2t$

a When $t = 0$, $x(0) = 3$, $y(0) = -4$ $\therefore$ $x(0) = 4$, $y(0) = 3$

$\therefore$ A is at $(3, -4)$ $\therefore$ B is at $(4, 3)$

b The velocity vector of A is $\begin{bmatrix}-1\\2\end{bmatrix}$ and the velocity vector of B is $\begin{bmatrix}-3\\-2\end{bmatrix}$.

c If the angle is θ,

$$\begin{aligned}\begin{bmatrix}-1\\2\end{bmatrix} \bullet \begin{bmatrix}-3\\-2\end{bmatrix} &= \sqrt{1+4}\sqrt{9+4}\cos\theta\\ \therefore\quad 3-4 &= \sqrt{5}\sqrt{13}\cos\theta\\ \therefore\quad \tfrac{-1}{\sqrt{65}} &= \cos\theta \qquad \text{and so} \quad \theta = 97.1^o\end{aligned}$$

d If D is the distance between them, then

$$\begin{aligned}D &= \sqrt{[(4-3t)-(3-t)]^2 + [(2t-4)-(3-2t)]^2}\\ &= \sqrt{[1-2t]^2 + [-7+4t]^2}\\ &= \sqrt{1-4t+4t^2+49-56t+16t^2}\\ &= \sqrt{20t^2-60t+50}\end{aligned}$$

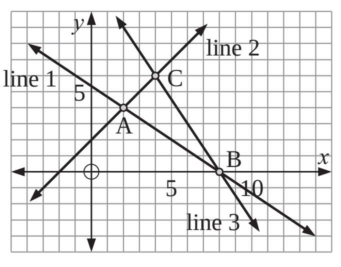
and D is a minimum when

$t = -\dfrac{b}{2a} = \frac{60}{40} = 1\frac{1}{2}$

i.e., $t = 1.5$ hours

EXERCISE 17E

1 **a**



b A is $(2, 4)$, B is $(8, 0)$, C is $(4, 6)$

c $BC = \sqrt{(8-4)^2 + (0-6)^2} = \sqrt{16+36} = \sqrt{52}$ units

$AC = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16} = \sqrt{52}$ units

$\therefore$ BC = AC and so ΔABC is isosceles.

d Line 1 and Line 2 meet at A.

$$\begin{aligned}\therefore\quad \begin{bmatrix}-1\\6\end{bmatrix} + \begin{bmatrix}3r\\-2r\end{bmatrix} &= \begin{bmatrix}0\\2\end{bmatrix} + s\begin{bmatrix}1\\1\end{bmatrix}\\ \therefore\quad \begin{bmatrix}3r-s\\-2r-s\end{bmatrix} &= \begin{bmatrix}1\\-4\end{bmatrix}\end{aligned}$$

i.e., $3r - s = 1$

and $2r + s = 4$

Adding, $5r = 5$ $\therefore$ $r = 1$

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\6\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$

Line 2 and Line 3 meet at C.

$$\begin{aligned}\therefore\quad \begin{bmatrix}0\\2\end{bmatrix} + s\begin{bmatrix}1\\1\end{bmatrix} &= \begin{bmatrix}10\\-3\end{bmatrix} + t\begin{bmatrix}-2\\3\end{bmatrix}\\ \therefore\quad \begin{bmatrix}s+2t\\s-3t\end{bmatrix} &= \begin{bmatrix}10\\-5\end{bmatrix}\end{aligned}$$

i.e., $s + 2t = 10$

$-s + 3t = 5$

Adding, $5t = 15$ $\therefore$ $t = 3$

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}10\\-3\end{bmatrix} + 3\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

Line 1 and Line 3 meet at B.

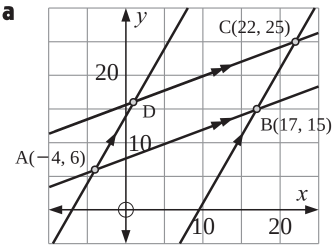
$$\begin{aligned}\therefore\quad \begin{bmatrix}-1\\6\end{bmatrix} + r\begin{bmatrix}3\\-2\end{bmatrix} &= \begin{bmatrix}10\\-3\end{bmatrix} + t\begin{bmatrix}-2\\3\end{bmatrix}\\ \therefore\quad \begin{bmatrix}3r+2t\\-2r-3t\end{bmatrix} &= \begin{bmatrix}11\\-9\end{bmatrix}\\ \therefore\quad 3r + 2t &= 11 \qquad \times 3\\ -2r - 3t &= -9 \qquad \times 2\\ \therefore\quad 9r + 6t &= 33\\ -4r - 6t &= -18\end{aligned}$$

Adding, $5r = 15$

$\therefore$ $r = 3$

So, $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\6\end{bmatrix} + 3\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}8\\0\end{bmatrix}$ ✓

2 **a**



Lines 3 and 4 meet at C.

$$\therefore \quad \begin{bmatrix}22\\25\end{bmatrix} + t\begin{bmatrix}-7\\-3\end{bmatrix} = \begin{bmatrix}22\\25\end{bmatrix} + n\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}-7t+u\\-3t+2u\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\therefore \quad \begin{aligned} -7t + u &= 0 \\ -3t + 2u &= 0 \end{aligned}$$

$$\therefore \quad \begin{aligned} 14t - 2u &= 0 \\ -3t + 2u &= 0 \end{aligned}$$

Adding, $\quad 11t = 0$

$$\therefore \quad t = 0$$

and $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}22\\25\end{bmatrix}$ ✓

b B(17, 15)
C(22, 25)
D(1, 16)

c Lines 1 and 4 meet at B

$$\therefore \quad \begin{bmatrix}-4\\6\end{bmatrix} + r\begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}22\\25\end{bmatrix} + u\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}7r+u\\3r+2u\end{bmatrix} = \begin{bmatrix}26\\19\end{bmatrix}$$

$$\therefore \quad \begin{aligned} 7r + u &= 26 \\ 3r + 2u &= 19 \end{aligned}$$

$$\therefore \quad \begin{aligned} -14r - 2u &= -52 \\ 3r + 2u &= 19 \end{aligned}$$

Adding, $\quad -11r = -33$

$$\therefore \quad r = 3$$

and $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-4\\6\end{bmatrix} + 3\begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}17\\15\end{bmatrix}$ ✓

Lines 2 and 3 meet at D

and $\begin{bmatrix}-4\\6\end{bmatrix} + s\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}22\\25\end{bmatrix} + t\begin{bmatrix}-7\\-3\end{bmatrix}$

$$\therefore \quad \begin{bmatrix}s+7t\\2s+3t\end{bmatrix} = \begin{bmatrix}26\\19\end{bmatrix}$$

$$\therefore \quad -2s - 14t = -52$$

$$\therefore \quad 2s + 3t = 19$$

Adding, $\quad -11t = -33$

$$\therefore \quad t = 3$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}22\\25\end{bmatrix} + 3\begin{bmatrix}-7\\-3\end{bmatrix} = \begin{bmatrix}1\\16\end{bmatrix} \quad ✓$$

3 **a** Lines 1 and 3 meet at A.

$$\therefore \quad \begin{bmatrix}0\\2\end{bmatrix} + r\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix} + t\begin{bmatrix}1\\7\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}2r-t\\r+t\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\therefore \quad \begin{aligned} 2r - t &= 0 \\ r + t &= 3 \end{aligned}$$

Adding, $\quad 3r = 3$

$$\therefore \quad r = 1$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

$\therefore$ A is (2, 3)

Lines 1 and 2 meet at B.

$$\therefore \quad \begin{bmatrix}0\\2\end{bmatrix} + r\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix} + s\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}2r+s\\r+2s\end{bmatrix} = \begin{bmatrix}8\\4\end{bmatrix}$$

$$\therefore \quad \begin{aligned} -4r - 2s &= -16 \\ r + 2s &= 4 \end{aligned}$$

Adding, $\quad -3r = -12$

$$\therefore \quad r = 4$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} + 4\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix}$$

$\therefore$ B is (8, 6)

Lines 2 and 3 meet at C.

$$\therefore \quad \begin{bmatrix}8\\6\end{bmatrix} + s\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix} + t\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}-s-t\\-2s+t\end{bmatrix} = \begin{bmatrix}-8\\-1\end{bmatrix}$$

$$\therefore \quad \begin{aligned} -s - t &= -8 \\ -2s + t &= -1 \end{aligned}$$

Adding, $\quad -3s = -9$

$$\therefore \quad s = 3$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix} + 3\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}$$

$\therefore$ C is (5, 0)

b A (2, 3), B(8, 6), C(5, 0)

$$\begin{aligned} \text{AB} &= \sqrt{(8-2)^2 + (6-3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \end{aligned}$$

$$\begin{aligned} \text{BC} &= \sqrt{(5-8)^2 + (0-6)^2} \\ &= \sqrt{9 + 36} \\ &= \sqrt{45} \end{aligned}$$

The two equal sides are AB and BC and they have length $\sqrt{45}$ units.

4 **a** Lines QP and PR meet at P

$$\therefore \begin{bmatrix}3\\-1\end{bmatrix} + r\begin{bmatrix}14\\10\end{bmatrix} = \begin{bmatrix}0\\18\end{bmatrix} + t\begin{bmatrix}5\\-7\end{bmatrix}$$

$$\therefore \begin{bmatrix}14r-5t\\10r+7t\end{bmatrix} = \begin{bmatrix}-3\\19\end{bmatrix}$$

$$\begin{aligned} \therefore\ 14r - 5t &= -3 \quad \times 7\\ 10r + 7t &= 19 \quad \times 5 \end{aligned}$$

$$\begin{aligned} \therefore\ 98r - 35t &= -21\\ 50r + 35t &= 95 \end{aligned}$$

Adding, $148r = 74$

$$\therefore\ r = \tfrac{1}{2}$$

$$\therefore \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}14\\10\end{bmatrix} = \begin{bmatrix}10\\4\end{bmatrix}$$

$\therefore$ P is (10, 4)

Lines QP and PR meet at Q

$$\begin{bmatrix}3\\-1\end{bmatrix} + r\begin{bmatrix}14\\10\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} + s\begin{bmatrix}17\\-9\end{bmatrix}$$

$$\therefore\ r\begin{bmatrix}14\\10\end{bmatrix} = s\begin{bmatrix}17\\-9\end{bmatrix}$$

$$\therefore\ r = s = 0$$

So, $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$

$\therefore$ Q is (3, −1)

Lines QR and PR meet at R

$$\therefore \begin{bmatrix}3\\-1\end{bmatrix} + s\begin{bmatrix}17\\-9\end{bmatrix} = \begin{bmatrix}0\\18\end{bmatrix} + t\begin{bmatrix}5\\-7\end{bmatrix}$$

$$\therefore \begin{bmatrix}17s-5t\\-9s+7t\end{bmatrix} = \begin{bmatrix}-3\\19\end{bmatrix}$$

$$\begin{aligned} \therefore\ 17s - 5t &= -3 \quad \times 7\\ -9s + 7t &= 19 \quad \times 5 \end{aligned}$$

$$\begin{aligned} \therefore\ 119s - 35t &= -21\\ -45s + 35t &= 95 \end{aligned}$$

Adding, $74s = 74$

$$\therefore\ s = 1$$

$$\therefore \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} + \begin{bmatrix}17\\-9\end{bmatrix} = \begin{bmatrix}20\\-10\end{bmatrix}$$

$\therefore$ R is (20, −10)

b $\overrightarrow{PQ} = \begin{bmatrix}3-10\\-1-4\end{bmatrix} = \begin{bmatrix}-7\\-5\end{bmatrix}$

$\overrightarrow{PR} = \begin{bmatrix}20-10\\-10-4\end{bmatrix} = \begin{bmatrix}10\\-14\end{bmatrix}$

and $\overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$

c $PQ \perp PR$ i.e., $\measuredangle QPR = 90^o$

d $$\begin{aligned} \text{Area} &= \tfrac{1}{2}\,|\,\overrightarrow{PQ}\,|\,|\,\overrightarrow{PR}\,|\\ &= \tfrac{1}{2}\sqrt{49+25}\sqrt{100+196}\\ &= 74 \text{ units}^2 \end{aligned}$$

5 **a** Lines 1 and 2 meet at B

$$\therefore \begin{bmatrix}2\\5\end{bmatrix} + r\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}18\\9\end{bmatrix} + s\begin{bmatrix}-8\\32\end{bmatrix}$$

$$\therefore \begin{bmatrix}4r+8s\\r-32s\end{bmatrix} = \begin{bmatrix}16\\4\end{bmatrix}$$

$$\begin{aligned} \therefore\ 4r + 8s &= 16\\ r - 32s &= 4 \end{aligned}$$

$$\begin{aligned} \therefore\ r + 2s &= 4\\ -r + 32s &= -4 \end{aligned}$$

Adding, $34s = 0$

$$\therefore\ s = 0$$

$\therefore \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}18\\9\end{bmatrix}$ $\therefore$ B is (18, 9)

Lines 2 and 3 meet at C

$$\therefore \begin{bmatrix}18\\9\end{bmatrix} + s\begin{bmatrix}-8\\32\end{bmatrix} = \begin{bmatrix}14\\25\end{bmatrix} + t\begin{bmatrix}-8\\-2\end{bmatrix}$$

$$\therefore \begin{bmatrix}-8s+8t\\32s+2t\end{bmatrix} = \begin{bmatrix}-4\\16\end{bmatrix}$$

$$\begin{aligned} 2s - 2t &= 1\\ 32s + 2t &= 16 \end{aligned}$$

Adding, $34s = 17$

$$\therefore\ s = \tfrac{1}{2}$$

$$\therefore \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}18\\9\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}-8\\32\end{bmatrix} = \begin{bmatrix}14\\25\end{bmatrix}$$

$\therefore$ C is (14, 25)

Lines 3 and 4 meet at D

$$\therefore \begin{bmatrix}14\\25\end{bmatrix} + t\begin{bmatrix}-8\\-2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + u\begin{bmatrix}-3\\12\end{bmatrix}$$

$$\therefore \begin{bmatrix}-8t+3u\\-2t-12u\end{bmatrix} = \begin{bmatrix}-11\\-24\end{bmatrix}$$

$$\begin{aligned} \therefore\ -8t + 3u &= -11\\ t + 6u &= 12 \end{aligned}$$

$$\begin{aligned} \therefore\ 16t - 6u &= 22\\ t + 6u &= 12 \end{aligned}$$

Adding, $17t = 34$

$$\therefore\ t = 2$$

$$\therefore \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}14\\25\end{bmatrix} + 2\begin{bmatrix}-8\\-2\end{bmatrix} = \begin{bmatrix}-2\\21\end{bmatrix}$$

$\therefore$ D is (−2, 21)

b $\overrightarrow{AC} = \begin{bmatrix}14-2\\25-5\end{bmatrix} = \begin{bmatrix}12\\20\end{bmatrix}$

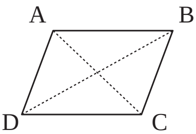
$\overrightarrow{DB} = \begin{bmatrix}18--2\\9-21\end{bmatrix} = \begin{bmatrix}20\\-12\end{bmatrix}$

i $|\,\overrightarrow{AC}\,| = \sqrt{12^2 + 20^2} = \sqrt{544}$ units

ii $|\,\overrightarrow{DB}\,| = \sqrt{20^2 + (-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

c The diagonals are perpendicular and equal in length, so, ABCD is a square.



EXERCISE 17F

1 **a** The vector equation is $\begin{bmatrix} x-1 \\ y-3 \\ z+7 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad t \in \mathcal{R}$

b The vector equation is $\begin{bmatrix} x-0 \\ y-1 \\ z-2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad t \in \mathcal{R}$

c Since the line is parallel to the X-axis, it has direction vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ the vector equation is $\begin{bmatrix} x+2 \\ y-2 \\ z-1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathcal{R}$

2 **a** The parametric equations are:

$x = 5 + (-1)t, \quad y = 2 + 2t, \quad z = -1 + 6t$

i.e., $x = 5 - t, \ y = 2 + 2t, \ z = -1 + 6t, \quad t \in \mathcal{R}$

b The parametric equations are:

$x = 0 + 2t, \quad y = 2 + (-1)t, \quad z = -1 + 3t$

i.e., $x = 2t, \ y = 2 - t, \ z = -1 + 3t, \quad t \in \mathcal{R}$

c Since the line is perpendicular to the XOY plane, it has direction vector [0, 0, 1].

$\therefore$ the parametric equations are:

$x = 3 + 0t, \quad y = 2 + 0t, \quad z = -1 + 1t$

i.e., $x = 3, \ y = 2, \ z = -1 + t, \quad t \in \mathcal{R}$

3 **a** $\overrightarrow{AB} = \begin{bmatrix} -1-1 \\ 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \quad x = 1 - 2t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in \mathcal{R}$

b $\overrightarrow{CD} = \begin{bmatrix} 3-0 \\ 1-1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \quad \therefore \quad x = 3t, \quad y = 1, \quad z = 3 - 4t, \quad t \in \mathcal{R}$

c $\overrightarrow{EF} = \begin{bmatrix} 1-1 \\ -1-2 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \quad \therefore \quad x = 1, \quad y = 2 - 3t, \quad z = 5, \quad t \in \mathcal{R}$

d $\overrightarrow{GH} = \begin{bmatrix} 5-0 \\ -1-1 \\ 3--1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \quad \therefore \quad x = 5t, \quad y = 1 - 2t, \quad z = -1 + 4t, \quad t \in \mathcal{R}$

4 Given $x = 1 - t, \quad y = 3 + t, \quad z = 3 - 2t$:

a The line meets the XOY plane when $z = 0$ i.e., $3 - 2t = 0 \quad \therefore \quad t = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, i.e., the point is $(-\frac{1}{2}, \frac{9}{2}, 0)$

b The line meets the YOZ plane when $x = 0$ i.e., $1 - t = 0$

$\therefore \quad t = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, i.e., the point is (0, 4, 1)

c The line meets the XOZ plane when $y = 0$ i.e., $3 + t = 0 \quad \therefore \quad t = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, i.e., the point is (4, 0, 9)

5 Given a line with equations $x = 2 - t$, $y = 3 + 2t$ and $z = 1 + t$ the distance to the point $(1, 0, -2)$ is $\sqrt{(2-t-1)^2 + (3+2t-0)^2 + (1+t+2)^2}$.

But this distance $= 5\sqrt{3}$ units

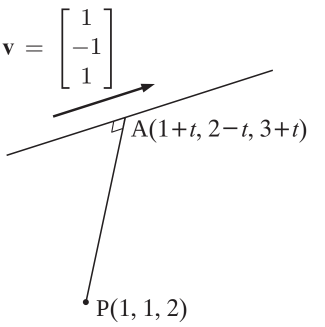
$$\begin{aligned}
\therefore\quad \sqrt{(1-t)^2 + (3+2t)^2 + (t+3)^2} &= 5\sqrt{3}\\
\therefore\quad (1-t)^2 + (3+2t)^2 + (t+3)^2 &= 75\\
\therefore\quad 1 - 2t + t^2 + 9 + 12t + 4t^2 + t^2 + 6t + 9 &= 75\\
\therefore\quad 6t^2 + 16t - 56 &= 0\\
\therefore\quad 3t^2 + 8t - 28 &= 0\\
\therefore\quad (3t+14)(t-2) &= 0\\
\therefore\quad t &= -\tfrac{14}{3} \quad\text{or}\quad t = 2
\end{aligned}$$

When $t = 2$, the point is $(0, 7, 3)$
and when $t = -\frac{14}{3}$, the point is $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$.

6 **a** Let $A(1+t, 2-t, 3+t)$ be a point on the given line.

Then $\overrightarrow{PA} = \begin{bmatrix} 1+t-1 \\ 2-t-1 \\ 3+t-2 \end{bmatrix} = \begin{bmatrix} t \\ 1-t \\ 1+t \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the direction vector of the line.

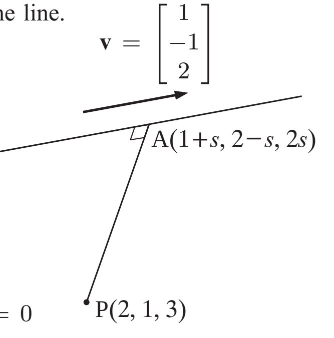


If $\overrightarrow{PA}$ and $\mathbf{v}$ are perpendicular, then $\overrightarrow{PA} \bullet \mathbf{v} = 0$

$$\begin{aligned}
\therefore\quad \begin{bmatrix} t \\ 1-t \\ 1+t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= 0\\
\therefore\quad t - 1(1-t) + 1(1+t) &= 0\\
\therefore\quad t - 1 + t + 1 + t &= 0\\
\therefore\quad 3t &= 0\\
\therefore\quad t &= 0
\end{aligned}$$

$\therefore$ A is at $(1, 2, 3)$ i.e., the foot of the perpendicular is $(1, 2, 3)$

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.
Then A is at $(1+s, 2-s, 2s)$ for some s.



Now $\overrightarrow{PA} = \begin{bmatrix} 1+s-2 \\ 2-s-1 \\ 2s-3 \end{bmatrix} = \begin{bmatrix} s-1 \\ 1-s \\ 2s-3 \end{bmatrix}$

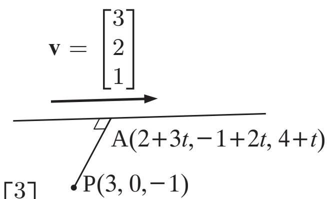
and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the direction vector of the line.

Since $\overrightarrow{PA} \bullet \mathbf{v} = 0$, $\begin{bmatrix} s-1 \\ 1-s \\ 2s-3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$

$$\begin{aligned}
\therefore\quad 1(s-1) - 1(1-s) + 2(2s-3) &= 0\\
\therefore\quad s - 1 - 1 + s + 4s - 6 &= 0\\
\therefore\quad 6s &= 8\\
\therefore\quad s &= \tfrac{4}{3}
\end{aligned}$$

$\therefore$ A is at $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$ i.e., the foot of the perpendicular is $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$

7 **a** Let A $(2+3t, -1+2t, 4+t)$ be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.



Now $\overrightarrow{PA} = \begin{bmatrix} 2+3t-3 \\ -1+2t-0 \\ 4+t-(-1) \end{bmatrix} = \begin{bmatrix} 3t-1 \\ 2t-1 \\ t+5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

is the direction vector of the line.

$\therefore$ since $\overrightarrow{PA} \bullet \mathbf{v} = 0$,

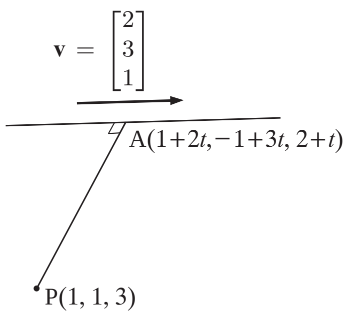
$$\begin{aligned} \begin{bmatrix} 3t-1 \\ 2t-1 \\ t+5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} &= 0 \\ \therefore \quad 3(3t-1)+2(2t-1)+1(t+5) &= 0 \\ \therefore \quad 9t-3+4t-2+t+5 &= 0 \\ \therefore \quad 14t &= 0 \\ \therefore \quad t &= 0 \end{aligned}$$

$\therefore$ A is at $(2, -1, 4)$

$$\begin{aligned} \therefore \text{ the distance } d = \sqrt{(2-3)^2+(-1-0)^2+(4-(-1))^2} &= \sqrt{1+1+25} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \text{ units} \end{aligned}$$

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.

Then A is at $(1+2t, -1+3t, 2+t)$ for some t.



Now $\overrightarrow{PA} = \begin{bmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t-2 \\ t-1 \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is the direction vector of the line.

$\therefore$ since $\overrightarrow{PA} \bullet \mathbf{v} = 0$,

$$\begin{aligned} \begin{bmatrix} 2t \\ 3t-2 \\ t-1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} &= 0 \\ \therefore \quad 4t+3(3t-2)+1(t-1) &= 0 \\ \therefore \quad 4t+9t-6+t-1 &= 0 \\ \therefore \quad 14t &= 7 \\ \therefore \quad t &= \tfrac{1}{2} \end{aligned}$$

$\therefore$ A is at $(2, \frac{1}{2}, \frac{5}{2})$

$$\begin{aligned} \therefore \text{ the distance } d = \sqrt{(2-1)^2+(\tfrac{1}{2}-1)^2+(\tfrac{5}{2}-3)^2} &= \sqrt{1+\tfrac{1}{4}+\tfrac{1}{4}} \\ &= \sqrt{\tfrac{3}{2}} \text{ units} \end{aligned}$$

EXERCISE 17G

1 **a** Line 1 has direction vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $1+2t = -2+3s$ $\quad$ $2-t = 3-s$ $\quad$ $3+t = 1+2s$

$\therefore$ $2t-3s = -3$ ...... (1) $\quad$ $\therefore$ $-t+s = 1$ ...... (2) $\quad$ $\therefore$ $t-2s = -2$ ...... (3)

Solving (2) and (3) simultaneously:

$$\begin{aligned} -t+s &= 1 \\ t-2s &= -2 \\ \hline -s &= -1 \end{aligned} \qquad \therefore \quad s=1 \quad \text{and} \quad t=0$$

and in (1), LHS $= 2t-3s = 2(0)-3(1) = -3$ ✓

$\therefore$ $s=1$, $t=0$ satisfies all three equations

$\therefore$ the two lines meet at (1, 2, 3) $\{$using $t = 0$ or $s = 1\}$

The acute angle between the lines has $\cos\theta = \dfrac{\left|\begin{bmatrix}2\\-1\\1\end{bmatrix} \bullet \begin{bmatrix}3\\-1\\2\end{bmatrix}\right|}{\sqrt{6}\sqrt{14}}$

$= \dfrac{|\,6+1+2\,|}{\sqrt{84}}$

$= \dfrac{9}{\sqrt{84}}$ and so $\theta \doteqdot 10.9^o$

b Line 1 has direction vector $\begin{bmatrix}2\\-12\\12\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}4\\3\\-1\end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $-1+2t = 4s-3$ $\quad$ $2-12t = 3s+2$ $\quad$ $4+12t = -s-1$

$\therefore$ $2t-4s = -2$ $\quad$ $-12t-3s = 0$ $\quad$ $12t+s = -5$ ...... (3)

$\therefore$ $t-2s = -1$ ...... (1) $\quad$ $s = -4t$ ...... (2)

Solving (1) and (2) simultaneously: $t-2(-4t) = -1$

$\therefore$ $9t = -1$

$\therefore$ $t = -\frac{1}{9}$ and so $s = \frac{4}{9}$

In (3), $12t+s = 12\left(-\frac{1}{9}\right)+\frac{4}{9} = -\frac{12}{9}+\frac{4}{9} = -\frac{8}{9}$, which is not -5.

Since the system is inconsistent, the lines don't intersect.

$\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{\left|\begin{bmatrix}2\\-12\\12\end{bmatrix} \bullet \begin{bmatrix}4\\3\\-1\end{bmatrix}\right|}{\sqrt{2^2+(-12)^2+12^2}\sqrt{4^2+3^2+(-1)^2}}$

$= \dfrac{|8-36-12|}{\sqrt{292}\sqrt{26}}$

$= \dfrac{40}{\sqrt{7592}}$ and so $\theta \doteqdot 62.7^o$

c Line 1 has direction vector $\begin{bmatrix}6\\8\\2\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\4\\1\end{bmatrix}$

As $\begin{bmatrix}6\\8\\2\end{bmatrix} = 2\begin{bmatrix}3\\4\\1\end{bmatrix}$ the two lines are parallel. Hence, $\theta = 0^o$.

d Line 1 has direction vector $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\-2\\2\end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $t = 1+3s$ ...... (1) $\quad$ $2-t = -2-2s$ $\quad$ $-2+t = 2s+\frac{1}{2}$

$\therefore$ $-t+2s = -4$ ...... (2) $\quad$ $t-2s = 2\frac{1}{2}$ ...... (3)

Solving (1) and (2) simultaneously, $-(1+3s)+2s = -4$

$\therefore$ $-1-3s+2s = -4$

$\therefore$ $-s = -3$

$\therefore$ $s = 3$ and so $t = 1+3(3) = 10$

Substituting in (3), $t-2s = 10-2(3) = 4 \neq 2\frac{1}{2}$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{\left|\begin{bmatrix}1\\-1\\1\end{bmatrix} \bullet \begin{bmatrix}3\\-2\\2\end{bmatrix}\right|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \dfrac{7}{\sqrt{3}\sqrt{17}}$

$\therefore \quad \theta \doteqdot 11.4^o$

e Line 1 has direction vector $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\-2\\1\end{bmatrix}$

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{ccc} 1+t = 2+3s & 2-t = 3-2s & 3+2t = s-5 \\ t-3s = 1 \text{ ...... } (1) & -t+2s = 1 \text{ ...... } (2) & 3t-s = -8 \text{ ...... } (3) \end{array}$$

Solving (1) and (2) simultaneously, $t-3s = 1$
$-t+2s = 1$

Adding, $-s = 2$

$\therefore \quad s = -2$ and $t-3(-2) = 1 \quad \therefore \quad t = -5$

Checking in (3), $2t-s = 2(-5)-(-2) = -10+2 = -8$ ✓

Since $s=-2$, $t=-5$ satisfies all three equations, the lines meet.

They meet at $x = 1+(-5)$, $y = 2-(-5)$, $z = 3+2(-5)$ i.e., at $(-4, 7, -7)$

The acute angle between the lines has $\cos\theta = \dfrac{\left|\begin{bmatrix}1\\-1\\2\end{bmatrix} \bullet \begin{bmatrix}3\\-2\\1\end{bmatrix}\right|}{\sqrt{1+1+4}\sqrt{9+4+1}}$

$$= \frac{|3+2+2|}{\sqrt{6}\sqrt{14}}$$

$$= \frac{7}{\sqrt{84}} \quad \text{and so} \quad \theta \doteqdot 40.2^o$$

f Line 1 has direction vector $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}4\\-2\\0\end{bmatrix}$

Now $\begin{bmatrix}4\\-2\\0\end{bmatrix} = -2\begin{bmatrix}-2\\1\\0\end{bmatrix}$, so the lines are parallel and hence $\theta = 0^o$.

REVIEW SET 17A

1 **a** The vector equation is
$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix} + t\begin{bmatrix}4\\-3\end{bmatrix}$

b The parametric equations are
$x = -6+4t$, $y = 3-3t$, $t \in \mathcal{R}$

2 The vector equation is $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\8\end{bmatrix} + t\begin{bmatrix}5\\4\end{bmatrix}$, $t \in \mathcal{R}$

3 $(-3, m)$ lies on the line, so $\begin{bmatrix}-3\\m\end{bmatrix} = \begin{bmatrix}18\\-2\end{bmatrix} + \begin{bmatrix}-7t\\4t\end{bmatrix}$

$\therefore \quad -3 = 18-7t$ and $m = -2+4t$

$\therefore \quad 7t = 21$

i.e., $t = 3$ and so $m = -2+4(3) = 10$

i.e., $m = 10$

4 **a** $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4, 3)$.

b $x(4) = -4 + 32 = 28$, $y(4) = 3 + 6(4) = 27$, so at $t = 4$, the position is $(28, 27)$.

c The velocity vector is $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$, so the speed is $\sqrt{8^2 + 6^2} = 10$ m/s **d** $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$

5 The direction vector is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$\therefore$ $2\sqrt{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ has length 20. So, the velocity vector is $\begin{bmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{bmatrix}$ or $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$

6 **a** **i** The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$ i.e., $-6\mathbf{i} + 10\mathbf{j}$

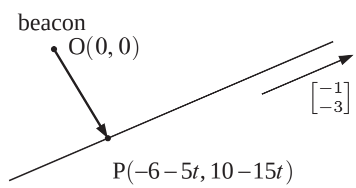
ii $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\therefore$ $5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$

$\therefore$ the direction vector is $-5\mathbf{i} - 15\mathbf{j}$

iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + t\begin{bmatrix} -5 \\ -15 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = -6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j}) = (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}$

b



$\overrightarrow{OP} = \begin{bmatrix} -6-5t \\ 10-15t \end{bmatrix}$ and $\overrightarrow{OP} \bullet \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0$

$\therefore$ $-1(-6 - 5t) - 3(10 - 15t) = 0$

$\therefore$ $6 + 5t - 30 + 45t = 0$

$\therefore$ $50t = 24$

$\therefore$ $t = 0.48$ h

(i.e., 28.8 min)

c When $t = 0.48$, $\overrightarrow{OP} = \begin{bmatrix} -6-5(0.48) \\ 10-15(0.48) \end{bmatrix} = \begin{bmatrix} -8.4 \\ 2.8 \end{bmatrix}$

and $\text{OP} = \sqrt{(-8.4)^2 + (2.8)^2} \doteqdot 8.85$ km

As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

7 **a** $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $t \geqslant 0$. When $t = 0$, the time is 2:17 pm

i.e., $x_1(t) = 2 + t$, $y_1(t) = 4 - 3t$

b Likewise, $x_2(t) = 11 - (t + 2)$, $y_2(t) = 3 + a(t + 2)$

i.e., $x_2(t) = 9 - t$ $y_2(t) = [3 + 2a] + at$

c They meet where $2 + t = 9 - t$ and $4 - 3t = [3 + 2a] + at$

$\therefore$ $2t = 7$

$\therefore$ $t = \frac{7}{2}$ $\therefore$ the time would be 2:17 pm plus $3\frac{1}{2}$ min i.e., 2:20:30 pm

d When $t = \frac{7}{2}$,

$4 - 3\left(\frac{7}{2}\right) = \left[3 + 2\left(\frac{7}{2}\right)\right] + a\left(\frac{7}{2}\right)$

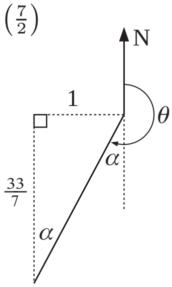
$\therefore$ $-\frac{13}{2} = 10 + \dfrac{7a}{2}$

$\therefore$ $-13 = 20 + 7a$

$\therefore$ $7a = -33$

$\therefore$ $a = -\frac{33}{7}$

Y18 has velocity vector $\begin{bmatrix} -1 \\ -\frac{33}{7} \end{bmatrix}$



with speed $= \sqrt{(-1)^2 + \left(-\frac{33}{7}\right)^2}$

$\doteqdot 4.82$ km/min

$\tan\alpha = \dfrac{1}{\frac{33}{7}} = \frac{7}{33}$

$\therefore$ $\alpha = \tan^{-1}\left(\frac{7}{33}\right) \doteqdot 12.0^o$

$\therefore$ the direction is $180^o + \alpha^o = 192^o$

i.e., the torpedo has speed 4.82 km/min and direction 192^o.

8 **a** Line 1 has direction vector $\begin{bmatrix}5\\-2\end{bmatrix}$ and Line 4 has direction vector $\begin{bmatrix}-5\\2\end{bmatrix}$.

Now $\begin{bmatrix}5\\-2\end{bmatrix} = -\begin{bmatrix}-5\\2\end{bmatrix}$, so Lines 1 and 4 are parallel, i.e., KL $\parallel$ MN.

b $\overrightarrow{KL} = \begin{bmatrix}5\\-2\end{bmatrix}$, $\overrightarrow{NK} = \begin{bmatrix}4\\10\end{bmatrix}$, $\overrightarrow{MN} = \begin{bmatrix}-5\\2\end{bmatrix}$ $\therefore$ $\overrightarrow{KL} \bullet \overrightarrow{NK} = 20 - 20 = 0$,

and $\overrightarrow{NK} \bullet \overrightarrow{MN} = -20 + 20 = 0$

$\therefore$ NK is perpendicular to both KL and MN.

c Lines 1 and 3 meet at K

$$\begin{aligned}
\therefore \quad \begin{bmatrix}2\\19\end{bmatrix} + p\begin{bmatrix}5\\-2\end{bmatrix} &= \begin{bmatrix}3\\7\end{bmatrix} + r\begin{bmatrix}4\\10\end{bmatrix}\\
\therefore \quad \begin{bmatrix}5p-4r\\-2p-10r\end{bmatrix} &= \begin{bmatrix}1\\-12\end{bmatrix}\\
\therefore \quad 5p - 4r &= 1 \qquad \times 5\\
2p + 10r &= 12 \qquad \times 2\\
\therefore \quad 25p - 20r &= 5\\
4p + 20r &= 24\\
\text{Adding,} \quad 29p \qquad &= 29\\
\therefore \quad p &= 1\\
\text{So,} \quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}2\\19\end{bmatrix} + \begin{bmatrix}5\\-2\end{bmatrix} = \begin{bmatrix}7\\17\end{bmatrix}
\end{aligned}$$

$\therefore$ K is (7, 17)

Lines 2 and 4 meet at M.

$$\begin{aligned}
\therefore \quad \begin{bmatrix}33\\-5\end{bmatrix} + r\begin{bmatrix}-11\\16\end{bmatrix} &= \begin{bmatrix}43\\-9\end{bmatrix} + s\begin{bmatrix}-5\\2\end{bmatrix}\\
\therefore \quad \begin{bmatrix}-11r+5s\\16r-2s\end{bmatrix} &= \begin{bmatrix}10\\-4\end{bmatrix}\\
\therefore \quad -11r + 5s &= 10 \qquad \times 2\\
16r - 2s &= -4 \qquad \times 5\\
\therefore \quad -22r + 10s &= 20\\
80r - 10s &= -20\\
\therefore \quad 58r \qquad &= 0\\
\therefore \quad r &= 0\\
\text{So,} \quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}33\\-5\end{bmatrix}
\end{aligned}$$

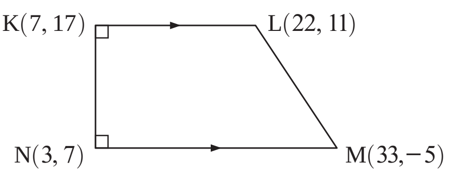
$\therefore$ M is (33, -5)

Lines 3 and 4 meet at N

$$\begin{aligned}
\therefore \quad \begin{bmatrix}3\\7\end{bmatrix} + r\begin{bmatrix}4\\10\end{bmatrix} &= \begin{bmatrix}43\\-9\end{bmatrix} + s\begin{bmatrix}-5\\2\end{bmatrix}\\
\therefore \quad \begin{bmatrix}4r+5s\\10r-2s\end{bmatrix} &= \begin{bmatrix}40\\-16\end{bmatrix}\\
\therefore \quad 4r + 5s &= 40 \qquad \times 2\\
10r - 2s &= -16 \qquad \times 5\\
\therefore \quad 8r + 10s &= 80\\
50r - 10s &= -80\\
\therefore \quad 58r \qquad &= 0\\
\therefore \quad r &= 0\\
\therefore \quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}3\\7\end{bmatrix}
\end{aligned}$$

i.e., N is (3, 7)

d



$$\begin{aligned}
\text{KL} &= \sqrt{(22-7)^2 + (11-17)^2}\\
&= \sqrt{225 + 36}\\
&= \sqrt{261} \text{ units}\\
\text{NK} &= \sqrt{(33-3)^2 + (-5-7)^2}\\
&= \sqrt{900 + 144}\\
&= \sqrt{1044} \text{ units}\\
\text{KN} &= \sqrt{(7-3)^2 + (17-7)^2}\\
&= \sqrt{16 + 100}\\
&= \sqrt{116}\\
\therefore \quad \text{area} &= \left(\frac{\sqrt{261} + \sqrt{1044}}{2}\right) \times \sqrt{116}\\
&= 261 \text{ units}^2
\end{aligned}$$

REVIEW SET 17B

1 **a** $\overrightarrow{AB} = \begin{bmatrix}-1-3\\2-2\\4--1\end{bmatrix} = \begin{bmatrix}-4\\0\\5\end{bmatrix}$ $\therefore$ the equation is $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\2\\-1\end{bmatrix} + t\begin{bmatrix}-4\\0\\5\end{bmatrix}$, $t \in \mathcal{R}$

b Any point on AB can be represented by a point $C(3-4t,\ 2,\ -1+5t)$.

If $AC = 2\sqrt{41}$, then

$$\begin{aligned}\sqrt{(3-4t-3)^2+(2-2)^2+(-1+5t-(-1)^2} &= 2\sqrt{41}\\ \therefore\ \sqrt{16t^2+0+25t^2} &= 2\sqrt{41}\\ \therefore\ 41t^2 &= 4\times 41\\ \therefore\ t^2 &= 4\\ \therefore\ t &= \pm 2\end{aligned}$$

So, C is $(-5,\ 2,\ 9)$ or $(11,\ 2,\ -11)$

2 **a** $\overrightarrow{AB} = \begin{bmatrix}0-3\\2--1\\-1-1\end{bmatrix} = \begin{bmatrix}-3\\3\\-2\end{bmatrix}$ $\quad\therefore\ \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix} + t\begin{bmatrix}-3\\3\\-2\end{bmatrix},\quad t\in\mathcal{R}$

b If P divides BA in the ratio $2:5$, then $\overrightarrow{BP}:\overrightarrow{PA} = 2:5$

If P is $(a,\ b,\ c)$ then $\begin{bmatrix}a-0\\b-2\\c--1\end{bmatrix} : \begin{bmatrix}3-a\\-1-b\\1-c\end{bmatrix} = 2:5$

$$5\begin{bmatrix}a\\b-2\\c+1\end{bmatrix} = 2\begin{bmatrix}3-a\\-1-b\\1-c\end{bmatrix}$$

$\therefore\ 5a = 6-2a,\quad 5b-10 = -2-2b,\quad 5c+5 = 2-2c$

$\therefore\ 7a = 6,\quad 7b = 8,\quad 7c = -3$

$\therefore\ a = \frac{6}{7},\quad b = \frac{8}{7},\quad c = -\frac{3}{7}$

So, P is $\left(\frac{6}{7},\ \frac{8}{7},\ \frac{-3}{7}\right)$

3 **a** $\overrightarrow{PQ} = \begin{bmatrix}4--1\\0-2\\-1-3\end{bmatrix} = \begin{bmatrix}5\\-2\\-4\end{bmatrix}$

b The x-axis has direction vector $\mathbf{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

and if the angle is θ,

$$\begin{aligned}\begin{bmatrix}5\\-2\\-4\end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix} &= \sqrt{25+4+16}\sqrt{1+0+0}\cos\theta\\ \therefore\ 5 &= \sqrt{45}\cos\theta\\ \therefore\ \cos\theta &= \tfrac{5}{\sqrt{45}}\\ \therefore\ \theta &\doteqdot 41.8^o\end{aligned}$$

4 **a** Let $\dfrac{x-8}{3} = \dfrac{y+9}{-16} = \dfrac{z-10}{7} = t$

$\therefore\ x = 8+3s,\quad y = -9-16s,\quad z = 10+7s$

Line 1 has direction vector $\begin{bmatrix}3\\-16\\7\end{bmatrix}$, and Line 2 has direction vector $\begin{bmatrix}3\\8\\-5\end{bmatrix}$

Since one vector is not a scalar multiple of the other, the vector lines are not parallel.

Now $8+3s = 15+3t,\qquad -9-16s = 29+8t,\qquad 10+7s = 5-5t$

$\therefore\ 3s-3t = 7\ ......\ (1)\qquad 16s+8t = -38\ ......\ (2)\qquad 7s+5t = -5\ ......\ (3)$

Solving (1) and (3),
$$\begin{aligned} 3s - 3t &= 7 \quad \times 5 \\ 7s + 5t &= -5 \quad \times 3 \\ \therefore\ 15s - 15t &= 35 \\ 21s + 15t &= -15 \\ \text{Adding,}\quad 36s &= 20 \\ \therefore\ s &= \tfrac{5}{9} \end{aligned} \qquad \text{and} \qquad \begin{aligned} 3\left(\tfrac{5}{9}\right) - 3t &= 7 \\ \therefore\ \tfrac{5}{3} - 3t &= 7 \\ \therefore\ 5 - 9t &= 21 \\ \therefore\ 9t &= -16 \\ \therefore\ t &= -\tfrac{16}{9} \end{aligned}$$

and in (2), $16s + 8t = 16\left(\frac{5}{9}\right) + 8\left(-\frac{16}{9}\right) = -5\frac{1}{3} \neq -38$

$\therefore$ the equations do not have a common solution and so the two lines are skew.

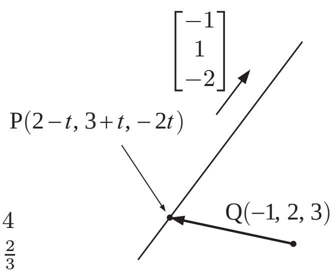
b If θ is the angle, $\begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix} = \sqrt{9+256+49}\sqrt{9+64+25}\cos\theta$

$$\therefore\ 9 - 128 - 35 = \sqrt{314}\sqrt{98}\cos\theta$$

$$\therefore\ \frac{-154}{\sqrt{314 \times 98}} = \cos\theta \quad \therefore\ \theta \doteqdot 28.6^o$$

5 If $2 - x = y - 3 = -\frac{1}{2}z = t$, then $x = 2 - t$, $y = 3 + t$, $z = -2t$. This has direction $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

$\overrightarrow{QP} = \begin{bmatrix} 2 - t - -1 \\ 3 + t - 2 \\ -2t - 3 \end{bmatrix} = \begin{bmatrix} 3 - t \\ 1 + t \\ -2t - 3 \end{bmatrix}$



In order that $\overrightarrow{QP} \perp \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$,

$\begin{bmatrix} 3 - t \\ 1 + t \\ -2t - 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = 0$
$$\begin{aligned} \therefore\ -3 + t + 1 + t + 4t + 6 &= 0 \\ \therefore\ 6t &= -4 \\ \therefore\ t &= -\tfrac{2}{3} \end{aligned}$$

So, P is $\left(2 - -\frac{2}{3}, 3 - \frac{2}{3}, -2(-\frac{2}{3})\right)$ i.e., $\left(\frac{8}{3}, \frac{7}{3}, \frac{4}{3}\right)$

6 **a** $\overrightarrow{PQ} = \begin{bmatrix} 3 - 2 \\ 4 - 0 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

and $|\overrightarrow{PQ}| = \sqrt{1 + 16 + 9} = \sqrt{26}$ units

b $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$\therefore\ x = 2 + t,\quad y = 4t,\quad z = 1 - 3t,\quad t \in \mathcal{R}$

7 **a** Let $\dfrac{x-3}{2} = y - 4 = \dfrac{z+1}{-2} = s \quad \therefore\ x = 3 + 2s,\quad y = 4 + s,\quad z = -1 - 2s,\quad s \in \mathcal{R}$

Line 1 has direction vector $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Since one of these vectors is not a scalar multiple of the other, the lines are not parallel.

Now $3 + 2s = -1 + 3t$, $\quad 4 + s = 2 + 2t$, $\quad -1 - 2s = 3 - t$

$\therefore\ 2s - 3t = -4$ ...... (1) $\quad s - 2t = -2$ ...... (2) $\quad -2s + t = 4$ ...... (3)

Solving (1) and (3),
$$\begin{aligned} 2s - 3t &= -4 \\ -2s + t &= 4 \\ \therefore\ -2t &= 0 \\ \therefore\ t &= 0 \quad \text{and} \quad s = -2 \end{aligned}$$

So, in (2), $s - 2t$
$= -2 - 2(0)$
$= -2$ ✓

$\therefore\ s = -2,\quad t = 0$ satisfies all three equations. $\therefore$ the lines intersect at $(-1, 2, 3)$.

b If θ is the angle between the lines, then
$$\begin{bmatrix}2\\1\\-2\end{bmatrix} \bullet \begin{bmatrix}3\\2\\-1\end{bmatrix} = \sqrt{4+1+4}\sqrt{9+4+1}\cos\theta$$
$$\therefore \quad 6+2+2 = 3\sqrt{14}\cos\theta$$
$$\therefore \quad \cos\theta = \tfrac{10}{3\sqrt{14}}$$

8 If A is the origin, AB the x-axis, AD the y-axis, and AP the z-axis, then Q is (4, 0, 7), D is (0, 10, 0) and M is (0, 5, 7).

Now $\overrightarrow{\mathrm{DQ}} = \begin{bmatrix}4-0\\0-10\\7-0\end{bmatrix} = \begin{bmatrix}4\\-10\\7\end{bmatrix}$ and $\overrightarrow{\mathrm{DM}} = \begin{bmatrix}0-0\\5-10\\7-0\end{bmatrix} = \begin{bmatrix}0\\-5\\7\end{bmatrix}$

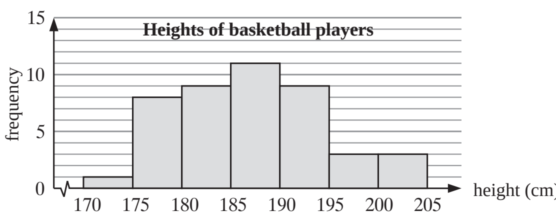
So, if the required angle is θ,
$$\begin{bmatrix}4\\-10\\7\end{bmatrix} \bullet \begin{bmatrix}0\\-5\\7\end{bmatrix} = \sqrt{16+100+49}\sqrt{0+25+49}\cos\theta$$
$$\therefore \quad 0+50+49 = \sqrt{165}\sqrt{74}\cos\theta$$
$$\therefore \quad \frac{99}{\sqrt{165\times 74}} = \cos\theta \qquad \text{and so} \quad \theta \doteqdot 26.4^o$$

Chapter 18
DESCRIPTIVE STATISTICS

EXERCISE 18A

1 **a** Heights can take any value from 170 cm to 205 cm, including decimal values such as 181.372 cm, i.e., any real number between 170 and 205.

b



c The modal class is the class occurring most often. This is '185 -'.

d The distribution is slightly positively skewed (more of a 'tail' to the right (positive)).

2 **a** The data is continuous numerical. Actual time is continuous and could be measured to the nearest millisecond. After it has been rounded to the nearest minute it becomes discrete numerical data.

b

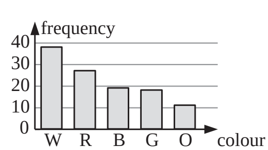
Stem	*Leaf*
0	3 6 8 8 8 8
1	0 0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

1 | 2 means 12 minutes

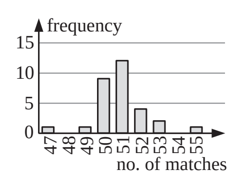
c The distribution is positively skewed (i.e., skewed to the high end).

d "The modal travelling time was between 10 and 20 minutes" if considering classes.
The mode is actually 10.

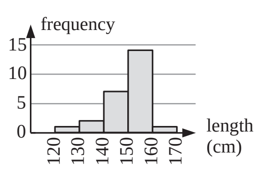
3 **a** The data is categorical, so a column graph should be used.



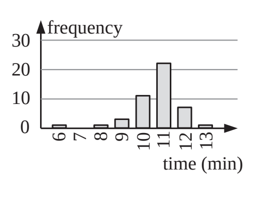
b The data is discrete numerical, so a column graph should be used.



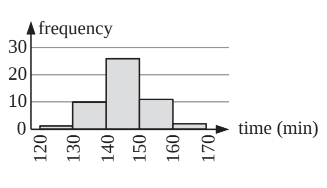
c The data is continuous, so a histogram should be used.



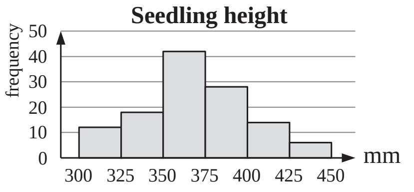
d The data is discrete numerical, so a column graph should be used.



e The data is continuous, so a histogram should be used.



4 **a**



b Number which are $\geqslant 400$ mm is $14 + 6 = 20$ seedlings.

c $\%$ between 349 and 400 $= \dfrac{42+28}{120} \times 100\%$

$= \dfrac{70}{120} \times 100\%$

$\doteqdot 58.3\%$

d **i** Number

$= \dfrac{12+18+42+28}{120} \times 1462$

$= \dfrac{100}{120} \times 1462$

$\doteqdot 1218$ seedlings

ii Number

$= \dfrac{28+14}{120} \times 1462$

$= \dfrac{42}{120} \times 1462$

$\doteqdot 512$ seedlings

5 **a** Number in survey $= 15 + 20 + 30 + 40 + 25 + 10 = 140$

b Number less than 180 cm $= 15 + 20 + 30 = 65$

c $\%$ which were $\geqslant 175$ cm $= \dfrac{40+25+10}{140} \times 100\%$

$= \dfrac{75}{140} \times 100\%$

$= 53.6\%$

EXERCISE 18B.1

1 **a** **i** mean $= \frac{2+3+3+3+4+....+9+9}{23}$

$= \frac{129}{23}$

$\doteqdot 5.61$

ii median $=$ 12th score (when in order)

$= 6$

iii mode $= 6$ (6 occurs most often)

b **i** mean $= \frac{10+12+12+15+....+20+21}{15}$

$= \frac{245}{15}$

$\doteqdot 16.3$

i median $=$ 8th score (when in order)

$= 17$

iii mode $= 18$

c **i** mean $= \frac{22.4+24.6+21.8+....+23.5}{11}$

$= \frac{273}{11}$

$\doteqdot 24.8$

ii median $=$ 6th score (when in order)

$= 24.9$

iii mode $= 23.5$

d **i** mean $= \frac{127+123+115+105+....+141}{21}$

$= \frac{2700}{21}$

$\doteqdot 128.6$

ii median $=$ 11th score (when in order)

$= 128$

iii mode $= 115, 127$ (bimodal)

2 **a** mean of set A

$= \dfrac{3+4+4+5+....+10}{13}$

$= 6.46$

mean of set B

$= \dfrac{3+4+4+5+....+15}{13}$

$= 6.85$

b median of set A $=$ 7th score $= 7$ median of set B $=$ 7th score $= 7$

c The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.

d The middle value of both data sets is the same, so the median is the same.

3 **a** mean $= \dfrac{23\,000 + 46\,000 + 23\,000 + ... + 32\,000}{10} = \$29\,300$

median $=$ middle score when in order of size $= \dfrac{\$23\,000 + \$24\,000}{2} = \$23\,500$

mode $= \$23\,000$

b The mode is unsatisfactory because it is the lowest salary. It does not take the higher values into account.

c The median is too close to the lower end of the distribution and it does not take the higher values into account. So the median is not a satisfactory measure of the middle.

4 **a** mean $= \dfrac{3 + 1 + 0 + 0 + ... + 1 + 0 + 0}{31} = \dfrac{99}{31} \doteqdot 3.19$

median $=$ 16th score (when in order) $= 0$

mode $= 0$ (most frequently occurring score)

b The median is not in the centre. It ignores the high upper values of the distribution.

c The mode is the lowest value. It does not take the higher values into account.

d Yes, 42 and 21.

e No, as this would ignore actual valid data.

5 **a** mean $= \dfrac{43 + 55 + 41 + 37}{4} = \dfrac{176}{4} = 44$

b another 44

c new mean $= \dfrac{43 + 55 + 41 + 37 + 25}{5} = 40.2$

d It will increase the new mean to 40.3 as 41 is greater than the old mean of 40.2.

$\left\{\dfrac{5 \times 40.2 + 41}{6} \doteqdot 40.3\right\}$

6 mean $= \dfrac{\text{total}}{10}$, $\therefore$ $11.6 = \dfrac{\text{total}}{10}$, $\therefore$ total $= 11.6 \times 10 = 116$

7 mean $= \dfrac{\text{total}}{12}$, $\therefore$ $262 = \dfrac{\text{total}}{12}$, $\therefore$ total $= 262 \times 12 = 3144$ km

8 mean $= \dfrac{\text{total}}{12}$, $\therefore$ $15\,467 = \dfrac{\text{total}}{12}$, $\therefore$ total $= 15\,467 \times 12 = \$185\,604$

9
$$\begin{aligned} \frac{5 + 9 + 11 + 12 + 13 + 14 + 17 + x}{8} &= 12 \\ \therefore \frac{81 + x}{8} &= 12 \\ \therefore 81 + x &= 96 \\ \therefore x &= 15 \end{aligned}$$

10
$$\begin{aligned} \frac{3 + 0 + a + a + 4 + a + 6 + a + 3}{9} &= 4 \\ \therefore \frac{4a + 16}{9} &= 4 \\ \therefore 4a + 16 &= 36 \\ \therefore 4a &= 20 \\ \therefore a &= 5 \end{aligned}$$

11
$$\frac{29 + 36 + 32 + 38 + 35 + 34 + 39 + x}{8} = 35 \qquad \therefore \frac{243 + x}{8} = 35$$
$$\therefore 243 + x = 280$$
$$\therefore x = 37, \quad \text{so her 8th result was 37}$$

12 **a** Total distance driven (first 5 days) $= 5 \times 256 = 1280$ km

b Total distance travelled (next 3 days) $= 172 \times 3 = 516$ km

c mean distance $= \dfrac{1280 + 516}{8}$ km $= 224.5$ km

13 Total for first 10 measurements $= 10 \times 15.7 = 157$

Total for next 20 measurements $= 20 \times 14.3 = 286$ $\quad \therefore$ mean $= \dfrac{157+286}{30} \doteqdot 14.8$

14 Total for first 10 innings $= 25.4 \times 10 = 254$ runs

$\therefore$ new average $= \dfrac{254+58+16}{12} \doteqdot 27.3$ runs

15 Scores were $5\ 7\ 9\ 9\ 10\ a\ b$ where $a \leqslant b$ say.

$$\text{mean} = \frac{5+7+9+9+10+a+b}{7} = 8$$
$$\therefore \frac{40+a+b}{7} = 8$$
$$\therefore 40+a+b = 56$$
$$\therefore a+b = 16 \quad \{a \leqslant 12, \quad b \leqslant 12\}$$

Possibilities are:

a	5	6	7	8
b	11	10	9	8
	×	×	✓	×

reject as modes are 8 and 9

reject as modes are 9 and 10

reject as modes are 5 and 9

So, the missing results are 7 and 9.

EXERCISE 18B.2

1 **a** The mode is 1 (occurs most often).

b The median is the average of the 15th and 16th scores $= \dfrac{1+1}{2} = 1$

c

x	f	fx
0	4	0
1	12	12
2	11	22
3	3	9
$\sum$	30	43

$$\text{mean} = \frac{\sum f x}{\sum f} = \frac{43}{30} \doteqdot 1.43$$

2 **a** **i**

x	f	fx
0	5	0
1	8	8
2	13	26
3	8	24
4	6	24
5	3	15
6	3	18
7	2	14
8	1	8
9	0	0
10	0	0
11	1	11
$\sum$	50	148

$$\text{mean} = \frac{\sum f x}{\sum f} = \frac{148}{50} = 2.96$$

ii median

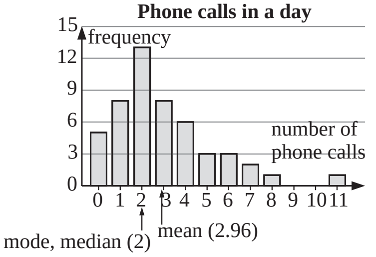
= average of 25th and 26th (when in order)

$= \dfrac{2+2}{2}$ $\left\{\begin{array}{l}\text{13 scores are 1 or 0} \\ \text{26 scores are 2, 1 or 0}\end{array}\right\}$

$= 2$

iii mode $= 2$ {occurs most often}

b



c The distribution is positively skewed. 11 is an outlier.

d The mean takes into account the larger numbers of phone calls.

e The mean as it best represents all the data.

3 **a**

Donation ($)	*Frequency*
1	7
2	9
3	2
4	4
5	8

b Total number of donations $= \sum f$

$= 7+9+2+4+8$

$= 30$

c **i**

x	f	$f\,x$
1	7	7
2	9	18
3	2	6
4	4	16
5	8	40
$\sum$	30	87

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{87}{30} = 2.9$$

$\therefore$ mean donation is \$2.90

ii median donation $=$ average of 15th and 16th (when in order)

$$= \frac{\$2 + \$2}{2} \quad \left\{\begin{array}{l}7 \text{ are } \$1 \\ 16 \text{ are } \$1 \text{ or } \$2\end{array}\right\}$$

$= \$2$

iii mode donation $= \$2$ {occurs most often}

b The mode. (highest column)

4 **a** **i** mode $= 49$ {occurs most often}

ii median $=$ average of 15th and 16th values (when in order)

$$= \frac{49+49}{2} \quad \{9 \text{ are } 47 \text{ or } 48 \text{ and the next } 11 \text{ are } 49\}$$

$= 49$

iii

x	f	$f\,x$
47	5	235
48	4	192
49	11	539
50	6	300
51	3	153
52	1	52
$\sum$	30	1471

$$\text{mean} = \frac{\sum f\,x}{\sum f}$$

$$= \frac{1471}{30}$$

$\doteqdot 49.0$

b No, as they claim the average (mean) is 50 matches/box.

c The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

5 **a** **i**

x	f	$f\,x$
1	5	5
2	28	56
3	15	45
4	8	32
5	2	10
6	1	6
$\sum$	59	154

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{154}{59} \doteqdot 2.61$$

ii mode $= 2$ {occurs most often}

iii median $=$ 30th score (when in order) $= 2$

b This school has more children per family (2.61) than the average Australian family (2.2).

c Positive as the higher values are more spread out.

d The mean is higher than the mode and median.

6 **a** **i** mean $= \dfrac{53 + 55 + 56 + 60 + ... + 91}{17} = \dfrac{1175}{17} \doteqdot 69.1$

ii median $=$ 9th score (when in order) $= 67$

iii mode $= 73$ (73 is the only one occurring more than once)

b **i** mean $= \dfrac{37 + 40 + 44 + 48 + ... + 81}{23} = \dfrac{1347}{23} \doteqdot 5.86$

ii median $=$ 12th score (when in order) $= 5.8$ **iii** mode $= 6.7$ {occurs most often}

7 **a** **Without fertiliser**

2	||
3	卌 卌 |
4	卌 卌 卌 ||||
5	卌 卌 卌 卌 卌 ||||
6	卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 |
7	卌 卌 卌 卌 卌
8	卌 卌 ||
9	|

x	f	$f\,x$	cf
2	2	4	2
3	11	33	13
4	19	76	32
5	29	145	61
6	51	306	112
7	25	175	137
8	12	96	149
9	1	9	150

i mean $= \dfrac{\sum f\,x}{\sum f} = \dfrac{845}{150} \doteqdot 5.63$ **ii** mode $= 6$ {occurs most often}

iii median $=$ average of 75th and 76th scores $= \dfrac{6+6}{2} = 6$

b **With fertiliser**

3	||||
4	卌 卌 |||
5	卌 卌 |
6	卌 卌 卌 卌 卌 |||
7	卌 卌 卌 卌 卌 卌 卌 卌 卌 |||
8	卌 卌 卌 卌 卌 ||
9	卌 卌 ||||
10	||||
13	|

x	f	$f\,x$	cf
3	4	12	4
4	13	52	17
5	11	55	28
6	28	168	56
7	48	336	104
8	27	216	131
9	14	126	145
10	4	40	149
13	1	13	150

i mean $= \dfrac{\sum f\,x}{\sum f} = \dfrac{1018}{150} \doteqdot 6.79$ **ii** mode $= 7$ {occurs most often}

iii median $=$ average of 75th and 76th scores $= \dfrac{7+7}{2} = 7$

c The mean best represents the centre for this data.

d Yes, as 6.79 is significantly greater than 5.63.

Note: The total yield of the crop may not have improved as, for example, the number of pods per plant may have decreased when using the fertiliser.

8 Mean for Team A $= \dfrac{91+76+104+....+82}{12} = 91.25$

Mean for Team B $= \dfrac{87+104+112+....+108}{12} = 91.75$

So, B has the higher average (but only just).

9 The mode(s) occurs most often.

a 49 **b** 144 and 147 (bimodal) **c** 25

10 **a** median $=$ 5th score (when in order) $= 29$

b median $= \dfrac{\text{5th score} + \text{6th score}}{2} = \dfrac{107+107}{2} = 107$

c median $= \dfrac{\text{6th score} + \text{7th score}}{2} = \dfrac{146+153}{2} = 149.5$

11

Siblings	f	fx
0	13	0
1	12	12
2	11	22
3	8	24
4	3	12
5	3	15
$\sum$	50	85

a modal number $= 0$

b mean number $= \dfrac{\sum fx}{\sum f} = \dfrac{85}{50} = 1.7$

c median number $= \dfrac{\text{25th} + \text{26th}}{2} = \dfrac{1+2}{2} = 1.5$

12 mean average rainfall $= \dfrac{16+34+38+....+52+21}{12} = \dfrac{974}{12} \doteqdot 81.2$ mm

13 **a** mean selling price $= \dfrac{146\,400+127\,600+211\,000+....+162\,500}{10} = \$163\,770$

median selling price $= \dfrac{\text{5th} + \text{6th}}{2} = \dfrac{146\,400+148\,000}{2} = \$147\,200$

These figures differ by \$16 570. There are more selling prices at the lower end of the market (i.e., smaller prices).

b **i** Use the mean as it tends to inflate the average house value of that district.

ii Use the median as you want to buy at the lowest price possible.

14

x	f	fx
32	6	192
33	8	264
34	9	306
35	13	455
36	10	360
37	3	111
38	2	76
$\sum$	51	1764

mean $= \dfrac{\sum fx}{\sum f} = \dfrac{1764}{51} \doteqdot 34.6$

mode $= 35$ {occurs most often}

median $=$ 26th score (when in order)
$= 35$ {23 are 34 or less
36 are 35 or less}

15 mean of all 27 is $\dfrac{12 \times 16.5 + 15 \times 18.6}{27} \doteqdot 17.7$

16 **a** mean birth weight $= \dfrac{75+70+80+ .... +83}{8} = \dfrac{567}{8} \doteqdot 70.9$ grams

b mean after 2 weeks $= \dfrac{210+200+200+ .... +230}{8} = \dfrac{1681}{8} \doteqdot 210$ grams

c mean increase $\doteqdot (210.13 - 70.88)$ grams $\doteqdot 139$ grams

17 15 of these   10.1, 10.4, 10.7, 10.9,   12 of these

$$\begin{aligned}\text{Median} &= \text{16th score (when in order)}\\ &= 10.1 \text{ cm}\end{aligned}$$

18 Total for first 14 matches $= 14 \times 16.5$ goals $= 231$ goals

$$\therefore \quad \text{new average} = \frac{231+21+24}{16} = \frac{276}{16} = 17.25 \text{ goals/game}$$

19 The measurements are $7, 9, 11, 13, 14, 17, 19, a, b$ where $a \leqslant b$

$$\text{mean} = \frac{7+9+11+13+ .... +a+b}{9} = \frac{90+a+b}{9}$$

$$\therefore \quad \frac{90+a+b}{9} = 12$$

$$\therefore \quad 90+a+b = 108$$

$$\therefore \quad a+b = 18$$

7 9 11 13 14 17 19

If $b \geqslant 13$, then $a \leqslant 5$ and the median $= 13$ $(\times)$

If $b = 12$, then $a = 6$ and the median $= 12$ $(\checkmark)$

The remaining cases are

a	7	8	9
b	11	10	9
median	11	11	11

So the other two data values are 6 and 12.

20 **a**

Brand A

x	f	$f\,x$
46	1	46
47	1	47
48	2	96
49	7	343
50	10	500
51	20	1020
52	15	780
53	3	159
55	1	55
$\sum$	60	3046

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{3046}{60} \doteqdot 50.8$$

Brand B

x	f	$f\,x$
48	3	144
49	17	833
50	30	1500
51	7	357
52	2	104
53	1	53
54	1	54
$\sum$	61	3045

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{3045}{61} \doteqdot 49.9$$

b Based on average contents the CPS should not prosecute either manufacturer. To the nearest match, the average contents of A is 51 and B, 50.

21 **Brand X**

x	f	$f\,x$
370	2	740
380	5	1900
390	32	12 480
400	35	14 000
410	20	8200
420	5	2100
430	1	430
$\sum$	100	39 850

mean weight

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{39\,850}{100}$$

$$\doteqdot 399 \text{ g}$$

Brand Y

x	f	$f\,x$
380	3	1140
390	21	8190
400	43	17 200
410	24	9840
420	7	2940
430	2	860
$\sum$	100	40 170

mean weight

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{40\,170}{100}$$

$$\doteqdot 402 \text{ g}$$

So, the magazine would recommend Brand Y.

22 **a** **i** median salary

$$= \frac{10\text{th} + 11\text{th}}{2} \quad \{\text{when in order}\}$$

$$= \frac{35\,000 + 28\,000}{2}$$

$$= \$31\,500$$

ii modal salary

$= \$28\,000$ {occurs most often}

iii

x	f	$f\,x$
50 000	1	50 000
42 000	3	126 000
35 000	6	210 000
28 000	10	280 000
$\sum$	20	666 000

mean

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{666\,000}{20}$$

$$= \$33\,300$$

b The mean as it is the highest value.

EXERCISE 18B.3

1 **a**

x (midpoint)	f	$f\,x$
3	7	21
8	12	96
13	15	195
18	10	180
23	11	253
$\sum$	55	745

$$\bar{x} = \frac{745}{55} \doteqdot 13.5$$

b

x (midpoint)	f	$f\,x$
41	2	82
44	1	44
47	5	235
50	6	300
53	12	636
56	3	168
$\sum$	29	1465

$$\bar{x} = \frac{1465}{29} \doteqdot 50.5$$

2

x (midpoint)	f	$f\,x$
4.5	2	9
14.5	5	72.5
24.5	7	171.5
34.5	27	931.5
44.5	9	400.5
$\sum$	50	1585

$$\therefore \text{ mean result} = \frac{1585}{50}$$

$$= 31.7$$

3 **a** $\bar{x} = \dfrac{15 + 8 + 6 + 10 + ... + 14 + 26}{50} = \dfrac{681}{50} \doteqdot 13.6$ goals

b **i**

Group	*Tally*	*midpoint* (x)	*frequency* (f)	$f\,x$
0 - 4	卌	2	5	10
5 - 9	卌 ||||	7	9	63
10 - 14	卌 卌 ||||	12	14	168
15 - 19	卌 卌 |||	17	13	221
20 - 24	卌 |	22	6	132
25 - 29	|||	27	3	81
		$\sum$	50	675

Approx. mean

$= \dfrac{675}{50}$

$= 13.5$ goals

ii

Group	*Tally*	*midpt.* (x)	*freq.* (f)	$f\,x$
0 - 8	卌 卌 |	4	11	44
9 - 16	卌 卌 卌 卌 |||	12.5	23	287.5
17 - 24	卌 卌 |||	20.5	13	266.5
25 - 30	|||	27.5	3	82.5
		$\sum$	50	680.5

Approx. mean

$= \dfrac{680.5}{50}$

$\doteqdot 13.6$ goals

c The approximations are very good as estimations of the actual mean, 13.6 goals.

4

midpoint (x)	f	$f\,x$
412	2	824
437	7	3059
462	15	6930
487	31	15 097
512	27	13 824
537	12	6444
562	4	2248
587	1	587
$\sum$	99	49 013

Approximate mean

$= \dfrac{\sum f\,x}{\sum f}$

$= \dfrac{49\,013}{99}$

$\doteqdot 495$ mm

5 **a** 70 **b** $\doteqdot 411\,000$ litres **c**

midpoint (x)	f	$f\,x$
2499.5	4	9998
3499.5	4	13 998
4499.5	9	40 495.5
5499.5	14	76 993
6499.5	23	149 488.5
7499.5	16	119 992
$\sum$	70	410 965

Approximate mean

$= \dfrac{\sum f\,x}{\sum f}$

$= \dfrac{410\,965}{70}$

$\doteqdot 5870$ litres

6 **a** $5 + 10 + 25 + 40 + 10 + 15 + 10 + 10 = 125$ people

b

midpoint (x)	*frequency* (f)	$f\,x$
85	5	425
95	10	950
105	25	2625
115	40	4600
125	10	1250
135	15	2025
145	10	1450
155	10	1550
$\sum$	125	14 875

Approximate mean

$= \dfrac{\sum f\,x}{\sum f}$

$= \dfrac{14\,875}{125}$

$\doteqdot 119$ people

c $\frac{15}{125}$ scored < 100

i.e., $\frac{3}{25}$ scored < 100

d 20% of 125 people = 25 people and 90 people scored < 130 for the test

$\therefore$ estimate is $130 + \frac{10}{15} \times 10 \doteqdot 137$ marks

7 **a** $10 + 15 + 20 + 30 + 15 + 5 = 95$ students

b

midpoint (x)	f	$f\,x$
47.5	10	475
52.5	15	787.5
57.5	20	1150
62.5	30	1875
67.5	15	1012.5
72.5	5	362.5
$\sum$	95	5662.5

Approx. mean

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{5662.5}{95}$$

$\doteqdot 59.6$ kg

c $10 + 15 = 25$ students

d $\dfrac{15+20}{95} \times 100\% \doteqdot 36.8\%$

e $\dfrac{10+15+20}{95} = \dfrac{45}{95}$

$= \frac{9}{19}$

($\doteqdot 47.4\%$)

EXERCISE 18C

1 **a** The total frequency $= 25 + 11 + 8 + 5 + 4 + 1 = 54$

$\therefore$ the median is the average of the 27th and 28th scores

$\therefore$ median $= \dfrac{2+2}{2} = 2$

b The total frequency $= 1 + 3 + 11 + 12 + 8 + 2 = 37$

$\therefore$ the median is the 19th score $\left(\dfrac{37+1}{2} = 19\right)$

$\therefore$ median $= 8$ {15 scores are 5, 6 or 7}

2 Total frequency $= 67 + 35 + 17 + 8 + 11 + 2 + 1 = 141$

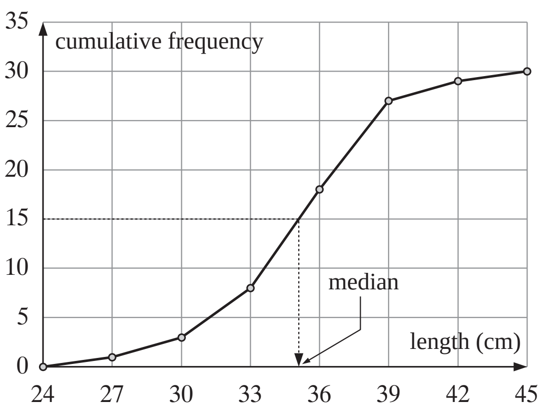
$\therefore$ the median is the 71st score $\left(\dfrac{141+1}{2} = 71\right)$

$\therefore$ median = 1 error {67 scores are 0s}

3 **a**

Length (x cm)	Freq.	C. freq.
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

b



c median $\doteqdot 35$

d There are 30 data values. So, the median is the average of the 15th and 16th scores (when in order).

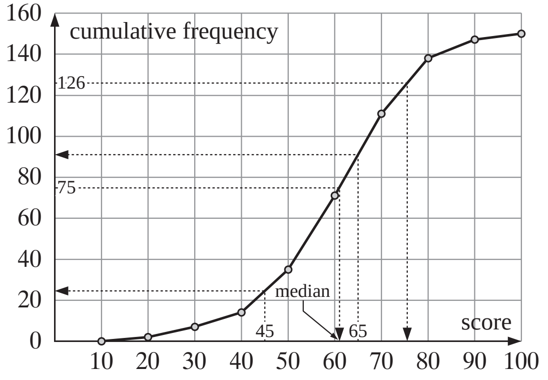
In order they are:

$24\ 27\ 28\ 30\ 31\ 31\ 32\ 32\ 33\ 33\ 33\ 33\ 34\ 34\ \underbrace{34\ 35}\ 35\ 35\ 36\ \ldots$ etc.

median $= \dfrac{34+35}{2} = 34.5$

So, the median from the graph is a good approximation.

4



a Total frequency $= 150$

$\therefore$ median $\doteqdot 61$ {from the graph}

b When the score $= 65$,

CF $\doteqdot 91$ {from the graph}

i.e., about 91 students scored 65 or less.

c $\doteqdot 36 + 40 = 76$ students scored between 50 and 70

d For a pass mark of 45, CF $\doteqdot 24.5$

$\therefore$ 24 or 25 students failed the exam.

e 84% of $150 = 126$

So, for a CF of 126, the score value is 76.

So, the minimum credit mark would be 76.

5 **a** Total frequency $= 1 + 1 + 0 + 3 + ... + 0 + 1 = 50$

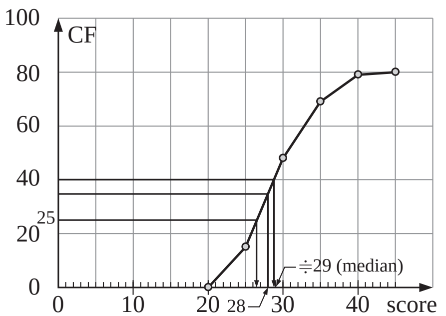
$\therefore$ the median is the average of the 25th and 26th scores

$$\left.\begin{array}{l}\text{23 are } 7\tfrac{1}{2} \text{ or less}\\ \text{40 are 8 or less}\end{array}\right\} \quad \therefore \quad \text{median} = \frac{8+8}{2} = 8$$

b **i** $13 + 17 + 7 + 2 + 0 + 1 = 40$ people

ii $1 + 1 + 0 + 3 + 5 + 13 + 17 = 40$ are 8 or less

6



a median $\doteqdot 29$ min {from the graph}

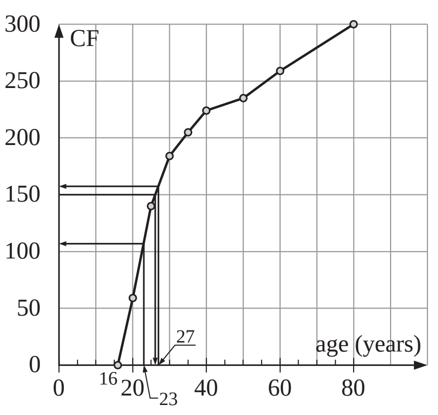
b when the score is 28, CF $\doteqdot 35$

i.e., 35 runners were faster than 28 min

c when CF $= 25$ {25 faster}

the score is $\doteqdot 26.5$ min

7



Total frequency $= 300$

a median $\doteqdot 26$ years {from the graph}

b when the age $= 23$, CF $\doteqdot 108$

and $\dfrac{108}{300} \times 100\% \doteqdot 36\%$

c **i** when age is 27, CF $\doteqdot 158$

$\therefore$ $P(\leqslant 27) = \dfrac{158}{300} = 0.527$

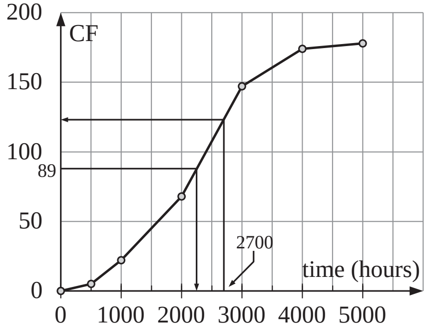
ii when age is 27, CF $\doteqdot 158$

when age is 28, CF $\doteqdot 167$

i.e., 9 died

$\therefore$ P(aged 27) $\doteqdot \dfrac{9}{300} \doteqdot 0.030$

8



Total frequency $= 178$

a median $\doteqdot 2300$ hours

b For a life of 2700 hours
CF $\doteqdot 123$

and $\dfrac{123}{178} \times 100\% \doteqdot 69\%$

So, about 69% have a life $\leqslant 2700$ h

c For a life of 1500 h, CF $\doteqdot 45$

For a life of 2500 h, CF $\doteqdot 107.5$

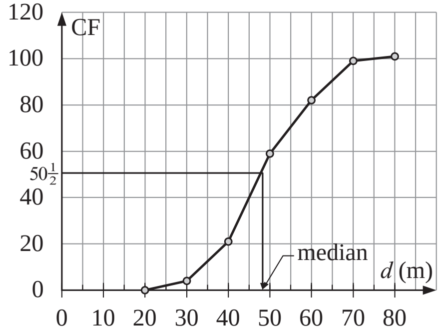
i.e., $107.5 - 45 = 62.5$

$\therefore$ 62 or 63 failed

d When life $= 800$, CF $\doteqdot 15$

$\therefore$ P(failed before 800) $\doteqdot \dfrac{15}{178} \doteqdot 0.084$

9



a median $= 50\frac{1}{2}$th score
$\doteqdot 48$ m

b When $d = 35$, CF $\doteqdot 12.5$
i.e., 12 or 13 students

c When $d = 45$, CF $\doteqdot 40$
When $d = 65$, CF $\doteqdot 90.5$
$\therefore$ number of students $\doteqdot 90.5 - 40$
$\doteqdot 50.5$

i.e., about 50 or 51 of them

d When $d = 53$, CF $\doteqdot 66$. So, $101 - 66 = 35$ were considered.

e When $d = 32$, CF $\doteqdot 7$. So, $P(\leqslant 32) = \frac{7}{101}$
$\doteqdot 0.07$ or 7%

EXERCISE 18D.1

1 **a** 2 3 3 3 4 4 4 5 5 5 5 6 6 6 6 6 7 7 8 8 8 9 9 $(n = 23)$

Q_1 (↑ second 4), median (↑ first 6), Q_3 (↑ second 7)

i median $= 6$

ii $Q_1 = 4$, $Q_3 = 7$

iii range $= 9 - 2 = 7$

iv IQR $= Q_3 - Q_1 = 7 - 4 = 3$

b 10 12 12 14 15 15 16 16 17 18 18 18 18 19 20 21 22 24 $(n = 18)$

Q_1 (↑ first 15), median (↑ between 17 and 18), Q_3 (↑ 19)

i median $= 17.5$

ii $Q_1 = 15$, $Q_3 = 19$

iii range $= 24 - 10 = 14$

iv IQR $= Q_3 - Q_1 = 19 - 15 = 4$

c $\underset{}{21.8}\ \ 22.4\ \ \underset{\substack{\uparrow \\ Q_1}}{23.5}\ \ 23.5\ \ 24.6\ \ \underset{\substack{\uparrow \\ \text{median}}}{24.9}\ \ 25.0\ \ 25.3\ \ \underset{\substack{\uparrow \\ Q_3}}{26.1}\ \ 26.4\ \ 29.5\quad (n = 11)$

i median $= 24.9$ **ii** $Q_1 = 23.5$, $Q_3 = 26.1$ **iii** range $= 29.5 - 21.8 = 7.7$ **iv** IQR $= Q_3 - Q_1 = 26.1 - 23.5 = 2.6$

d $103\ \ 105\ \ 115\ \ 115\ \ \overbrace{119\ \ 123}^{Q_1}\ \ 124\ \ 125\ \ 127\ \ 127\ \ \underset{\substack{\uparrow \\ \text{median}}}{128}\ \ 129\ \ 130\ \ 133\ \ 135\ \ \underbrace{140\ \ 141}_{Q_3}\quad (n = 21)$
$142\ \ 145\ \ 146\ \ 148$

i median $= 128$ **ii** $Q_1 = 121$, $Q_3 = 140.5$ **iii** range $= 148 - 103 = 45$ **iv** IQR $= Q_3 - Q_1 = 140.5 - 121 = 19.5$

2 $\underset{\substack{\uparrow \\ \text{min}}}{0}\ \ 0\ \ 0\ \ 0.8\ \ \underbrace{1.4\ \ 1.5}_{\substack{\uparrow \\ Q_1}}\ \ 1.6\ \ 1.9\ \ 2.1\ \ \underbrace{2.2\ \ 2.7}_{\substack{\uparrow \\ \text{median}}}\ \ 3.0\ \ 3.4\ \ 3.6\ \ \underbrace{3.8\ \ 3.8}_{\substack{\uparrow \\ Q_3}}\ \ 4.5\ \ 4.8\ \ 5.2\ \ \underset{\substack{\uparrow \\ \text{max}}}{5.2}$

a median $= \dfrac{2.2 + 2.7}{2} = 2.45$ $\quad Q_1 = 1.45$, $Q_3 = 3.8$ **b** range $= 5.2 - 0 = 5.2$ $\quad$ IQR $= Q_3 - Q_1 = 3.8 - 1.45 = 2.35$

c **i** the median, i.e., 2.45 min **ii** Q_3, i.e., 3.8 min

iii The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes. The waiting time was spread over 5.2 minutes.

3 $\underset{\substack{\uparrow \\ \text{min}}}{3}\ \ 4\ \ 7\ \ 9\ \ 10\ \ \underset{\substack{\uparrow \\ Q_1}}{13}\ \ 14\ \ 16\ \ 17\ \ 18\ \ 20\ \ \underset{\substack{\uparrow \\ \text{median}}}{20}\ \ 23\ \ 25\ \ 26\ \ 29\ \ 29\ \ \underset{\substack{\uparrow \\ Q_3}}{29}\ \ 31\ \ 33\ \ 37\ \ 38\ \ \underset{\substack{\uparrow \\ \text{max}}}{42}\quad (n = 23)$

a min $= 3$ **b** max $= 42$ **c** median $= 20$ **d** $Q_1 = 13$ **e** $Q_3 = 29$

f range $= 42 - 3 = 39$ **g** IQR $= Q_3 - Q_1 = 29 - 13 = 16$

4 $\overset{\substack{\text{min} \\ \downarrow}}{109}\ \ 111\ \ 113\ \ 114\ \ \underbrace{114\ \ 118}_{\substack{\uparrow \\ Q_1}}\ \ 119\ \ 122\ \ 122\ \ \underbrace{124\ \ 124}_{\substack{\uparrow \\ \text{median}}}\ \ 126\ \ 128\ \ 129\ \ \underbrace{129\ \ 131}_{\substack{\uparrow \\ Q_3}}\ \ 132\quad (n = 20)$
$135\ \ 138\ \ \underset{\substack{\uparrow \\ \text{max}}}{138}$

a **i** median $= 124$ cm **ii** $Q_3 = 130$ cm, $Q_1 = 116$ cm

b **i** .... 124 cm tall **ii** .... 130 cm tall

c **i** range $= 138 - 109 = 29$ cm **ii** IQR $= Q_3 - Q_1 = 130 - 116 = 14$ cm

d the IQR, i.e., over 14 cm

5 **a** **Without fertiliser** - See **Exercise 18B.2** solution to question **7**.

i range $= 9 - 2 = 7$ **ii** median $= 6$

iii lower quartile $=$ 38th score $= 5$ **iv** upper quartile $=$ 113th score $= 7$

v interquartile range $= 7 - 5 = 2$

b **With fertiliser**

i range $= 13 - 3 = 10$ **ii** median $= 7$

iii lower quartile $=$ 38th score $= 6$ **iv** upper quartile $=$ 113th score $= 8$

v interquartile range $= 8 - 6 = 2$

EXERCISE 18D.2

1 **a** **i** median $= 35$ **ii** max. value $= 78$ **iii** min. value $= 13$

iv $Q_3 = 53$ **v** $Q_1 = 26$

b **i** range $= 78 - 13 = 65$ **ii** IQR $= Q_3 - Q_1 = 53 - 26 = 27$

2 **a** highest mark $= 98$ **b** lowest mark $= 25$ **c** the median which is 70

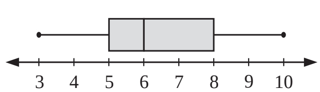
d Q_3 which is 85 **e** $Q_1 = 55$ and $Q_3 = 85$ **f** range $= 98 - 25 = 73$

g IQR $= Q_3 - Q_1 = 85 - 55 = 30$

3 **a** **i** $\underset{\text{min}}{\underset{\uparrow}{3}}\ 4\ 5\ \underset{Q_1}{\underset{\uparrow}{5}}\ 5\ 6\ \underbrace{6\ \ 6}_{\substack{\uparrow\\ \text{median}}}\ 7\ 7\ \underset{Q_3}{\underset{\uparrow}{8}}\ 8\ 9\ \underset{\text{max}}{\underset{\uparrow}{10}}$

So, min $= 3$, $Q_1 = 5$, median $= 6$, $Q_3 = 8$, max $= 10$

ii



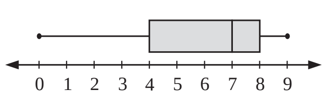
iii range $= 10 - 3$
$= 7$

iv IQR $= Q_3 - Q_1$
$= 8 - 5$
$= 3$

b **i** $\underset{\text{min}}{\underset{\uparrow}{0}}\ 1\ 2\ 3\ \underset{Q_1}{\underset{\uparrow}{4}}\ 5\ 6\ 6\ 7\ \underset{\text{median}}{\underset{\uparrow}{7}}\ 7\ 8\ 8\ 8\ \underset{Q_3}{\underset{\uparrow}{8}}\ 8\ 8\ 9\ \underset{\text{max}}{\underset{\uparrow}{9}}$

So, min $= 0$, $Q_1 = 4$, median $= 7$, $Q_3 = 8$, max $= 9$

ii



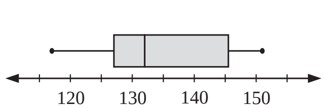
iii range $= 9 - 0$
$= 9$

iv IQR $= Q_3 - Q_1$
$= 8 - 4$
$= 4$

c **i** $\overset{\substack{\text{min}\\ \downarrow}}{117}\ 120\ 123\ \underbrace{126\ \ 128}_{\substack{\uparrow\\ Q_1}}\ 130\ 131\ 131\ \underbrace{131\ \ 133}_{\substack{\uparrow\\ \text{median}}}\ 135\ 135\ 137\ \underbrace{144\ \ 147}_{\substack{\uparrow\\ Q_3}}\ 147$
$149\ 149\ \underset{\text{max}}{\underset{\uparrow}{151}}$

So, min $= 117$, $Q_1 = 127$, median $= 132$, $Q_3 = 145.5$, max $= 151$

ii



iii range
$= 151 - 117$
$= 34$

iv IQR $= Q_3 - Q_1$
$= 145.5 - 127$
$= 18.5$

4 **a**

Stem	*Leaf*
88	3 5 9
89	2 2 5 7 8 8 9 9 9
90	0 0 1 1 2 3 3 3 5 5 6 6 6
91	0 1 1 1 2 3 3 4 6 7 7 8 9
92	0 1 4 7

min, Q_1, median, Q_2, max

88 | 3 represents 883 g

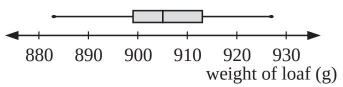
$n = 42$, so, median $=$ ave. of 21st and 22nd $= \dfrac{905+905}{2} = 905$

$Q_1 =$ average of 10th and 11th $= \dfrac{899+899}{2} = 899$

$Q_3 =$ average of 31st and 32nd $= \dfrac{913+913}{2} = 913$

So, min $= 883$ g, $Q_1 = 899$ g, median $= 905$ g, $Q_3 = 913$ g, max $= 927$ g

b



c **i** $\text{IQR} = Q_3 - Q_1 = 913 - 899 = 14$ g
ii range $=$ max $-$ min $= 927 - 883 = 44$ g

d **i** the median, i.e., 905 g **ii** less than 900 g $\Rightarrow$ $\leqslant 899$ g, so 25%
iii the IQR, i.e., 14 g **iv** Q_1 or less, i.e., 899 g

e slightly skewed to the left, i.e., a little negatively skewed

5 **a**

Statistic	*Year* 9	*Year* 12
min value	1	6
Q_1	5	10
median	7.5	14
Q_3	10	16
max value	12	17.5

b For year 9 group
i range $= 12 - 1 = 11$
ii IQR $= 10 - 5 = 5$

For year 12 group
i range $= 17.5 - 6 = 11.5$
ii IQR $= 16 - 10 = 6$

c **i** True, indicated by the median.
ii True, as Q_1 for year 9 $= 5$ and min for year 12 $= 6$.

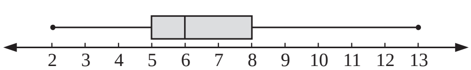
6 2 3 3 4 4 4 4 5 5 5 5 5 5 5 6 6 6 6 6 6 7 7 7 7 8 8 8 9 9 9 10 12 13

min (2), Q_1 (5, 5), median (6), Q_3 (8, 8), max (13)

a median $= 6$, $Q_1 = 5$, $Q_3 = 8$

b IQR $= 8 - 5 = 3$

c



7 **a**

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

min (33); median is one of these (36); max (40)

There are 47 scores
$\therefore$ median $=$ 24th $\left(\dfrac{47+1}{2} = 24\right)$

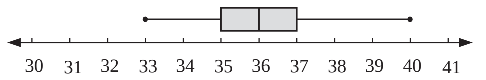
14 scores are 35 or less
27 scores are 36 or less
$\therefore$ median is 36

$Q_1 =$ 12th $\left(\dfrac{47+1}{4} = 12\right)$
$= 35$

$Q_3 =$ 36th
$= 37$

So, min $= 33$, $Q_1 = 35$, median $= 36$, $Q_3 = 37$, max $= 40$

b **i** range $= 40 - 33 = 7$
ii IQR $= 37 - 35 = 2$



8 **a** For $h = 5$, CF $\doteqdot 10$ $\therefore$ 10 seedlings have height 5 cm or less

b For $h = 8$, CF $\doteqdot 43$ $\therefore$ % taller than 8 cm $= \dfrac{60 - 43}{60} \times 100\% \doteqdot 28.3\%$

c Approx. median occurs at CF $= 30$, i.e., median $\doteqdot 7$ cm.

d $\text{IQR} = Q_3 - Q_1 = (h \text{ when CF} = 45) - (h \text{ when CF} = 15)$
$\doteqdot 8.4 - 5.8$
$\doteqdot 2.6$ cm

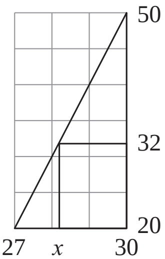
e 90th percentile occurs when CF $= 90\%$ of $60 = 54$
$\therefore$ 90th percentile $= 10$

This means that 90% of the seedlings have a height of 10 cm or less.

9 **a** The lower quartile occurs when CF $= 25\%$ of $80 = 20$
$\therefore$ $Q_1 = 27$ min

b The median occurs when CF $= 50\%$ of $80 = 40$
$\therefore$ median $= 29$ min

c The upper quartile occurs when CF $= 75\%$ of $80 = 60$
$\therefore$ $Q_3 \doteqdot 31\frac{1}{2}$ min



d $\text{IQR} = Q_3 - Q_1 = 31\frac{1}{2} - 27 = 4\frac{1}{2}$ min

e For the 40th percentile, CF $= 40\%$ of $80 = 32$

From the similar Δs, $\dfrac{x - 27}{12} = \dfrac{3}{30}$

$\therefore$ $x - 27 = 1.2$

$\therefore$ $x = 28.2$

So about 28 min 10 sec.

10 **a** Highest CF value $= 480$ $\therefore$ 480 sat for the exam.

b Highest mark $= 120$ {assuming 'best' student got full marks}

c The median occurs when CF $\doteqdot 50\%$ of $480 = 240$
$\therefore$ median $\doteqdot 84$ marks

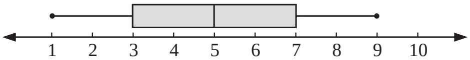
d $\text{IQR} = Q_3 - Q_1 = (\text{mark when CF} = 360) - (\text{mark when CF} = 120)$
$\doteqdot 99 - 71$
$\doteqdot 28$ marks

e For the 85th percentile, CF $= 85\%$ of $480 = 408$
$\therefore$ 85th percentile $\doteqdot 107$ marks

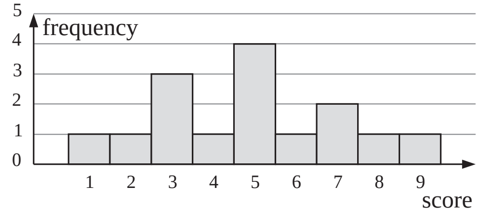
EXERCISE 18E.1

1 **a** $\overline{x} \doteqdot 4.87$, $\text{Min}_x = 1$, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 7$, $\text{Max}_x = 9$

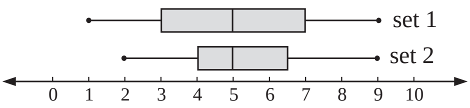
b



c

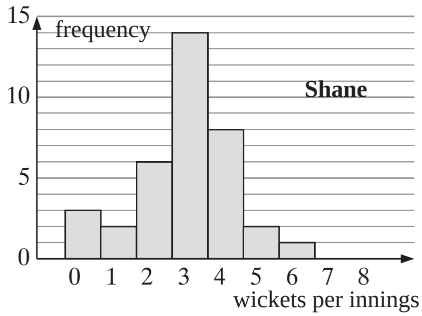


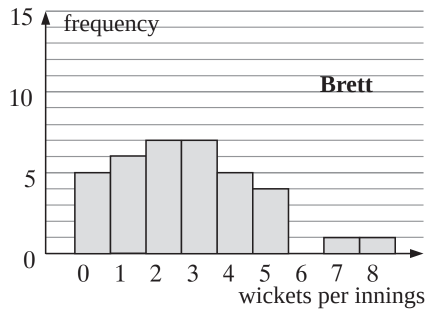
d $\overline{x} \doteqdot 5.24$, $\text{Min}_x = 2$, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6.5$, $\text{Max}_x = 9$



EXERCISE 18E.2

1 **a** discrete **c**





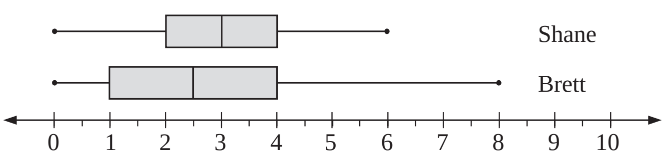
d There are no outliers for Shane. Brett has outliers of 7 and 8 which must not be removed.

e Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.

f Shane has a higher mean ($\doteqdot$ 2.89 wickets) compared with Brett ($\doteqdot$ 2.67 wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has a bi-modal distribution of 2 and 3 (7 times each).

g Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.

h



j Generally, Shane takes more wickets than Brett and is a more consistent bowler.

2 **a** continuous

c For the 'New type' globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d

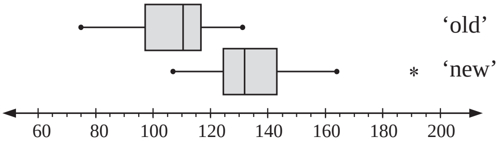
	Old type	*New type*
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The mean and median are $\doteq$ 25% and $\doteq$ 19% higher for the 'new type' of globe compared with the 'old type'.

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).

The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed. For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

3

	Set 1	*Set 2*
Mean	21.95	21.81
Min_x	21.6	21.5
Q_1	21.9	21.7
Median	21.9	21.8

	Set 1	*Set 2*
Q_3	22.0	21.9
Max_x	22.2	22.2
Range	0.6	0.7
IQR	0.1	0.2

Set 2 has more data points than *set 1*.

The 5-number summary for each set of data is reasonably similar, however the spread of the middle 50% of data for *set 2* is double that for *set 1*.

EXERCISE 18F

1 a **Sally**

x	$(x-\bar{x})^2$
23	4
17	64
31	36
25	0
25	0
19	36
28	9
32	49
200	198

$$\bar{x} = \frac{\sum x}{n} = \frac{200}{8} = 25 \qquad s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{198}{8}} \doteq 4.97$$

Joanne

x	$(x-\bar{x})^2$
9	$(21.5)^2$
29	$(1.5)^2$
41	$(10.5)^2$
26	$(4.5)^2$
14	$(16.5)^2$
44	$(13.5)^2$
38	$(7.5)^2$
43	$(12.5)^2$
244	1262

$$\bar{x} = \frac{\sum x}{n} = \frac{244}{8} = 30.5 \qquad s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{1262}{8}} \doteq 12.56$$

b The standard deviation. The lower the value of s, the more consistent.

2 a Glen has mean $= \dfrac{0+10+1+9+11+0+8+5+6+7}{10} = \dfrac{57}{10} = 5.7$

Shane has mean $= \dfrac{4+3+4+1+4+11+7+6+12+5}{10} = \dfrac{57}{10} = 5.7$

Glen's range $= 11 - 0 = 11$ Shane's range $= 12 - 1 = 11$

b We suspect Glen's as he has more high and low values.

c **Glen**

x	$(x-\bar{x})^2$
0	$(5.7)^2$
10	$(4.3)^2$
1	$(4.7)^2$
9	$(3.3)^2$
11	$(5.3)^2$
0	$(5.7)^2$
8	$(2.3)^2$
5	$(0.7)^2$
6	$(0.3)^2$
7	$(1.3)^2$
	154.9

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{154.9}{10}} \doteqdot 3.9$$

greater variability

Shane

x	$(x-\bar{x})^2$
4	$(1.7)^2$
3	$(2.7)^2$
4	$(1.7)^2$
1	$(4.7)^2$
4	$(1.7)^2$
11	$(5.3)^2$
7	$(1.3)^2$
6	$(0.3)^2$
12	$(6.3)^2$
5	$(0.7)^2$
	108.1

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{108.1}{10}} \doteqdot 3.29$$

d The standard deviation as it takes all values into account, not just the lowest and highest.

3 **a** We suspect variability in standard deviation since the factors may change every day.

b **i** sample mean **ii** sample standard deviation **c** less variability

4 **a**

x	$(x-\bar{x})^2$
79	10^2
64	5^2
59	10^2
71	2^2
68	1^2
68	1^2
74	5^2
483	256

$$\bar{x} = \frac{\sum x}{n} = \frac{483}{7} = 69$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{256}{7}} \doteqdot 6.05$$

b

x	$(x-\bar{x})^2$
89	10^2
74	5^2
69	10^2
81	2^2
78	1^2
78	1^2
84	5^2
553	256

$$\bar{x} = \frac{\sum x}{n} = \frac{553}{7} = 79$$

$$s \doteqdot 6.05$$

c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.

5 **a**

x	$(x-\bar{x})^2$
0.8	$(0.21)^2$
1.1	$(0.09)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
1.2	$(0.19)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
0.7	$(0.31)^2$
1.0	$(0.01)^2$
1.1	$(0.09)^2$
10.4	0.289

$$\bar{x} = \frac{\sum x}{n} = \frac{10.1}{10} = 1.01 \text{ kg}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{0.289}{10}} \doteqdot 0.17 \text{ kg}$$

b

x	$(x-\bar{x})^2$
1.6	$(0.42)^2$
2.2	$(0.18)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
2.4	$(0.38)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
1.4	$(0.62)^2$
2.0	$(0.02)^2$
2.2	$(0.18)^2$
20.2	1.156

$$\bar{x} = \frac{\sum x}{n} = \frac{20.2}{10} = 2.02 \text{ kg}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{1.156}{10}} = 0.34 \text{ kg}$$

c Doubling the values doubles the mean and standard deviation.

6
$$\begin{aligned}\text{mean} &= \frac{3+7+a+4+b}{5}\\ &= \frac{14+a+b}{5}\\ \therefore\ \frac{14+a+b}{5} &= 5\\ \therefore\ 14+a+b &= 25\\ \therefore\ a+b &= 11\\ \therefore\ b &= 11-a \quad ..... (1)\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{(3-5)^2+(7-5)^2+(a-5)^2+(4-5)^2+(b-5)^2}{5}}\\ &= \sqrt{\frac{4+4+(a-5)^2+1+(b-5)^2}{5}}\\ &= \sqrt{\frac{9+(a-5)^2+(6-a)^2}{5}} \quad \{\text{using (1)}\}\\ &= \sqrt{\frac{9+a^2-10a+25+36-12a+a^2}{5}}\\ &= \sqrt{\frac{2a^2-22a+70}{5}}\end{aligned}$$

So, $\sqrt{\dfrac{2a^2-22a+70}{5}} = \sqrt{2}$

$$\begin{aligned}\therefore\ 2a^2-22a+70 &= 10\\ \therefore\ 2a^2-22a+60 &= 0\\ \therefore\ a^2-11a+30 &= 0\\ \therefore\ (a-5)(a-6) &= 0\end{aligned}$$

$\therefore\ a = 5$ or 6 and corresponding values of b are 6 or 5 and as $a > b$, $a = 6$ and $b = 5$

7

x	f	$f\,x$	$f(x-\bar{x})^2$
0	14	0	41.61
1	18	18	9.44
2	13	26	0.99
3	5	15	8.14
4	3	12	15.54
5	2	10	21.46
6	2	12	36.57
7	1	7	27.84
$\sum$	58	100	161.59

$$\begin{aligned}\bar{x} &= \frac{\sum f\,x}{\sum f}\\ &= \frac{100}{58}\\ &\doteqdot 1.724 \text{ children}\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}\\ &\doteqdot \sqrt{\frac{161.59}{58}}\\ &\doteqdot 1.67 \text{ children}\end{aligned}$$

8 **a**

x	f	$f\,x$	$f(x-\bar{x})^2$
0	1	0	26.01
1	0	0	0
2	1	2	9.61
3	1	3	4.41
4	2	8	2.42
5	6	30	0.06
6	5	30	4.05
7	3	21	10.83
8	1	8	8.41
$\sum$	20	102	65.80

$$\begin{aligned}\bar{x} &= \frac{\sum f\,x}{\sum f}\\ &= \frac{102}{20}\\ &= 5.1\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}\\ &= \sqrt{\frac{65.80}{20}}\\ &\doteqdot 1.81\end{aligned}$$

b

x	f	$f\,x$	$f(x-\bar{x})^2$
11	2	22	24.22
12	1	12	6.15
13	4	52	8.76
14	5	70	1.15
15	6	90	1.62
16	4	64	9.24
17	2	34	12.70
18	1	18	12.39
$\sum$	25	362	76.23

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{362}{25} = 14.48 \doteqdot 14.5$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{76.25}{25}} \doteqdot 1.75$$

9

x	f	$f\,x$	$f(x-\bar{x})^2$
33	1	33	18.24
35	5	175	25.78
36	7	252	11.31
37	13	481	0.95
38	12	456	6.38
39	8	312	23.92
40	2	80	14.90
$\sum$	48	1789	101.48

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{1789}{48} \doteqdot 37.3$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{101.48}{48}} \doteqdot 1.45$$

10

Midpoint (x)	f	$f\,x$	$f(x-\bar{x})^2$
40.5	1	40.5	52.85
42.5	1	42.5	27.77
44.5	3	133.5	32.08
46.5	7	325.5	11.29
48.5	11	533.5	5.86
50.5	5	252.5	37.26
52.5	2	105	44.75
$\sum$	30	1433	211.86

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{1433}{30} = 47.7666\ldots. \doteqdot 47.8 \text{ cm}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \doteqdot \sqrt{\frac{211.86}{30}} \doteqdot 2.657\ldots. \doteqdot 2.66 \text{ cm}$$

11

Midpoint (x)	f	$f\,x$	$f(x-\bar{x})^2$
364.995	17	6204.9	10 885.83
374.995	38	14 249.8	8901.23
384.995	47	18 094.8	1322.72
394.995	57	22 514.7	1256.45
404.995	18	7289.9	3886.97
414.995	10	4149.9	6098.43
424.995	10	4250.0	12 037.43
434.995	3	1305.0	5992.93
$\sum$	200	78 059	50 381.99

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{78\,059}{200} \doteqdot \$390.30$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{n}} = \sqrt{\frac{50\,381.99}{200}} \doteqdot \$15.87$$

12 **a** $\bar{x} = 26.5,\ s \doteqdot 2.29$ **b** $\bar{x} \doteqdot 13.2,\ s \doteqdot 4.65$ **c** $\bar{x} \doteqdot 170,\ s \doteqdot 9.84$ **d** $\bar{x} = 0,\ s = 2$

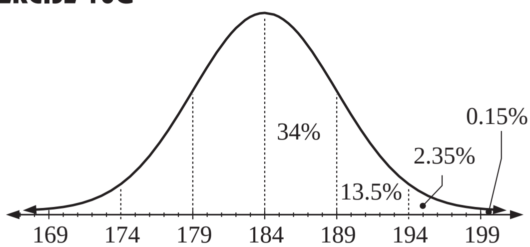
13 $\bar{x} = 55$ L, $s \doteqdot 10.9$ L

14 **a** $\bar{x} \doteqdot 33.7,\ s \doteqdot 1.11$ **b** $\bar{x} \doteqdot 17.4,\ s \doteqdot 0.123$ **c** $\bar{x} \doteqdot 34.6,\ s \doteqdot 10.4$

15 $\bar{x} \doteqdot \$18.60,\ s \doteqdot \$8.33$

EXERCISE 18G

1



a $50\% + 34\% = 84\%$

So, 16% are above 189.

b $50\% + 34\% = 84\%$

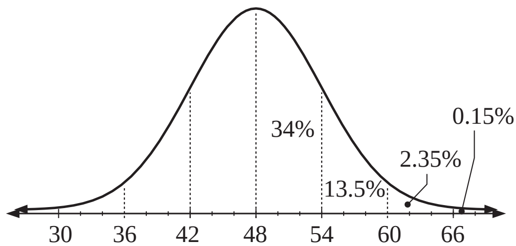
So, 84% are above 179.

c $2 \times 34\% + 2 \times 13.5\% + 2.35\%$

$\doteqdot 97.4\%$

d 0.15%

2



$100\% - 50\% - 34\% = 16\%$

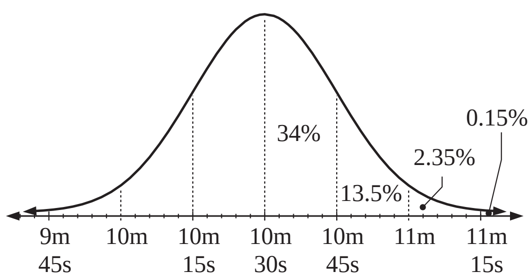
and $\quad 20 \times 16\%$

$= 20 \times 0.16$

$= 3.2$

On 3 occasions.

3



a $\quad 200 \times (50 + 34 + 13.5)\%$

$= 200 \times 95.5\%$

$\doteqdot 195$

$\therefore \quad 200 - 195 = 5$ took > 11 minutes

b $\quad 200 \times (0.15 + 2.35 + 13.5)\%$

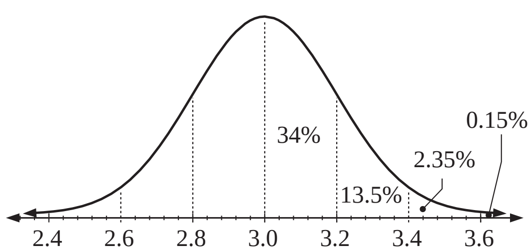
$= 200 \times 16\%$

$= 32$ lifesavers

c $\quad 200 \times 68\%$

$= 136$ lifesavers

4



a $\quad 545 \times (50 + 34)\%$

$= 545 \times 0.84$

$\doteqdot 458$ babies

b $\quad 545 \times (68 + 13.5)\%$

$= 545 \times 81.5\%$

$\doteqdot 444$ babies

REVIEW SET 18A

1 **a** **i** many **ii** n.a. **iii** categorical

b **i** many **ii** breaths per minute **iii** quantitative discrete

c **i** infinitely many **ii** centimetres, metres, etc. **iii** quantitative continuous

2 **a** Diameter of bacteria colonies

0	4 8 9
1	3 5 5 7
2	1 1 5 6 8 8
3	0 1 2 3 4 5 5 6 6 7 7 9
4	0 1 2 7 9

leaf unit: 0.1 cm

b Using technology

i median $= 3.15$ cm

ii range $= 4.9 - 0.4 = 4.5$ cm

c The distribution has a large tail on the left. So, it is negatively skewed.

3 **a** Using technology

	Girls	*Boys*
shape	pos. skewed	approx. symm.
centre (median)	36.3 sec	34.9 sec
spread (range)	7.7 sec	4.9 sec

b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 **a** highest $= 97.5$ m, lowest $= 64.6$ m

b The range $= 97.5 - 64.5 = 33$

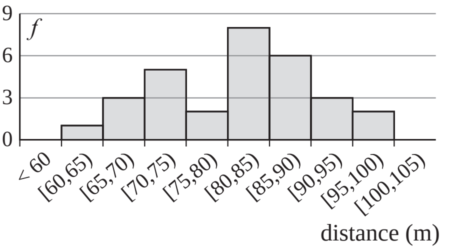
So, if intervals of length 5 are used we need about 7 of them.

We choose 60 -, 65 -, 70 -, 75 -, 80 -, etc.

c

A frequency distribution table for distances thrown by Thabiso		
distance (m)	tally	freq. (f)
60 -	|	1
65 -	|||	3
70 -	~~||||~~	5
75 -	||	2
80 -	~~||||~~ |||	8
85 -	~~||||~~ |	6
90 -	|||	3
$95 < 100$	||	2
	Total	30

d **i/ii** **Frequency histogram displaying the distance Thabiso throws a cricket ball**



e Using technology

i $\bar{x} \doteqdot 81.1$ m **ii** median $\doteqdot 83.0$ m

5 **a** Reading from the box-plots

	A	B
Min	11	11.2
Q_1	11.6	12
Median	12	12.6
Q_3	12.6	13.2
Max	13	13.8

b **i** range of A
$= 13 - 11$
$= 2$

range of B
$= 13.8 - 11.2$
$= 2.6$

ii IQR of A
$= 12.6 - 11.6$
$= 1.0$

IQR of B
$= 13.2 - 12$
$= 1.2$

c **i** The members in squad A generally ran faster times.

ii The times in squad B were more varied.

6 Using technology **a** **i** 101.5 **ii** 98 **iii** 105.5 **b** 7.5 **c** $\bar{x} = 100.2$, $s \doteqdot 7.59$

7 **a** The mean length is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

b Similar reasons as for **a**.

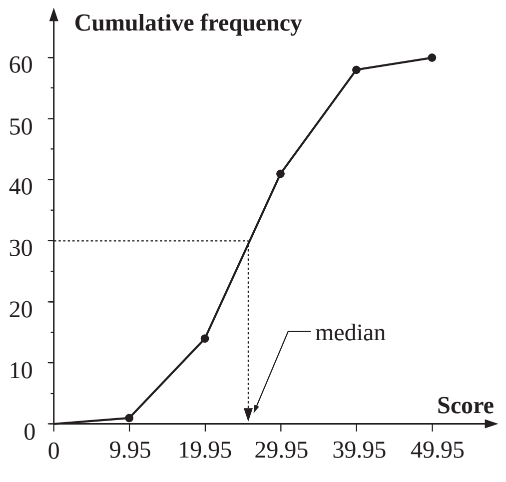
8 This question could be done using technology or

litres (x)	f	$f\,x$	$f(x-\bar{x})^2$
17	5	85	1299.27
22	13	286	1607.50
27	17	459	636.72
32	29	928	36.38
37	27	999	406.47
42	18	756	1419.38
47	7	329	1348.58
$\sum$	116	3842	6754.30

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{3842}{116} \doteq 33.12 \doteq 33.1 \text{ litres}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \doteq \sqrt{\frac{6754.30}{116}} \doteq 7.63 \text{ litres}$$

9 **a**



b median $\doteq$ 25.9 (see graph)

c $\text{IQR} = Q_3 - Q_1$
$= $ (score for CF of 45) $-$ (score for CF of 15)
$\doteq 32 - 20 \doteq 12$

d

	f	midpt x	$f\,x$	$(x-\bar{x})^2$
0 – 9.9	1	4.95	4.95	441
10 – 19.9	13	14.95	194.35	121
20 – 29.9	27	24.95	673.65	1
30 – 39.9	17	34.95	594.15	81
40 – 49.9	2	44.95	89.9	361
	60		1557	

$$\bar{x} = \frac{\sum f\,x}{\sum x} = \tfrac{1557}{60} \doteq 26.0 \qquad s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\tfrac{4140}{60}} \doteq 8.31$$

10
$$\frac{6+x+9+y+3+11}{6} = 7 \quad \{\text{as the mean is } 7\}$$
$$\therefore\ 29+x+y = 42$$
$$\therefore\ x+y = 13$$
$$\therefore\ y = 13-x \quad .... (1)$$

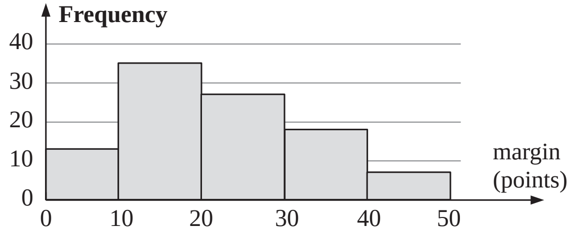
Also
$$\sqrt{\frac{(6-7)^2+(x-7)^2+(9-7)^2+(y-7)^2+(3-7)^2+(11-7)^2}{6}} = \frac{5}{\sqrt{3}}$$
$$\therefore\ \frac{1+(x-7)^2+4+(y-7)^2+16+16}{6} = \frac{25}{3}$$
$$\therefore\ 37+(x-7)^2+(6-x)^2 = 50$$
$$\therefore\ 37+x^2-14x+49+36-12x+x^2-50 = 0$$
$$\therefore\ 2x^2-26x+72 = 0$$
$$\therefore\ x^2-13x+36 = 0$$
$$\therefore\ (x-9)(x-4) = 0$$
$$\therefore\ x = 9 \text{ or } 4$$

and corresponding y-values are 4 or 9

But $x < y$ $\therefore$ $x = 4$, $y = 9$.

REVIEW SET 18B

1



2 Using technology:

a $x \doteqdot 3.69$, mode $= 4$, median $= 4$ **b** $\bar{x} = 56$, bi-modal (58 and 63), median $= 58$

3 Using technology, $\bar{x} \doteqdot 49.6$, $s \doteqdot 1.60$ does not justify claim. A much greater sample is needed.

4 Use technology or

Midpoint (x)	f	$f\,x$
274.5	14	3843
324.5	34	11 033
374.5	68	25 466
424.5	72	30 564
474.5	54	25 623
524.5	23	12 063.5
574.5	7	4021.5
$\sum$	272	112 613.5

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{112\,613.5}{272}$$

$\doteqdot$ 414 customers

5 116 118 120 122 127 128 132 135 ($n = 8$)

min: 116, Q_1: between 118 and 120, median: between 122 and 127, Q_3: between 128 and 132, max: 135

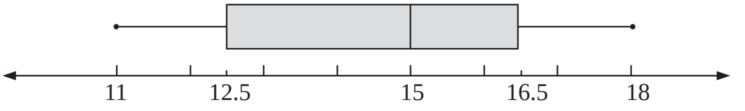
range $= 135 - 116$
$= 19$

$$Q_1 = \frac{118 + 120}{2} = 119 \qquad Q_3 = \frac{128 + 132}{2} = 130$$

$s \doteqdot 6.38$ {technology}

6 11 12 12 13 14 14 15 15 15 16 17 17 18

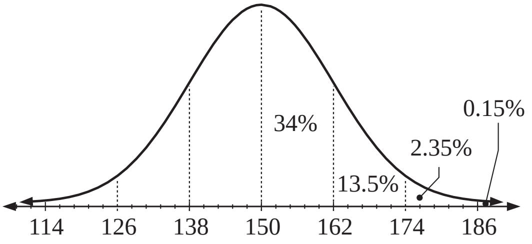
min: 11, Q_1: between 12 and 13, median: 15, Q_3: between 16 and 17, max: 18



7 Using technology with x values 74.995, 84.995, 94.995, etc.

$\bar{x} = \$103.50$ and $s = \$19.40$

8



a 68% lie here

b $2 \times (34 + 13.5) = 95\%$ lie here

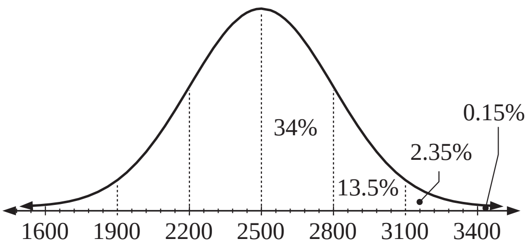
c $2 \times 34 + 13.5 = 81.5\%$ lie here

d 13.5% lie here

9 Using technology

a $\bar{x} \doteqdot 46.0, \quad s \doteqdot 1.50$ **b** $\bar{x} \doteqdot 149.5, \quad s \doteqdot 14.0$

10



a $(50 - 34 - 13.5)\% = 2.5\%$ days are less than 1900

b $(50 + 34)\% = 84\%$

c $(2 \times 34 + 13.5)\% = 81.5\%$

11 $\dfrac{3 + a + 6 + b + 13}{5} = 6.8 \quad \therefore \quad \dfrac{a + b + 22}{5} = 6.8$

$$\therefore \quad a + b + 22 = 34$$
$$\therefore \quad a + b = 12$$
$$\therefore \quad b = 12 - a \quad ..... (1)$$

Now $\sqrt{\dfrac{(3-6.8)^2 + (a-6.8)^2 + (6-6.8)^2 + (b-6.8)^2 + (13-6.8)^2}{5}} = \sqrt{12.56}$

$$\therefore \quad \frac{(-3.8)^2 + (a-6.8)^2 + (-0.8)^2 + (12-a-6.8)^2 + (6.2)^2}{5} = 12.56$$

$$\therefore \quad 53.52 + (a-6.8)^2 + (5.2-a)^2 = 62.8$$

$$\therefore \quad 53.52 + a^2 - 13.6a + 46.24 + 27.04 - 10.4a + a^2 - 62.8 = 0$$

$$\therefore \quad 2a^2 - 24a + 64 = 0$$

$$\therefore \quad a^2 - 12a + 32 = 0$$

$$\therefore \quad (a-4)(a-8) = 0$$

$$\therefore \quad a = 4 \text{ or } 8 \quad \text{and} \quad b = 8 \text{ or } 4$$

But $a > b, \quad \therefore \quad a = 8, \quad b = 4$

Chapter 19
PROBABILITY

EXERCISE 19A

1 **a** P(inside a square)
$\doteqdot \dfrac{113}{145}$
$\doteqdot 0.7793\ldots.$
$\doteqdot 0.78$

b P(on a line)
$\doteqdot \dfrac{32}{145}$
$\doteqdot 0.2206\ldots.$
$\doteqdot 0.22$

2 Total frequency $= 17 + 38 + 19 + 4 = 78$

a P(20 to 39 seconds)
$= \dfrac{38}{78}$
$\doteqdot 0.487$

b P($>$ 60 seconds)
$= \dfrac{4}{78}$
$\doteqdot 0.051$

c P(between 20 and 59 sec (inc))
$= \dfrac{38+19}{78}$
$\doteqdot 0.731$

3

Calls/day	*No. of days*
0	2
1	7
2	11
3	8
4	7
5	4
6	3
7	0
8	1

a Survey lasted $2 + 7 + 11 + 8 + 7 + 4 + 3 + 0 + 1$
$= 43$ days

b **i** P(0 calls)
$= \dfrac{2}{43}$
$\doteqdot 0.047$

ii P($\geqslant$ 5 calls)
$= \dfrac{4+3+0+1}{43}$
$\doteqdot 0.186$

iii P($<$ 3 calls)
$= \dfrac{2+7+11}{43}$
$\doteqdot 0.465$

4 Total frequency
$= 37 + 81 + 48 + 17 + 6 + 1$
$= 190$

a P(4 days gap)
$= \dfrac{17}{190}$
$\doteqdot 0.089$

b P(at least 4 days gap)
$= \dfrac{17+6+1}{190}$
$\doteqdot 0.126$

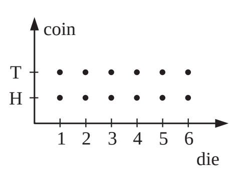
EXERCISE 19B

1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}

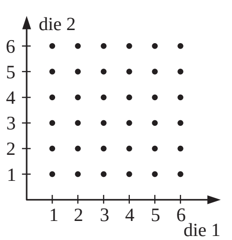
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

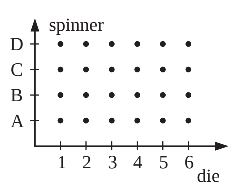
2 **a**



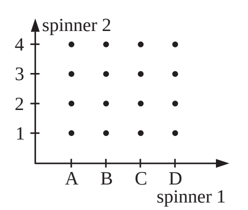
b



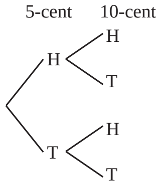
c



d



3 **a**



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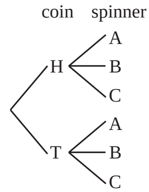

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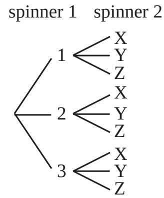
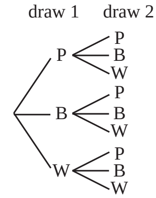


d



EXERCISE 19C

1 Total number of marbles $= 5 + 3 + 7 = 15$

a P (red) $= \frac{3}{15}$ $= \frac{1}{5}$

b P (green) $= \frac{5}{15}$ $= \frac{1}{3}$

c P (blue) $= \frac{7}{15}$

d P (not red) $= \dfrac{5+7}{15}$ $= \frac{12}{15}$ or $= \frac{4}{5}$

e P (neither green nor blue) $=$ P (red) $= \frac{1}{5}$

f P (green or red) $= \dfrac{5+3}{15}$ $= \frac{8}{15}$

2 **a** 8 are brown and so 4 are white.

b **i** P (brown) $= \frac{8}{12}$ $= \frac{2}{3}$

ii P (white) $= \frac{4}{12}$ $= \frac{1}{3}$

3 8 have *one*
19 have *two*
5 have *three*

a P (no first name) $= \frac{0}{32}$ $= 0$

b P (one first name) $= \frac{8}{32}$ $= \frac{1}{4}$

c P (two first names) $= \frac{19}{32}$

d P (three first names) $= \frac{5}{32}$

4 **a** P (multiple of 4)
$=$ P (4, 8, 12, 16, 20, 24, 28, 32, 36)
$= \frac{9}{36}$
$= \frac{1}{4}$

b P (between 6 and 9 inclusive)
$=$ P (6, 7, 8 or 9)
$= \frac{4}{36}$
$= \frac{1}{9}$

c P (> 20)
$=$ P (21, 22, 23, 24, ..., 35, 36)
$= \dfrac{36-20}{36}$
$= \frac{16}{36}$
$= \frac{4}{9}$

d P (9)
$= \frac{1}{36}$

e P (multiple of 13)
$=$ P (13 or 26)
$= \frac{2}{36}$
$= \frac{1}{18}$

f P (odd multiple of 3)
$=$ P (3, 9, 15, 21, 27, or 33)
$= \frac{6}{36}$
$= \frac{1}{6}$

5 **a** P (on Tuesday) $= \frac{1}{7}$

b P (on a weekend) $= \frac{2}{7}$

c P (in July) $= \dfrac{4 \times 31}{365 \times 3 + 366}$ {over a 4 year period} $= \dfrac{124}{1461}$

d P (in January or February) $= \dfrac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366}$ {over a 4 year period} $= \dfrac{237}{1461}$

6 Let A denote Antti, K denote Kai and N denote Neda.
Possible orders are: {AKN, ANK, KAN, KNA, NAK, NKA}

a P (A in middle) $= \frac{2}{6}$ $= \frac{1}{3}$

b P (A at left end) $= \frac{2}{6}$ $= \frac{1}{3}$

c P (A at right end) $= \frac{2}{6}$ $= \frac{1}{3}$

d P (K and N are together) $= \frac{4}{6} = \frac{2}{3}$

7 Let G denote 'a girl' and B denote 'a boy'.

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

b **i** P (all boys) = P (BBB) $= \frac{1}{8}$

ii P (all girls) = P (GGG) $= \frac{1}{8}$

iii P (boy, then girl, then girl) = P (BGG) $= \frac{1}{8}$

iv P (2 girls and a boy) = P (GGB or GBG or BGG) $= \frac{3}{8}$

v P (girl is eldest) = P (GGG or GBG or GBB or GGB) $= \frac{4}{8}$ $= \frac{1}{2}$

vi P (at least one boy) $= \frac{7}{8}$ {all except GGG}

8 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

b **i** P (A sits on one end) $= \frac{12}{24}$ $= \frac{1}{2}$

ii P (B sits on one of the two middle seats) $= \frac{12}{24}$ $= \frac{1}{2}$

iii P (A and B are together) $= \frac{12}{24}$ $= \frac{1}{2}$

iv P (A, B and C are together) $= \frac{12}{24}$ $= \frac{1}{2}$

9 {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, THTH, HTTH, TTHH, TTTH, TTHT, THTT, HTTT, TTTT}

a P (all heads) $= \frac{1}{16}$

b P (2H and 2T) $= \frac{6}{16}$ $= \frac{3}{8}$

c P (more H than T) $= \frac{5}{16}$

d P (at least one T)
$= \frac{15}{16}$ {excludes HHHH}

e P (exactly one H) $= \frac{4}{16}$
$= \frac{1}{4}$

10 P (hits the bulls-eye) $= \dfrac{\text{area of bulls-eye}}{\text{area of whole target}}$

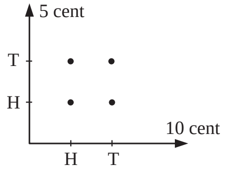
$= \dfrac{\pi \times 20^2}{\pi \times 30^2}$

$= \dfrac{400\pi}{900\pi}$

$= \frac{4}{9}$

EXERCISE 19D

1



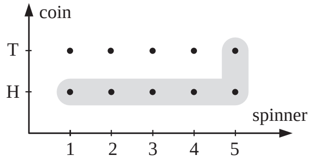
a P (2 heads)
$= \frac{1}{4}$

b P (2 tails)
$= \frac{1}{4}$

c P (exactly 1 head)
= P (HT or TH)
$= \frac{2}{4}$ or $\frac{1}{2}$

d P (at least one H)
= P (HT or TH or HH)
$= \frac{3}{4}$

2 **a**



b There are $2 \times 5 = 10$ possible outcomes.

c **i** P (T and 3)
$= \frac{1}{10}$

ii P (H and even)
= P (H2 or H4)
$= \frac{2}{10}$
$= \frac{1}{5}$

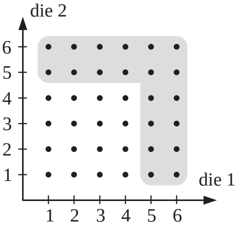
iii P (an odd)
= P (H1, T1, H3, T3, H5, T5)
$= \frac{6}{10}$
$= \frac{3}{5}$

iv P (H or 5)
$= \frac{6}{10}$
$= \frac{3}{5}$
{those shaded}

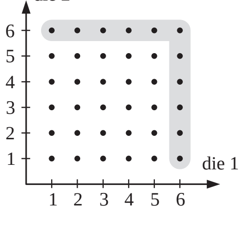
3 **a** P (two 3s)
= P ((3, 3))
$= \frac{1}{36}$

b P (5 and a 6)
= P ((5, 6), (6, 5))
$= \frac{2}{36}$
$= \frac{1}{18}$

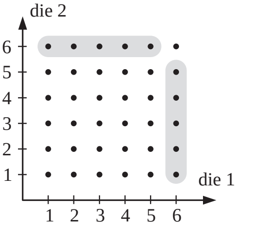
c P (5 or a 6)
$= \frac{20}{36}$
$= \frac{5}{9}$



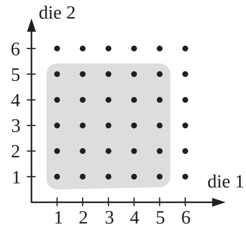
d P (at least one 6)
$= \frac{11}{36}$



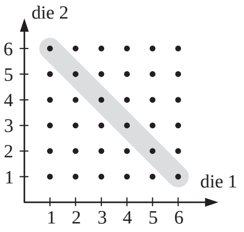
e P (exactly one 6)
$= \frac{10}{36}$
$= \frac{5}{18}$



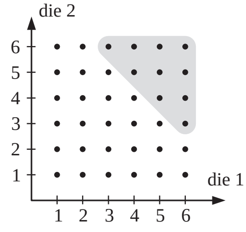
f P(no sixes)
$= \frac{25}{36}$



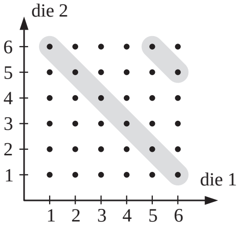
g P(sum of 7)
$= \frac{6}{36}$
$= \frac{1}{6}$



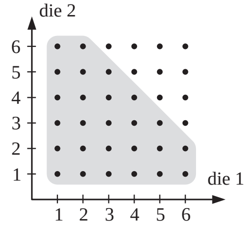
h P(sum > 8)
$= \frac{10}{36}$
$= \frac{5}{18}$



i P(sum of 7 or 11)
$= \dfrac{6+2}{36}$
$= \frac{2}{9}$



j P(sum no more than 8)
$=$ P(sum $\leqslant 8$)
$= \frac{26}{36}$
$= \frac{13}{18}$



EXERCISE 19E.1

1 **a** P(rains on any one day)
$= \frac{6}{7}$

b P(rains on 2 successive days)
$=$ P(R and R)
$= \frac{6}{7} \times \frac{6}{7}$
$= \frac{36}{49}$

c P(rains on 3 successive days)
$=$ P(R and R and R)
$= \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7}$ or $\frac{216}{343}$

2 **a** P(H, then H, then H)
$=$ P(H and H and H)
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b P(T, then H, then T)
$=$ P(T and H and T)
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

3 Let A be the event of photocopier A malfunctioning and
B be the event of photocopier B malfunctioning.

a P(both malfunction)
$=$ P(A and B)
$= 0.08 \times 0.12$
$= 0.0096$

b P(both work)
$=$ P(A' and B')
$= 0.92 \times 0.88$
$= 0.8096$

4 **a** P(they will be happy)
$=$ P(B, then G, then B, then G)
$=$ P(B and G and B and G)
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{16}$

b P(they will be unhappy)
$= 1 -$ P(they will be happy)
$= 1 - \frac{1}{16}$
$= \frac{15}{16}$

5 Let J be the event of Jiri hitting the target and B be the event of Benita hitting the target.

a P (both hit)
$= \text{P(JB)}$
$= 0.7 \times 0.8$
$= 0.56$

b P (both miss)
$= \text{P(J}'\text{B}')$
$= 0.3 \times 0.2$
$= 0.06$

c P (J hits and B misses)
$= \text{P(JB}')$
$= 0.7 \times 0.2$
$= 0.14$

d P (B hits and J misses) $= \text{P(BJ}')$
$= 0.8 \times 0.3$
$= 0.24$

6 Let H be the event the archer hits the target.
$\therefore$ $\text{P(H)} = \frac{2}{5}$, $\text{P(H}') = \frac{3}{5}$

a P (3 hits)
$= \text{P(HHH)}$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
$= \frac{8}{125}$

b P (2 hits then a miss)
$= \text{P(HHH}')$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}$
$= \frac{12}{125}$

c P (all misses)
$= \text{P(H}'\text{H}'\text{H}')$
$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
$= \frac{27}{125}$

EXERCISE 19E.2

1 **a** P (both red)
$= \text{P(RR)}$
$= \frac{7}{10} \times \frac{6}{9}$
$= \frac{7}{15}$

b P (GR)
$= \frac{3}{10} \times \frac{7}{9}$
$= \frac{7}{30}$

c P (a green and a red)
$= \text{P(GR or RG)}$
$= \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9}$
$= \frac{7}{15}$

2 **a** P (all strawberry creams)
$= \text{P(SSS)}$
$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$
$= \frac{14}{55}$

b P (none is a strawberry cream)
$= \text{P(S}'\text{S}'\text{S}')$
$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$
$= \frac{1}{55}$

3 **a** P (wins first prize)
$= \frac{3}{100}$

b P (wins 1st and 2nd)
$= \text{P(WW)}$
$= \frac{3}{100} \times \frac{2}{99}$
$\doteqdot 0.000\,606$

c P (wins all 3)
$= \text{P(WWW)}$
$= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$
$\doteqdot 0.000\,006\,18$

d P (wins none of them)
$= \text{P(W}'\text{W}'\text{W}')$
$= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$
$\doteqdot 0.912$

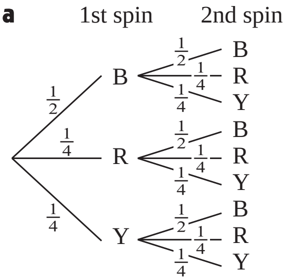
4 **a** P (contains the captain)
$= \text{P(CC}'\text{C}' \text{ or C}'\text{CC}' \text{ or C}'\text{C}'\text{C)}$
$= \frac{1}{7} \times \frac{6}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{1}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{5}{6} \times \frac{1}{5}$
$= 3\left(\dfrac{30}{7 \times 6 \times 5}\right)$
$= \frac{3}{7}$

b P (does not contain captain or vice captain)
$= \text{P(OOO)}$
$= \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5}$
$= \frac{60}{210}$
$= \frac{2}{7}$

$\therefore$ P (does contain captain or vice captain)
$= 1 - \frac{2}{7}$
$= \frac{5}{7}$

EXERCISE 19F

1 **a**



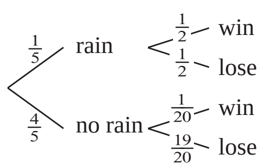
b P (both black)
$= \text{P}(\text{BB})$
$= \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{4}$

c P (both yellow)
$= \text{P}(\text{YY})$
$= \frac{1}{4} \times \frac{1}{4}$
$= \frac{1}{16}$

d P (both different)
$= \text{P}(\text{BR or BY or RB or RY or YB or YR})$
$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4}$
$= \frac{4}{8} + \frac{2}{16}$
$= \frac{5}{8}$

e P (B appears on either spin)
$= \text{P}(\text{BB or BR or BY or RB or YB})$
$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}$
$= 4(\frac{1}{8}) + \frac{1}{4}$
$= \frac{3}{4}$

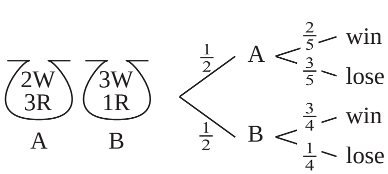
2



P (M wins)
$= \text{P}(\text{rain and win or no rain and win})$
$= \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{1}{20}$
$= \frac{1}{10} \times \frac{10}{10} + \frac{4}{100}$
$= \frac{14}{100}$
$= \frac{7}{50}$

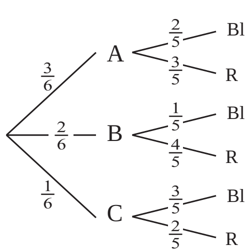
3 P (next is spoiled) $=$ P (from A and spoiled or from B and spoiled)
$= 0.4 \times 0.05 + 0.6 \times 0.02$
$= 0.020 + 0.012$
$= 0.032 \quad (3.2\%)$

4



P (red)
$= \text{P}(\text{A and red or B and red})$
$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{4}$
$= \frac{3}{10} + \frac{1}{8}$
$= \frac{17}{40}$

5



a P (blue) $=$ P (A and Bl or B and Bl or C and Bl)
$= \frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5}$
$= \frac{11}{30}$

b P (red) $= 1 -$ P (blue)
$= 1 - \frac{11}{30}$
$= \frac{19}{30}$

EXERCISE 19G

1

2P
5G

a P (different colours)
$= \text{P (PG or GP)}$
$= \frac{2}{7} \times \frac{5}{7} + \frac{5}{7} \times \frac{2}{7}$
$= \frac{20}{49}$

b P (different colours)
$= \text{P (PG or GP)}$
$= \frac{2}{7} \times \frac{5}{6} + \frac{5}{7} \times \frac{2}{6}$
$= \frac{20}{42}$
$= \frac{10}{21}$

2 **a** P (both odd)
$= \text{P (odd and odd)}$
$= \frac{3}{5} \times \frac{2}{4}$
$= \frac{3}{10}$

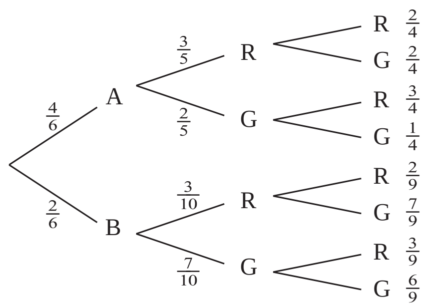
b P (both even)
$= \text{P (even and even)}$
$= \frac{2}{5} \times \frac{1}{4}$
$= \frac{1}{10}$

c P (one odd and other even)
$= 1 - \text{P (both odd)} - \text{P (both even)}$
$= 1 - \frac{3}{10} - \frac{1}{10}$
$= \frac{6}{10}$
$= \frac{3}{5}$

3

3R
2G
A

3R
7G
B

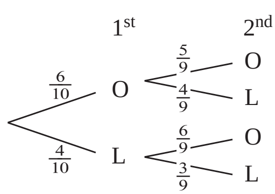


a P (both green)
$= \text{P (AGG or BGG)}$
$= \frac{4}{6} \times \frac{2}{5} \times \frac{1}{4} + \frac{2}{6} \times \frac{7}{10} \times \frac{6}{9}$
$= \frac{1}{15} + \frac{7}{45}$
$= \frac{3}{45} + \frac{7}{45}$
$= \frac{10}{45}$
$= \frac{2}{9}$

b P (different in colour)
$= 1 - \text{P (both green)} - \text{P (both red)}$
$= 1 - \frac{2}{9} - \text{P (ARR or BRR)}$
$= \frac{7}{9} - (\frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{2}{6} \times \frac{3}{10} \times \frac{2}{9})$
$= \frac{7}{9} - (\frac{1}{5} + \frac{1}{45})$
$= \frac{7}{9} - (\frac{9}{45} + \frac{1}{45})$
$= \frac{7}{9} - \frac{2}{9}$
$= \frac{5}{9}$

4

6O
4L



a P (both O)
$= \frac{6}{10} \times \frac{5}{9}$
$= \frac{1}{3}$

b P (both L)
$= \frac{4}{10} \times \frac{3}{9}$
$= \frac{2}{15}$

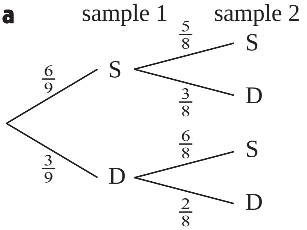
c P (OL)
$= \frac{6}{10} \times \frac{4}{9}$
$= \frac{4}{15}$

d P (LO)
$= \frac{4}{10} \times \frac{6}{9}$
$= \frac{4}{15}$

$\frac{1}{3} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15}$
$= \frac{5}{15} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15}$
$= \frac{15}{15}$ which is 1

The answer must be 1 as the four categories **a**, **b**, **c**, **d** are all the possibilities that could occur.

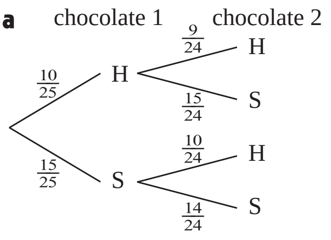
5 **a**



b P (both have two yolks)
$= \text{P (DD)}$
$= \frac{3}{9} \times \frac{2}{8}$
$= \frac{1}{12}$

c P (both have one yolk)
$= \text{P (SS)}$
$= \frac{6}{9} \times \frac{5}{8}$
$= \frac{10}{24}$
$= \frac{5}{12}$

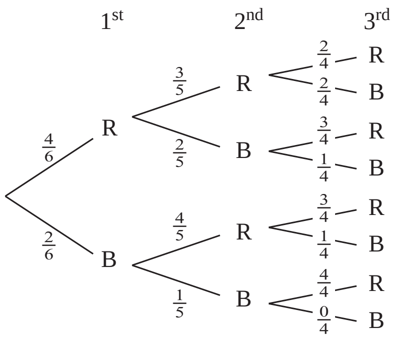
6 **a**



b P (both hard)
$= \text{P (HH)}$
$= \frac{10}{25} \times \frac{9}{24}$
$= \frac{3}{20}$

c P (both soft)
$= \text{P (SS)}$
$= \frac{15}{25} \times \frac{14}{24}$
$= \frac{7}{20}$

7

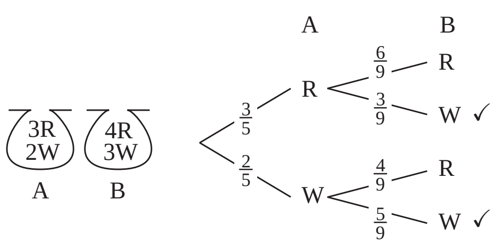


a P (all red)
$= \text{P (RRR)}$
$= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4}$
$= \frac{1}{5}$

b P (only two are red)
$= \text{P (RRB or RBR or BRR)}$
$= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{4}{6} \times \frac{2}{5} \times \frac{3}{4} + \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4}$
$= 3 \times \left(\frac{24}{6\times5\times4}\right)$
$= \frac{3}{5}$

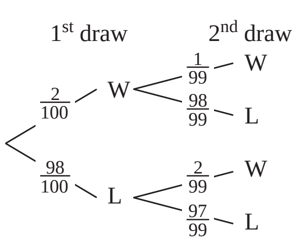
c P (at least two are red)
$= \text{P (all red or only two are red)}$
$= \frac{1}{5} + \frac{3}{5}$ {from **a** and **b**}
$= \frac{4}{5}$

8



P (marble from B is W) $= \text{P (RW or WW)}$
$= \frac{3}{5} \times \frac{3}{9} + \frac{2}{5} \times \frac{5}{9}$
$= \frac{19}{45}$
{paths ticked}

9



a P (wins both)
$= \text{P (WW)}$
$= \frac{2}{100} \times \frac{1}{99}$
$\doteqdot 0.000\,202$

b P (wins neither)
$= \text{P (LL)}$
$= \frac{98}{100} \times \frac{97}{99}$
$\doteqdot 0.960$

c $\quad \text{P(wins at least one prize)} = 1 - \text{P(wins neither)}$
$= 1 - \frac{98}{100} \times \frac{97}{99}$
$\doteqdot 0.0398$

EXERCISE 19H

1 **a** The rule is: *add* the two terms directly above the new row with 1s at either end.
b 1 7 21 35 35 21 7 1 **c** 1 7 21 35 35 21 7 1

2

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\ 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \end{array}$$

a $C_2^3 = 3$ **b** $C_5^7 = 21$
c $C_2^5 = 10$ **d** $C_1^8 = 8$
e $C_0^4 = 1$ **f** $C_6^6 = 1$

3 **a** $C_5^9 = 126$ **b** $C_3^{14} = 364$ **c** $C_1^{40} = 40$ **d** $C_2^3 = 3$
e $C_{10}^{40} = 847\,660\,528$ **f** $C_{20}^{40} \doteqdot 1.3784 \times 10^{11}$

4 **a**

HHHHH	HHHHT	HHHTT	HHTTT	HTTTT	TTTTT
	HHHTH	HHTHT	HTHTT	THTTT	
	HHTHH	HTHHT	THHTT	TTHTT	
	HTHHH	THHHT	HTTHT	TTTHT	
	THHHH	HHTTH	THTHT	TTTTH	
		HTHTH	TTHHT		
		THHTH	HTTTH		
		HTTHH	THTTH		
		THTHH	TTHTH		
		TTHHH	TTTHH		

b **i** 1 **ii** 5 **iii** 10 **iv** 10 **v** 5 **vi** 1

5 **a** $C_{10}^{18} = 43\,758$ **b** $C_{14}^{23} = 817\,190$

6 **a**
$$\begin{aligned} &(p+q)^3 \\ &= (p+q)(p+q)^2 \\ &= (p+q)(p^2+2pq+q^2) \\ &= p^3+2p^2q+pq^2 \\ &\qquad + p^2q+2pq^2+q^3 \\ &= p^3+3p^2q+3pq^2+q^3 \end{aligned}$$
Coefficients are: 1, 3, 3, 1

b
$$\begin{aligned} &(p+q)^4 \\ &= (p+q)(p+q)^3 \\ &= (p+q)(p^3+3p^2q+3pq^2+q^3) \quad \{\text{from } \mathbf{a}\} \\ &= p^4+3p^3q+3p^2q^2+pq^3 \\ &\qquad + p^3q+3p^2q^2+3pq^3+q^4 \\ &= p^4+4p^3q+6p^2q^2+4pq^3+q^4 \end{aligned}$$
Coefficients are: 1, 4, 6, 4, 1

c From **a** and **b** we notice that for $(p+q)^n$ the coefficients are the nth row of Pascal's triangle. The powers of p decrease by 1 as the powers of q increase by 1.

i $p^5+5p^4q+10p^3q^2+10p^2q^3+5pq^4+q^5$

ii $p^6+6p^5q+15p^4q^2+20p^3q^3+15p^2q^4+6pq^5+q^6$

d
$$\begin{aligned} C_0^4 + C_1^4\,p^3q + C_2^4\,p^2q^2 + C_3^4\,pq^3 + C_4^4\,q^4 &= (p+q)^4 \quad \{\text{Binomial expansion in reverse}\} \\ &= 1^4 \\ &= 1 \end{aligned}$$

7 **a** $(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b $\text{P(3 heads)} = 4p^3q$

$= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ $\{\text{as } p = q = \frac{1}{2}\}$

$= \frac{1}{4}$

8 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b P(4H and 1T)

$= 5p^4q$

$= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$

$= \frac{5}{32}$

c P(2H and 3T)

$= 10p^2q^3$

$= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$

$= \frac{10}{32}$

$= \frac{5}{16}$

9 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b $\text{P(S)} = \frac{2}{3}$, $\text{P(S}') = \frac{1}{3}$ $\quad$ S$'$ represents a non-strawberry cream (or an almond centre)

i P(all S)

$= \left(\frac{2}{3}\right)^4$

$= \frac{16}{81}$

ii P(two of each)

$= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$

$= \frac{8}{27}$

iii P(at least 2 strawberry creams)

= P(all S or 3S, 1T or 2S, 2T)

$= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$

$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$

$= \frac{72}{81}$

$= \frac{8}{9}$

10 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b **i** P(2 ‘flat backs’)

= P(2Fs and 3F$'$s)

$= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

$= \frac{135}{512}$

ii P(at least 3 are ‘flat backs’)

= P(3F, 2F$'$ or 4F, 1F$'$ or 5F)

$= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

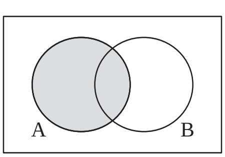
$= \dfrac{90+15+1}{4^5}$

$= \frac{106}{1024}$

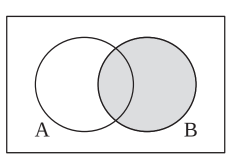
$= \frac{53}{512}$

EXERCISE 19I

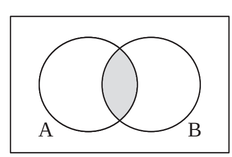
1 **a**



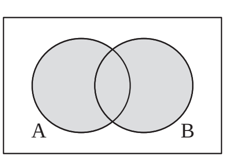
b



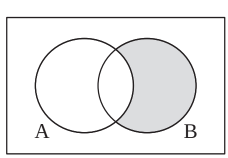
c



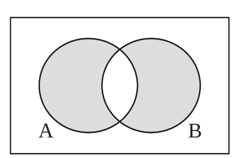
d



e



f



2 **a** Total number in the class $= 3+4+5+17 = 29$

b Number who study both $= 17$ {the intersection}

c Number who study at least one $= 5+17+4 = 26$ {the union}

d Number who study only Chem. $= 5$

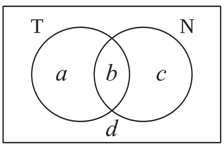
3 **a** Total number in the survey $= 37+9+15+4 = 65$

b Number who liked both $= 9$ {the intersection}

c Number who liked neither $= 4$

d Number who liked exactly one $= 37+15 = 52$

4

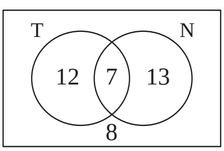


T represents those playing tennis
N represents those playing netball

$$\begin{aligned}\therefore\quad a+b+c+d &= 40\\ a+b &= 19\\ b+c &= 20\\ d &= 8\end{aligned}$$

$$\begin{aligned}\text{So,}\quad a+b+c &= 32\\ \therefore\quad 19+c &= 32 \quad\text{and}\quad a+20=32\\ \therefore\quad c &= 13 \quad\text{and}\quad a=12\end{aligned}$$

$$\begin{aligned}\text{Hence,}\quad 12+b+13+8 &= 40\\ \therefore\quad b &= 7\end{aligned}$$



a P (plays tennis)

$= \dfrac{12+7}{40}$

$= \frac{19}{40}$

b P (does not play netball)

$= \dfrac{12+8}{40}$

$= \frac{1}{2}$

c P (plays at least one)

$= \dfrac{12+7+13}{40}$

$= \frac{32}{40}$

$= \frac{4}{5}$

d P (plays one and only one)

$= \dfrac{12+13}{40}$

$= \frac{25}{40}$

$= \frac{5}{8}$

e P (plays netball, but not tennis)

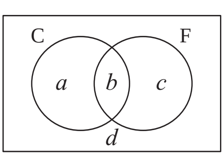
$= \frac{13}{40}$

f P (plays tennis given plays netball)

$= \dfrac{7}{7+13}$

$= \frac{7}{20}$

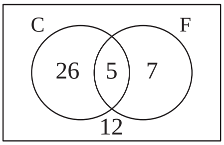
5



C represents men who gave chocolates
F represents men who gave flowers

$$\begin{aligned}\therefore\quad a+b+c+d &= 50\\ a+b &= 31\\ b+c &= 12\\ b &= 5\end{aligned}$$

$$\begin{aligned}\text{Thus}\quad c=7,\quad a=26\quad\text{and}\quad 26+5+7+d &= 50\\ \therefore\quad d &= 12\end{aligned}$$



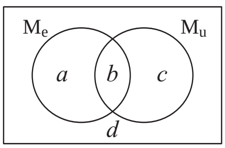
a P (C or F)
$= \dfrac{26+5+7}{50}$
$= \frac{38}{50}$
$= \frac{19}{25}$

b P (C but not F)
$= \frac{26}{50}$
$= \frac{13}{25}$

c P (neither C nor F)
$= \frac{12}{50}$
$= \frac{6}{25}$

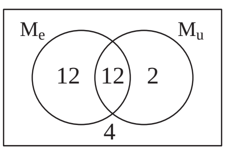
d P (F given that C′)
$= \dfrac{7}{7+12}$
$= \frac{7}{19}$

6



$$\begin{aligned} a+b+c+d &= 30 \\ a+b &= 24 \\ b &= 12 \\ a+b+c &= 26 \end{aligned}$$

$$\begin{aligned} \therefore\quad 26+d &= 30 \quad \text{i.e.,}\quad d=4 \\ 24+c &= 26 \quad \text{i.e.,}\quad c=2 \\ \text{and}\quad a+12+2 &= 26 \quad \therefore\quad a=12 \end{aligned}$$



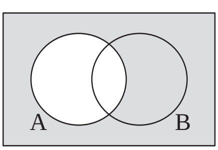
a P (Mu)
$= \frac{14}{30}$
$= \frac{7}{15}$

b P (Mu, but not Me)
$= \frac{2}{30}$
$= \frac{1}{15}$

c P (neither Mu nor Me)
$= \frac{4}{30}$
$= \frac{2}{15}$

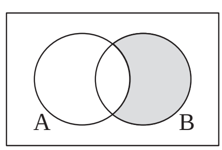
d P (Me given Mu)
$= \frac{12}{14}$
$= \frac{6}{7}$

7 **a**



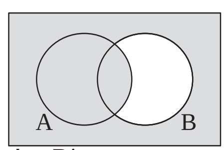
A′

b



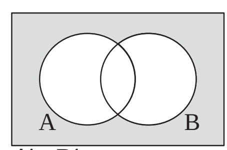
$A' \cap B$

c



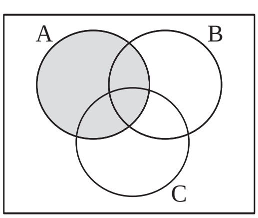
$A \cup B'$

d

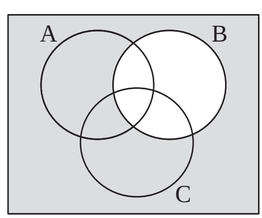


$A' \cap B'$

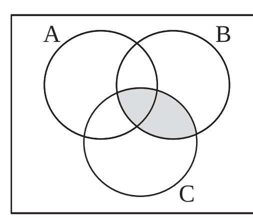
8 **a**



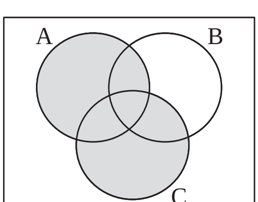
b



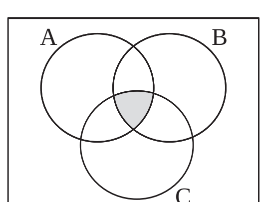
c



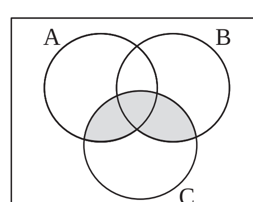
d



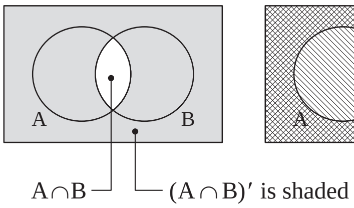
e

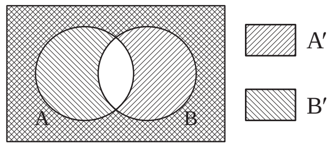


f



9 **a**

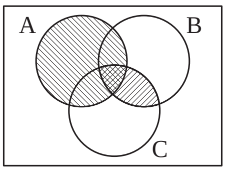


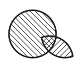


So $A' \cup B'$ is the region containing either type of shading.

Thus, as the regions are the same, $(A \cap B)' = A' \cup B'$ is verified.

b

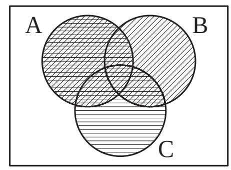


$A \cup (B \cap C)$ consists of the shaded region 

As the two regions are identical

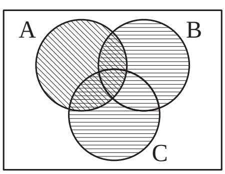
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

is verified.

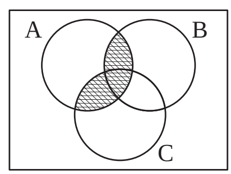


$(A \cup B) \cap (A \cup C)$ consists of the 'double shaded' region.

c



$A \cap (B \cup C)$ consists of the double shaded region 



$(A \cap B) \cup (A \cap C)$ consists of the region shaded. (all forms  and )

As the regions are identical, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is verified.

10 **a** $A = \{7, 14, 21, 28, 35, ..., 98\}$
$B = \{5, 10, 15, 20, 25, ..., 95\}$

i as $98 = 7 \times 14$, $n(A) = 14$ **ii** as $95 = 5 \times 19$, $n(B) = 19$

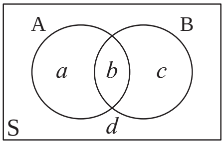
iii $A \cap B = \{35, 70\}$ $\therefore$ $n(A \cap B) = 2$

iv $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$

$\therefore$ $n(A \cup B) = 31$

b $n(A) + n(B) - n(A \cap B)$
$= 14 + 19 - 2$
$= 31$
$= n(A \cup B)$ ✓

11



$$\begin{aligned} & n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B}) \\ &= a+b+b+c-b \\ &= a+b+c \\ &= n(\text{A} \cup \text{B}) \end{aligned}$$

12 **a** **i** P(B) $= \dfrac{n(\text{B})}{n(\text{S})} = \dfrac{b+c}{a+b+c+d}$

ii P(A and B) $= \dfrac{n(\text{A} \cap \text{B})}{n(\text{S})} = \dfrac{b}{a+b+c+d}$

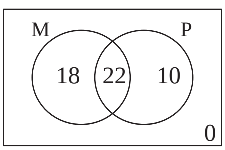
iii P(A or B) $= \dfrac{n(\text{A} \cup \text{B})}{n(\text{S})} = \dfrac{a+b+c}{a+b+c+d}$

iv P(A) + P(B) − P(A and B) $= \dfrac{a+b+b+c-b}{a+b+c+d} = \dfrac{a+b+c}{a+b+c+d}$

b P(A or B) = P(A) + P(B) − P(A and B) {using **iii** and **iv**}

EXERCISE 19J

1 **a**

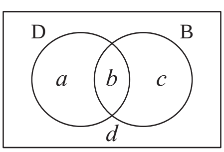


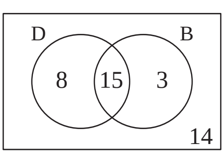
So 22 study both.

b **i** P(M but not P) $= \frac{18}{50} = \frac{9}{25}$

ii P(P given M) $= \dfrac{22}{18+22} = \frac{22}{40} = \frac{11}{20}$

2





$$\begin{aligned} a+b+c+d &= 40 \quad \text{.... (1)} \\ a+b &= 23 \quad \text{.... (2)} \\ b+c &= 18 \quad \text{.... (3)} \\ a+b+c &= 26 \quad \text{.... (4)} \end{aligned}$$

$\therefore\ d = 14$ {using (1) and (4)}

$23 + c = 26$ and $a + 18 = 26$

$\therefore\ c = 3$ and $a = 8$

Thus $b = 18 - c = 15$

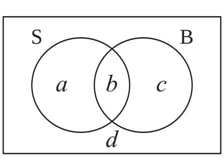
a P(D and B) $= \frac{15}{40} = \frac{3}{8}$

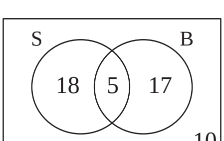
b P(neither D nor B) $= \frac{14}{40} = \frac{7}{20}$

c P(D, but not B) $= \frac{8}{40} = \frac{1}{5}$

d P(B given D) $= \frac{15}{23}$

3





$$\begin{aligned} a+b+c+d &= 50 \\ a+b &= 23 \\ b+c &= 22 \\ b &= 5 \end{aligned}$$

$\therefore\ c = 17,\ a = 18$

and $18 + 5 + 17 + d = 50$

$\therefore\ d = 10$

a P(not B) $= \text{P}(\text{B}') = \frac{28}{50} = \frac{14}{25}$

b P(B or S) $= \dfrac{18+5+17}{50} = \frac{40}{50} = \frac{4}{5}$

c P(neither B nor S) $= \frac{10}{50} = \frac{1}{5}$

d P(B, given S)

$= \dfrac{5}{18+5}$

$= \frac{5}{23}$

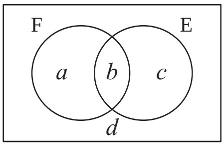
e P(S, given B′)

$= \dfrac{18}{18+10}$

$= \frac{18}{28}$

$= \frac{9}{14}$

4



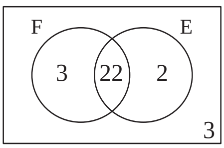
$$\begin{aligned} a+b+c+d &= 30 \\ a+b &= 25 \\ b+c &= 24 \\ d &= 3 \end{aligned}$$

$\therefore\ a+b+c = 27$

and so, $25+c = 27$ and $a+24 = 27$

$\therefore\ c = 2$ and $a = 3$

Also, $b = 25-a = 22$



a P(E, given F)

$= \dfrac{22}{3+22}$

$= \frac{22}{25}$

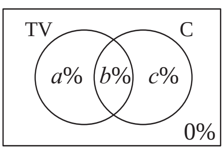
b P(F, given E′)

$= \dfrac{3}{3+3}$

$= \frac{3}{6}$

$= \frac{1}{2}$

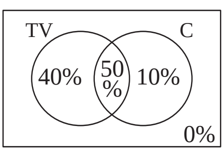
5



$$\begin{aligned} a+b+c &= 100 \\ a+b &= 90 \\ b+c &= 60 \end{aligned}$$

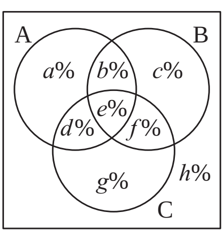
$\therefore\ c = 10$ and $a = 40$

$\therefore\ b = 50$



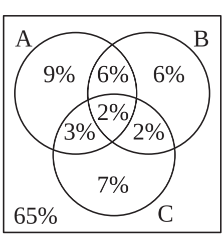
$\text{P(TV, given C)} = \dfrac{50}{50+10} = \frac{5}{6}$

6



$$\begin{aligned} a+b+c+d+e+f+g+h &= 100 \\ a+b+d+e &= 20 \\ b+c+e+f &= 16 \\ d+e+f+g &= 14 \\ b+e &= 8 \\ d+e &= 5 \\ e+f &= 4 \\ e &= 2 \end{aligned}$$

i.e., $e=2$, $f=2$, $d=3$, $b=6$, $\left\{\begin{aligned} a+6+3+2 &= 20 \\ 6+c+2+2 &= 16 \\ 3+2+2+g &= 14 \end{aligned}\right.$ $\therefore$ $\left\{\begin{aligned} a &= 9 \\ c &= 6 \\ g &= 7 \end{aligned}\right.$



a P(none)

$= \frac{65}{100}$

$= \frac{13}{20}$

b P(at least one)

$= 1 - \text{P(none)}$

$= 1 - \frac{13}{20}$

$= \frac{7}{20}$

c P(exactly one)

$= \dfrac{9+6+7}{100}$

$= \frac{22}{100}$

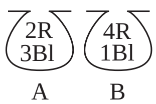
$= \frac{11}{50}$

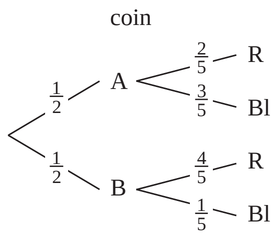
d $\mathrm{P(A\ or\ B)} = \frac{9+6+6+3+2+2}{100} = \frac{28}{100} = \frac{7}{25}$

e $\mathrm{P(A,\ given\ at\ least\ one)} = \frac{9+6+2+3}{35} = \frac{20}{35} = \frac{4}{7}$

f $\mathrm{P(C,\ given\ A\ or\ B\ or\ both)} = \frac{3+2+2}{9+6+6+3+2+2} = \frac{7}{28} = \frac{1}{4}$

7

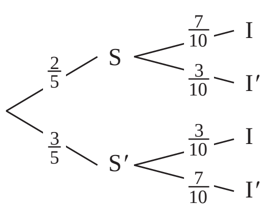




a $\mathrm{P(R)} = \frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{4}{5} = \frac{6}{10} = \frac{3}{5}$

b $\mathrm{P(B|R)} = \frac{\mathrm{P(B\cap R)}}{\mathrm{P(R)}} = \frac{\frac{1}{2}\times\frac{4}{5}}{\frac{3}{5}} = \frac{2}{3}$

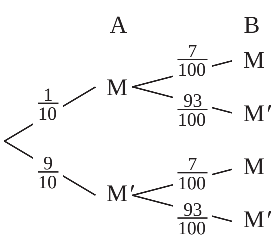
8



a $\mathrm{P(I)} = \frac{2}{5}\times\frac{7}{10}+\frac{3}{5}\times\frac{3}{10} = \frac{23}{50}$ (or 0.46)

b $\mathrm{P(S|I)} = \frac{\mathrm{P(S\cap I)}}{\mathrm{P(I)}} = \frac{\frac{2}{5}\times\frac{7}{10}}{\frac{23}{50}} = \frac{14}{23}$

9



$\mathrm{P(B\,|\,at\ least\ one\ malfunctions)}$

$$= \frac{\mathrm{P(B\cap at\ least\ one\ malfunctions)}}{\mathrm{P(at\ least\ one\ malfunctions)}}$$

$$= \frac{\frac{1}{10}\times\frac{7}{100}+\frac{9}{10}\times\frac{7}{100}}{\frac{1}{10}\times\frac{7}{100}+\frac{1}{10}\times\frac{93}{100}+\frac{9}{10}\times\frac{7}{100}}$$

$$= \frac{7+63}{7+93+63}$$

$$= \frac{70}{163}$$

10 $\mathrm{P(B)} = 0.5,\quad \mathrm{P(G)} = 0.6,\quad \mathrm{P(G\,|\,B)} = 0.9$

a P(both eat)

$= \mathrm{P(B\cap G)}$

$= \mathrm{P(G\,|\,B)} \times \mathrm{P(B)}$ $\quad \left\{\text{as}\quad \mathrm{P(G\,|\,B)} = \frac{\mathrm{P(G\cap B)}}{\mathrm{P(B)}}\right\}$

$= 0.9\times 0.5$

$= 0.45$

b $\mathrm{P(B\,|\,G)} = \frac{\mathrm{P(B\cap G)}}{\mathrm{P(G)}} = \frac{0.45}{0.6} = 0.75$

c P(at least one eats)

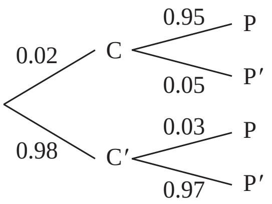
$= \mathrm{P(B\cup G)}$

$= \mathrm{P(B)} + \mathrm{P(G)} - \mathrm{P(B\cap G)}$

$= 0.5 + 0.6 - 0.45$

$= 0.65$

11



a $\text{P}(\text{P})$
$= 0.02 \times 0.95 + 0.98 \times 0.03$
$= 0.0484$

b $\text{P}(\text{C} \mid \text{P})$
$= \dfrac{\text{P}(\text{C} \cap \text{P})}{\text{P}(\text{P})}$
$= \dfrac{0.02 \times 0.95}{0.0484}$
$\doteqdot 0.3926$

12 The coins are H, H T, T and H, T.

Any one of these 6 faces could be seen uppermost, $\therefore$ P(falls H) $= \frac{3}{6} = \frac{1}{2}$

$$\text{Now} \quad \text{P(HH coin} \mid \text{falls H)} = \frac{\text{P(HH coin} \cap \text{falls H)}}{\text{P(falls H)}}$$
$$= \frac{\text{P(HH)}}{\text{P(falls H)}}$$
$$= \frac{\frac{1}{3}}{\frac{1}{2}}$$
$$= \frac{2}{3}$$

EXERCISE 19K

1 $\text{P}(\text{R} \cap \text{S})$
$= \text{P}(\text{R}) + \text{P}(\text{S}) - \text{P}(\text{R} \cup \text{S})$
$= 0.4 + 0.5 - 0.7$
$= 0.2$

Also, $\text{P}(\text{R}) \times \text{P}(\text{S})$
$= 0.4 \times 0.5$
$= 0.2$

So, $\text{P}(\text{R} \cap \text{S}) = \text{P}(\text{R}) \times \text{P}(\text{S}) \Rightarrow$ R and S are independent events.

2 **a** $\text{P}(\text{A} \cap \text{B})$
$= \text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A} \cup \text{B})$
$= \frac{2}{5} + \frac{1}{3} - \frac{1}{2}$
$= \frac{7}{30}$

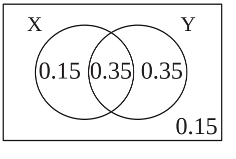
b $\text{P}(\text{B}|\text{A})$
$= \dfrac{\text{P}(\text{B} \cap \text{A})}{\text{P}(\text{A})}$
$= \dfrac{\frac{7}{30}}{\frac{2}{5}}$
$= \frac{7}{12}$

c $\text{P}(\text{A}|\text{B})$
$= \dfrac{\text{P}(\text{A} \cap \text{B})}{\text{P}(\text{B})}$
$= \dfrac{\frac{7}{30}}{\frac{1}{3}}$
$= \frac{7}{10}$

A and B are not independent as $\text{P}(\text{A} \cap \text{B}) = \frac{7}{30}$ whereas $\text{P}(\text{A}) \times \text{P}(\text{B}) = \frac{2}{15}$ i.e., $\text{P}(\text{A} \cap \text{B}) \neq \text{P}(\text{A}) \times \text{P}(\text{B})$

3 **a** As X and Y are independent
$\text{P}(\text{X} \cap \text{Y}) = \text{P}(\text{X}) \times \text{P}(\text{Y})$
$= 0.5 \times 0.7$
$= 0.35$
i.e., P (both X and Y) $= 0.35$

b P(X or Y)
$= \text{P}(\text{X} \cup \text{Y})$
$= \text{P}(\text{X}) + \text{P}(\text{Y}) - \text{P}(\text{X} \cap \text{Y})$
$= 0.5 + 0.7 - 0.35$
$= 0.85$



c P (neither X nor Y)
$= 0.15$

d P (X but not Y)
$= 0.15$

e $\text{P}(\text{X}|\text{Y}) = \dfrac{\text{P}(\text{X} \cap \text{Y})}{\text{P}(\text{Y})} = \dfrac{0.35}{0.70} = \frac{1}{2}$

4 P (at least one solves it)
$= 1 - \text{P (no-one solves it)}$
$= 1 - \text{P}(\text{A}' \text{ and } \text{B}' \text{ and } \text{C}')$
$= 1 - \frac{2}{5} \times \frac{1}{3} \times \frac{1}{2}$
$= 1 - \frac{1}{15}$
$= \frac{14}{15}$

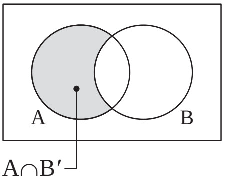
5 **a** $\text{P (at least one 6)} = 1 - \text{P(no 6s)}$
$= 1 - \text{P}(6' \text{ and } 6' \text{ and } 6')$
$= 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$
$= 1 - \frac{125}{216}$
$= \frac{91}{216}$

b $\text{P (at least one 6 in } n \text{ throws)} = 1 - (\frac{5}{6})^n$

So we want $1 - (\frac{5}{6})^n > 0.99$
$\therefore \quad -(\frac{5}{6})^n > -0.01$
$\therefore \quad (\frac{5}{6})^n < 0.01$
$\therefore \quad n \log(\frac{5}{6}) < \log(0.01)$
$\therefore \quad n > \dfrac{\log(0.01)}{\log(\frac{5}{6})}$
{as $\log(\frac{5}{6}) < 0$}
$\therefore \quad n > 25.2585....$
i.e., $n = 26$

6 A and B are independent $\Rightarrow$ $\text{P(A} \cap \text{B)} = \text{P(A) P(B)}$ ...... (1)

Now $\text{P}(\text{A} \cap \text{B}') = \text{P(A} \cup \text{B)} - \text{P(B)}$ {see diagram}
$= \text{P(A)} + \text{P(B)} - \text{P(A} \cap \text{B)} - \text{P(B)}$
$= \text{P(A)} - \text{P(A} \cap \text{B)}$
$= \text{P(A)} - \text{P(A) P(B)}$ {from (1)}
$= \text{P(A)}[1 - \text{P(B)}]$
$= \text{P(A)} \times \text{P}(\text{B}')$



$\therefore$ A and B′ are also independent.

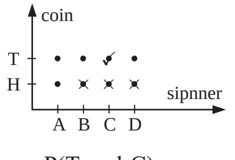
REVIEW SET 19A

1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB,
BACD, BADC, BCAD, BCDA, BDAC, BDCA,
CABD, CADB, CBAD, CBDA, CDAB, CDBA,
DABC, DACB, DBAC, DBCA, DCAB, DCBA

a There are 24 possible orderings.
$\therefore$ P (A is next to C)
$= \frac{12}{24}$ {12 have A next to C}
$= \frac{1}{2}$

b P (exactly one person between A and C)
$= \frac{8}{24}$ {8 have one person between A and C}
$= \frac{1}{3}$

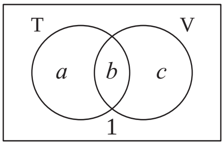
2



a Consonants are B, C and D
$\therefore$ P(H and a consonant)
$= \frac{3}{8}$ {those with $\times$}

b P(T and C)
$= \frac{1}{8}$ {those with a ✓}

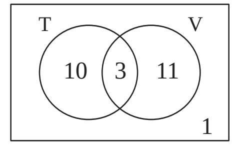
c P(T or vowel)
$= \text{P(T or A)}$
$= \text{P(T)} + \text{P(A)} - \text{P(T and A)}$
$= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$
$= \frac{5}{8}$

3 

$$\begin{aligned} a+b+c &= 24 \\ a+b &= 13 \\ b+c &= 14 \\ \text{Also} \quad b &= 13-a \\ &= 3 \end{aligned}$$

$\therefore \quad 13+c = 24 \quad$ and $\quad a+14 = 24$

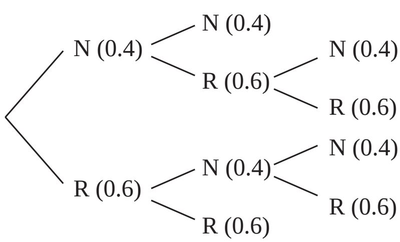
$\therefore \quad c = 11 \quad$ and $\quad a = 10$



a P(T and V)
$= \frac{3}{25}$

b P(at least one)
$= 1 - \text{P(neither)}$
$= 1 - \frac{1}{25}$
$= \frac{24}{25}$

c $\text{P}(\text{V} \mid \text{T}')$
$= \dfrac{11}{11+1}$
$= \frac{11}{12}$

4 

P(Niklas wins)
$= (0.4)(0.4) + (0.4)(0.6)(0.4) + (0.6)(0.4)(0.4)$
$= 0.352$

5 $\text{P(M)} = \frac{3}{5}, \quad \text{P(W)} = \frac{2}{3}$

a P(M and W)
$= \frac{3}{5} \times \frac{2}{3}$ {assuming independence}
$= \frac{2}{5}$

b P(at least one)
$= \text{P(M or W)}$
$= \text{P(M)} + \text{P(W)} - \text{P(M and W)}$
$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$
$= \frac{13}{15}$

c $\text{P(M}'$ and W)
$= (1 - \frac{3}{5}) \times \frac{2}{3}$
$= \frac{2}{5} \times \frac{2}{3}$
$= \frac{4}{15}$

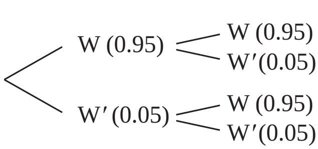
6 A repetition of 5 independent events $\therefore$ binomial model applies.
$\text{P(M)} = \frac{4}{5}, \quad \text{P(M}') = \frac{1}{5}$

a P(M wins 3 games)
$= C^5_3 (\frac{4}{5})^3 (\frac{1}{5})^2$
$= 0.2048$
$\doteqdot 0.205$

b P(M wins 4 or 5 games)
$= C^5_4 (\frac{4}{5})^4 (\frac{1}{5})^1 + C^5_5 (\frac{4}{5})^5$
$\doteqdot 0.737$

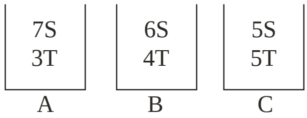
7 **a** P(wins first 3 prizes)
$= \text{P(WWW)}$
$= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$
$\doteqdot 1.93 \times 10^{-7}$

b P(wins at least one of the 3 prizes)
$= 1 - \text{P(wins none of them)}$
$= 1 - \text{P(W}'\text{W}'\text{W}')$
$= 1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}$
$\doteqdot 0.0239$

8 

P(works on at least one day)
$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$
$= 0.9975$

9



$P(A) = \frac{3}{6}, \quad P(B) = \frac{2}{6}, \quad P(C) = \frac{1}{6}$

a $$\begin{aligned} P(S) &= P(\text{A and S or B and S or C and S}) \\ &= \tfrac{3}{6} \times \tfrac{7}{10} + \tfrac{2}{6} \times \tfrac{6}{10} + \tfrac{1}{6} \times \tfrac{5}{10} \\ &= \tfrac{38}{60} \\ &= \tfrac{19}{30} \end{aligned}$$

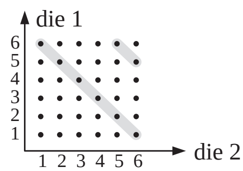
b $$\begin{aligned} P(B \mid S) &= \frac{P(B \cap S)}{P(S)} \\ &= \frac{\frac{2}{6} \times \frac{6}{10}}{\frac{38}{60}} \\ &= \tfrac{12}{38} \\ &= \tfrac{6}{19} \end{aligned}$$

REVIEW SET 19B

1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGB, GGGG

$$\begin{aligned} &P(\text{2B and 2G}) \\ &= \frac{6}{16} \longleftarrow \text{6 have 2B and 2G} \\ &= \tfrac{3}{8} \end{aligned}$$

2

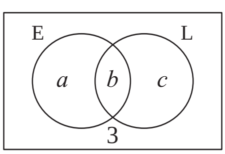


There are 36 possible outcomes.

a $$\begin{aligned} &P(\text{sum of 7 or 11}) \\ &= \tfrac{8}{36} \quad \{\text{those shaded}\} \\ &= \tfrac{2}{9} \end{aligned}$$

b $$\begin{aligned} &P(\text{sum of at least 8}) \\ &= \tfrac{1+2+3+4+5}{36} \\ &= \tfrac{15}{36} \\ &= \tfrac{5}{12} \end{aligned}$$

3



$$\begin{aligned} a + b + c &= 37 \\ a + b &= 22 \\ b + c &= 25 \end{aligned}$$

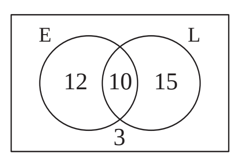
$\therefore \; 22 + c = 37$ and $a + 25 = 37$

$\therefore \; c = 15$ and $a = 12$

Hence, $b = 22 - a = 10$

a $$\begin{aligned} &P(\text{E and L}) \\ &= \tfrac{10}{40} \\ &= \tfrac{1}{4} \end{aligned}$$

b $$\begin{aligned} &P(\text{at least one}) \\ &= \tfrac{12+10+15}{40} \\ &= \tfrac{37}{40} \end{aligned}$$



c $P(E \mid L) = \frac{10}{15+10} = \frac{10}{25} = \frac{2}{5}$

4 Multiples of 6 are: 6, 12, 18, 24, ..., 96 $\leftarrow 6 \times 16$ i.e., 16 of them
Multiples of 8 are: 8, 16, 24, 32, ..., 88, 96 $\leftarrow 8 \times 12$ i.e., 12 of them
Multiples of 6 and 8 are: 24, 48, 72, 96 i.e., 4 of them

The integers could be 1, 2, 3, 4, ..., 99 {between 0 and 100} i.e., 99 of them

$$\begin{aligned} \therefore \; P(M_6 \text{ or } M_8) &= P(M_6) + P(M_8) - P(M_6 \text{ and } M_8) \\ &= \tfrac{16}{99} + \tfrac{12}{99} - \tfrac{4}{99} \\ &= \tfrac{24}{99} \\ &= \tfrac{8}{33} \end{aligned}$$

5 **a** P(both blue)
$= \text{P(BB)}$
$= \frac{5}{12} \times \frac{4}{11}$
$= \frac{5}{33}$

b P(both same colour)
$= \text{P(BB or RR or YY)}$
$= \frac{5}{12} \times \frac{4}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11}$
$= \frac{19}{66}$

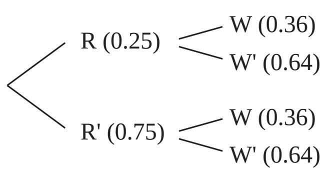
c P(at least one R)
$= 1 - \text{P(no reds)}$
$= 1 - \text{P}(\text{R}'\text{R}')$
$= 1 - \frac{9}{12} \times \frac{8}{11}$
$= 1 - \frac{6}{11}$
$= \frac{5}{11}$

d P(exactly one Y)
$= \text{P}(\text{YY}' \text{ or } \text{Y}'\text{Y})$
$= \frac{4}{12} \times \frac{8}{11} + \frac{8}{12} \times \frac{4}{11}$
$= \frac{16}{33}$

6 **a** Two events are independent if the occurrence of one does not influence the occurrence of the other. For A and B independent, $\text{P(A)} \times \text{P(B)} = \text{P(A and B)}$

b Two events, A and B, are disjoint if they have no common outcomes,
i.e., $\text{P(A and B)} = 0$ and so $\text{P(A or B)} = \text{P(A)} + \text{P(B)}$

7



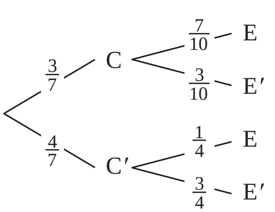
a P(W and R)
$= 0.25 \times 0.36$
$= 0.09$

b P(W or R)
$= \text{P(W)} + \text{P(R)} - \text{P(W and R)}$
$= 0.36 + 0.25 - 0.09$
$= 0.52$

or $\text{P(W or R)} = 1 - \text{P}(\text{W}'\text{R}')$
$= 1 - 0.64 \times 0.75$
$= 0.52$

8 $\text{P(A)} = 0.1$, $\text{P(B)} = 0.2$, $\text{P(C)} = 0.3$ $\therefore$ P(group solves it) $= \text{P(at least one solves it)}$
$= 1 - \text{P(no-one solves it)}$
$= 1 - \text{P}(\text{A}' \text{ and } \text{B}' \text{ and } \text{C}')$
$= 1 - (0.9 \times 0.8 \times 0.7)$
$= 0.496$

9



a $\text{P(E)} = \frac{3}{7} \times \frac{7}{10} + \frac{4}{7} \times \frac{1}{4}$
$= \frac{3}{10} + \frac{1}{7}$
$= \frac{31}{70}$

b $\text{P(C} \mid \text{E)} = \dfrac{\text{P(C and E)}}{\text{P(E)}}$
$= \dfrac{\frac{3}{7} \times \frac{7}{10}}{\frac{31}{70}}$
$= \frac{21}{31}$

10 **a** $$\left(\tfrac{3}{5} + \tfrac{2}{5}\right)^4 = \underbrace{\left(\tfrac{3}{5}\right)^4}_{4\text{B}} + \underbrace{4\left(\tfrac{3}{5}\right)^3\left(\tfrac{2}{5}\right)}_{3\text{B}\ 1\text{B}'} + \underbrace{6\left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^2}_{2\text{B}\ 2\text{B}'} + \underbrace{4\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)^3}_{1\text{B}\ 3\text{B}'} + \underbrace{\left(\tfrac{2}{5}\right)^4}_{4\text{B}'}$$

$\text{P(B)} = \frac{12}{20}$
$= \frac{3}{5}$
$\therefore \text{P}(\text{B}') = \frac{2}{5}$

b **i** P(2 Blue inks)
$= \text{P}(2\text{B and } 2\text{B}')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2$
$= \dfrac{6 \times 9 \times 4}{5^4}$
$= \frac{144}{625}$

ii P(at most 2 Blue inks)
$= \text{P}(2\text{B and } 2\text{B}' \text{ or } 1\text{B and } 3\text{B}' \text{ or } 4\text{B}')$
$= 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2 + 4\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$
$= \dfrac{6 \times 9 \times 4 + 4 \times 3 \times 8 + 16}{625}$
$= \frac{328}{625}$

Chapter 20

INTRODUCTION TO CALCULUS

EXERCISE 20A.1

1 **a** 67 beats/minute means that every minute Ozair's heart beats 67 times.

b Number of beats $= 67 \times 60 = 4020$ beats

2 **a** Number of words $= 14 \times 380$
$= 5320$ words

$\therefore$ rate $= \dfrac{8}{5320}$
$\doteqdot 0.0015$ errors/word

b errors/100 words $= \dfrac{8}{53.2}$
$= 0.15$

3 Paul's hourly rate $= \dfrac{\$148.20}{12}$
$= \$12.35$/hour

Marita's hourly rate $= \dfrac{\$157.95}{13}$
$= \$12.15$/hour

$\therefore$ Paul's rate is better

4 **a** Tyre wear $= 8.0 - 2.3$
$= 5.7$ mm

$\therefore$ wearing rate $= \dfrac{5.7}{32\,178}$ mm/km
$= 1.77 \times 10^{-4}$ mm/km

b wearing rate $= \dfrac{5.7}{3.2178}$
$= 1.77$ mm / $10\,000$ km

5 **a** The time for 11:43 am to 12:39 pm is 56 mins.

$\therefore$ speed $= \dfrac{71}{\frac{56}{60}}$ km/hour
$\doteqdot 76.07$ kmph

b speed $= 76.07 \times \left(\frac{1000}{3600}\right)$ m/s
$= 21.1$ m/s

EXERCISE 20A.2

1 **a** Average speed from Tailem Bend to Nhill

$$= \frac{\text{distance travelled}}{\text{time taken}}$$
$$= \frac{(324 - 98)\text{ km}}{\left(\dfrac{204-63}{60}\right)\text{ h}}$$
$$= \frac{226 \times 60}{141}$$
$$= 96.17 \text{ kmph}$$

b Average speed from Horsham to Melbourne

$$= \frac{729 - 431\text{ km}}{\left(\dfrac{534-261}{60}\right)\text{ h}}$$
$$= \frac{298 \times 60}{273}$$
$$= 65.5 \text{ kmph}$$

2 **a** 800 m to the newsagency

b $m = \dfrac{y\text{-step}}{x\text{-step}} = \dfrac{500-0}{4-0} = 125$

c Average speed $= \dfrac{(500-0)\text{ m}}{(4-0)\text{ min}} = 125$ m/minute

d The *slope* represents the *average walking speed*.

e Paul stayed between the 12th and 20th minutes i.e., 8 minutes.

f Average speed $= \dfrac{(800-0)\text{ m}}{(32-20)\text{ min}}$ {return journey is from (20, 800) to (32, 0)}

$= 66.7$ m/minute

g Total distance travelled $= 1600$ m

$= 1.6$ km (to the shop and return)

3 Average water loss $= \dfrac{\text{amount of water lost}}{\text{time taken}}$

$= \dfrac{53.8-48.2}{11}$

$= 0.509$ million kL/day

$= 509\,000$ kL/day

4 **a** First quarter: Used 106.8 kL Number of days $= 31+28+31$

$\therefore$ rate $= \dfrac{106.8}{90}$

$= 1.19$ kL/day

b First six months: Used

$$\begin{array}{r} 106.8 \\ +\ \ 79.4 \\ \hline 186.2 \end{array} \text{ kL}$$

Number of days $= 90+30+31+30$

$= 181$ days

$\therefore$ rate $= \dfrac{186.2}{181}$

$= 1.03$ kL/day

c The whole year: Used $= 106.8+79.4+81.8+115.8$

$= 383.8$ kL

$\therefore$ rate $= \dfrac{383.8}{365}$ kg

$= 1.05$ kL/day

EXERCISE 20A.3

1 **a** In the first 4 seconds: average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

$= \dfrac{(2.4-2)\text{ m}}{(4-0)\text{ sec}}$

$= 0.1$ m/s

b In the last 4 seconds: average speed $= \dfrac{(6-2.4)\text{ m}}{(4-0)\text{ sec}}$

$= 0.9$ m/s

c In the 8 second period: average speed $= \dfrac{(6-2)\text{ m}}{(8-0)\text{ sec}}$

$= 0.5$ m/s

2 **a** **i** Initially there are 35 beetles. After a 10 g dose there are 3 beetles.

$$\text{rate of change} = \frac{\text{change in population}}{\text{change in dose}} = \frac{(35-3)\text{ beetles}}{(0-10)\text{ gm}} = -3.2 \text{ beetles/gm}$$

i.e., rate of decrease = 3.2 beetles/gram

ii $$\text{rate of change} = \frac{(26-8)\text{ beetles}}{(4-8)\text{ beetles}} = -4.5 \text{ beetles/gm}$$

i.e., rate of decrease = 4.5 beetles/gram

b A dose of 0 to 1 g has little or no effect.
A dose of 1 to 8 g has good or considerable effect (rapid decrease).
A dose of 8 to 14 g has an effect, but less rapid than the 1 to 8 rate.

EXERCISE 20B.1

1 **a** The tangent passes through (2, 3) and (0, 1) $\therefore$ slope of tangent $\doteqdot \dfrac{(3-1)\text{ m}}{(2-0)\text{ s}}$

$\doteqdot \frac{2}{2}$ m/s

$\doteqdot 1$ m/s

b The tangent passes through (3.5, 6) and (2.2, 2) $\therefore$ slope of tangent $\doteqdot \dfrac{(6-2)\text{ km}}{(3-2.2)\text{ h}}$

$\doteqdot \frac{4}{1.3}$ km/h

$\doteqdot 3$ km/h

c The tangent passes through (20, 2700) and (40, 3700) $\therefore$ slope of tangent

$\doteqdot \dfrac{(3.7-2.7)\text{ thousands of \$}}{(40-20)\text{ items}}$

$\doteqdot 0.05$ thousands of \$/item

$\doteqdot$ \$50/item

d The tangent passes through (0, 35) and (7, 0) $\therefore$ slope of tangent $\doteqdot \dfrac{(35-0)\text{ bats}}{(0-7)\text{ weeks}}$

$\doteqdot -5$ bats/week

i.e., the population is decreasing at the rate of 5 bats/week after 5 weeks.

2 **a** Originally (when $x=0$) the volume was 8200 L.

b After 1 hour ($x=1$) the volume was 3000 L.

c The tangent at (0, 8.2) passes through (1, 0)

$\therefore$ slope of tangent $= \dfrac{(8.2-0)\text{ kL}}{(0-1)\text{ h}}$

$= -8.2$ kL i.e., loses 8200 L/hour

d After 1 hour, $x=1$ and the tangent at (1, 3) passes through (0, 6)

$\therefore$ slope of tangent $= \dfrac{(6-3)\text{ kL}}{(0-1)\text{ h}} = -3$ kL/h i.e., the rate of loss is 3000 L/hour

EXERCISE 20B.2

1 Let M $(1.5+h,\ (15+h)^2)$ be a point on $y=x^2$ which is close to F $(1.5,\ 1.5^2)$.

$$\begin{aligned}\therefore\ \text{slope of MF} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \frac{(1.5+h)^2-(1.5)^2}{(1.5+h)-1.5}\\ &= \frac{2.25+h^2+3h-2.25}{h}\\ &= \frac{h^2+3h}{h}\\ &= h+3 \qquad \{\text{as } h\neq 0\}\end{aligned}$$

As M approaches F, h approaches 0, $\therefore$ $h+3$ approaches 3, $\therefore$ slope of the tangent $=3$

2 **a**

$$\begin{aligned}&(x+h)^3\\ &= (x+h)^2(x+h)\\ &= (x^2+2xh+h^2)(x+h)\\ &= x^3+2x^2h+h^2x+hx^2+2xh^2+h^3\\ &= x^3+3x^2h+3xh^2+h^3\end{aligned}$$

b

$$\begin{aligned}&(1+h)^3\\ &= 1^3+3(1)^2h+3(1)h^2+h^3\\ &= 1+3h+3h^2+h^3\end{aligned}$$

c M $(1+h,\ (1+h)^3)$ is the point on $y=x^3$ which is close to F(1, 1).

d

$$\begin{aligned}&\text{slope of chord MF}\\ &= \frac{y\text{-step}}{x\text{-step}}\\ &= \frac{(1+h)^3-1}{(1+h)-1}\\ &= \frac{1+3h+3h^2+h^3-1}{h}\\ &= \frac{3h+3h^2+h^3}{h}\\ &= 3+3h+h^2 \qquad \{\text{as } h\neq 0\}\end{aligned}$$

e As M approaches F, h approaches 0

$\therefore$ slope MF approaches 3

$\therefore$ slope of tangent $=3$

3 **b** from **2 a** $(2+h)^3 = 2^3+3(2)^2h+3(2)h^2+h^3$
$= 8+12h+6h^2+h^3$

c M $(2+h,\ (2+h)^3)$ is the point on $y=x^3$ which is close to F(2, 8).

d

$$\begin{aligned}&\text{slope of chord MF}\\ &= \frac{y\text{-step}}{x\text{-step}}\\ &= \frac{(2+h)^3-2^3}{2+h-2}\\ &= \frac{8+12h+6h^2+h^3-8}{h}\\ &= \frac{12h+6h^2+h^3}{h}\\ &= 12+6h+h^2 \qquad \{\text{as } h\neq 0\}\end{aligned}$$

e as M approaches F,
h approaches 0,
slope approaches 12
$\therefore$ slope of tangent $=12$

4 **a** $\dfrac{1}{x+h}-\dfrac{1}{x}=\dfrac{x(1)-1(x+h)}{(x+h)x}$

$=\dfrac{x-x-h}{x(x+h)}$

$=-\dfrac{h}{x(x+h)}$

b M is $\left(2+h,\ \dfrac{1}{2+h}\right)$

$\therefore$ y-coordinate is $\dfrac{1}{2+h}$

$\therefore$ slope of MF $=\dfrac{y\text{-step}}{x\text{-step}}$

$=\dfrac{\dfrac{1}{2+h}-\dfrac{1}{2}}{(2+h)-2}$

$=\dfrac{\dfrac{-h}{2(2+h)}}{h}$ {using **a** with $x=2$}

$=\dfrac{-h}{2h(2+h)}$

$=\dfrac{-1}{2(2+h)}$ {as $h\neq 0$}

c F is the point where $x=2$. Now as h approaches 0, M approaches F, and the slope MF approximates the slope of the tangent at M.
But as h approaches 0, the slope MF approaches $-\frac{1}{4}$, $\therefore$ slope of tangent $=-\frac{1}{4}$

d At the point where $x=3$, $y=\frac{1}{3}$.
Also, the point with x-coordinate $3+h$ has y-coordinate $\dfrac{1}{3+h}$.
Using the method in **b**, the slope of the line between these points is $-\dfrac{1}{3(3+h)}$, $h\neq 0$.
As h approaches 0, the slope of the line approximates the slope of the tangent when $x=3$, and this is $-\frac{1}{9}$.

5 **a** $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}=\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}$

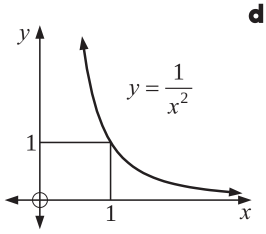
$=\dfrac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}$

$=\dfrac{-2xh-h^2}{x^2(x+h)^2}$

b When $x=2$,

$\dfrac{1}{(2+h)^2}-\dfrac{1}{4}=\dfrac{-4h-h^2}{4(2+h)^2}$

c



d Let M be close to $\text{F}\left(2,\ \dfrac{1}{2^2}\right)$ i.e., $\left(2+h,\ \dfrac{1}{(2+h)^2}\right)$

$\therefore$ slope MF $=\dfrac{y\text{-step}}{x\text{-step}}=\dfrac{\dfrac{1}{(2+h)^2}-\dfrac{1}{2^2}}{(2+h)-2}$

$=\dfrac{\dfrac{-4h-h^2}{4(2+h)^2}}{h}$ {using **b**}

$=\dfrac{-4h-h^2}{4h(2+h)^2}$

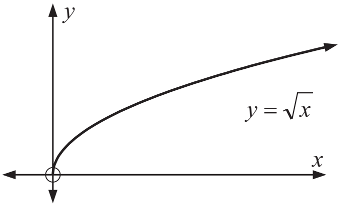
$=\dfrac{-4-h}{4(2+h)^2}$ {as $h\neq 0$}

as h approaches 0, slope MF approaches $\frac{-4}{16}$
i.e., $-\frac{1}{4}$ $\therefore$ slope of tangent $=-\frac{1}{4}$

6 **a**
$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{1}{\sqrt{x+h}+\sqrt{x}} \qquad h \neq 0$$

b When $x = 9$,
$$\frac{\sqrt{9+h}-3}{h} = \frac{1}{\sqrt{9+h}+3} \quad \ldots\ldots (*)$$

c



d Slope MF
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{\sqrt{9+h}-\sqrt{9}}{(9+h)-9}$$
$$= \frac{\sqrt{9+h}-3}{h}$$
$$= \frac{1}{\sqrt{9+h}+3} \qquad \{\text{using } *\}$$

$\therefore$ as h approaches 0, slope MF approaches $\frac{1}{6}$, $\quad\therefore$ slope of tangent $= \frac{1}{6}$

e $N = \sqrt{t}$ for $t \geqslant 4$ is the same graph as $y = \sqrt{x}$ for $x \geqslant 4$,

$\therefore$ rate after 9 days is the same as the slope of the tangent in **d**, i.e., $\frac{1}{6}$

$\therefore$ the population is increasing at a rate of $\frac{1}{6}$ (thousand insects/day) $= 167$ insects/day

REVIEW SET 20

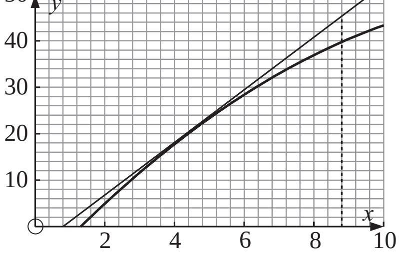
1 **a** When $t = 1$, $d = 40$ and when $t = 4$, $d = 70$

$\therefore$ average speed on $1 \leqslant t \leqslant 4$ is
$$\frac{\text{distance travelled}}{\text{time taken}}$$
$$= \frac{70-40}{4-1}$$
$$= \frac{30}{3}$$
$$= 10 \text{ m/s}$$

b When $t = 1$, $d = 40$ and when $t = 10$, $d \doteqdot 81$

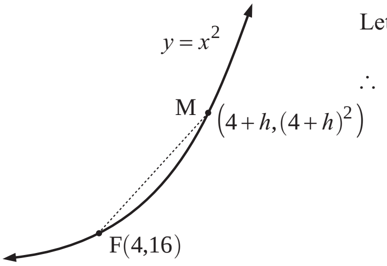
$$\therefore \text{ average speed } = \frac{81-40}{10-1} = \frac{41}{9} \doteqdot 4.6 \text{ m/s}$$

2 **a**



b slope
$$= \frac{y\text{-step}}{x\text{-step}}$$
$$\doteqdot \frac{46-0}{8.8-0.8}$$
$$\doteqdot \frac{46}{8}$$
$$\doteqdot 5.8$$

3



Let M $(4+h,\ (4+h)^2)$ be a point close to F(4, 16).

$$\therefore \quad \text{slope of MF} = \frac{(4+h)^2 - 16}{4+h-4}$$
$$= \frac{16 + 8h + h^2 - 16}{h}$$
$$= \frac{8h + h^2}{h}$$
$$= 8 + h \quad (\text{as} \quad h \neq 0)$$

Now as M approaches F, $h \to 0$ and $8 + h$ approaches 8.

$\therefore$ the tangent at F has slope 8.

4 **a**
$$f(x+h)$$
$$= (2(x+h)+3)^2$$
$$= ((2x+2h)+3)^2$$
$$= (2x+2h)^2 + 2(2x+2h)3 + 3^2$$
$$= 4x^2 + 8xh + 4h^2 + 12x + 12h + 9$$

b
$$\frac{f(x+h) - f(x)}{h}$$
$$= \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{12x} + 12h \cancel{+9} - \cancel{4x^2} - \cancel{12x} \cancel{-9}}{h}$$
$$= \frac{8xh + 4h^2 + 12h}{h}$$
$$= 8x + 4h + 12 \qquad \{\text{as} \quad h \neq 0\}$$

c $\dfrac{f(x+h) - f(x)}{h}$ is the slope of the secant (chord) AB.

d **i** As $h \to 0$, $\dfrac{f(x+h) - f(x)}{h} = 8x + 4h + 12$ approaches $8x + 12$.

ii As $h \to 0$, $\dfrac{f(1+h) - f(1)}{h} = 8 + 4h + 12$ approaches 20.

e **i** $8x + 12$ gives us the slope of the tangent at a point with x-coordinate x.

ii 20 is the slope of the tangent at $x = 1$.

5 **a** Average speed on $2 \leqslant t \leqslant 5$ is
$$\frac{s(5) - s(2)}{5-2} \text{ m/s}$$
$$= \frac{(25+20) - (4+8)}{3} \text{ m/s}$$
$$= \frac{33}{3} \text{ m/s}$$
$$= 11 \text{ m/s}$$

b Average speed on $2 \leqslant t \leqslant 2+h$ is
$$\frac{s(2+h) - s(2)}{2+h-2} \text{ m/s}$$
$$= \frac{(2+h)^2 + 4(2+h) - 12}{h} \text{ m/s}$$
$$= \frac{4 + 4h + h^2 + 8 + 4h - 12}{h} \text{ m/s}$$
$$= \frac{8h + h^2}{h} \text{ m/s}$$
$$= (8+h) \text{ m/s}$$

c as $h \to 0$, $8 + h \to 8$

$$\therefore \quad \frac{s(2+h) - s(2)}{h} \to 8 \text{ m/s}$$

8 m/s is the instantaneous velocity at $t = 2$ seconds.

Chapter 21

DIFFERENTIAL CALCULUS

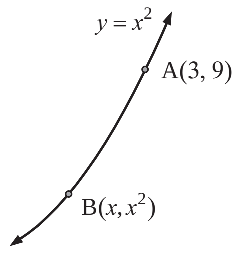
EXERCISE 21A

1 **a** Slope AB $= \dfrac{x^2-9}{x-3}$

$= \dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= x+3 \quad \text{(provided } x \neq 3)$

$\therefore \lim\limits_{x\to 3}(\text{slope AB}) = 6 \quad \text{...... (1)}$



But, as B $\to$ A, i.e., $x \to 3$ and $\lim\limits_{x\to 3}(\text{slope AB}) = \text{slope of tangent at A}$

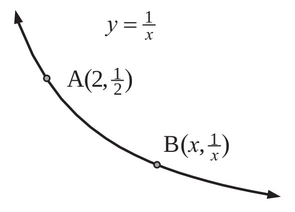
$\therefore$ slope of tangent $= 6$

b Slope AB $= \dfrac{\dfrac{1}{x}-\dfrac{1}{2}}{x-2}$

$= \dfrac{2-x}{2x(x-2)}$

$= \dfrac{-1\cancel{(x-2)}}{2x\cancel{(x-2)}}$

$= -\dfrac{1}{2x} \quad \text{(provided } x \neq 2)$



$\therefore \lim\limits_{x\to 2}(\text{slope AB}) = -\frac{1}{4}$

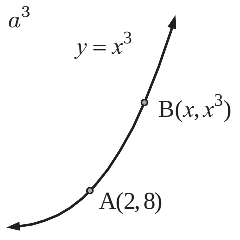
But, as B $\to$ A, i.e., $x \to 2$ and $\lim\limits_{x\to 2}(\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent $= -\frac{1}{4}$

2 **a** $(x-a)(x^2+ax+a^2)$

$= x^3 + ax^2 + a^2x - ax^2$
$\quad -a^2x - a^3$

$= x^3 - a^3$



b Slope AB $= \dfrac{x^3-8}{x-2}$

$= \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$

$= x^2+2x+4 \quad \text{(provided } x \neq 2)$

$\therefore \lim\limits_{x\to 2}(\text{slope of AB}) = 2^2+4+4 = 12$

But, as B $\to$ A, i.e., $x \to 2$,

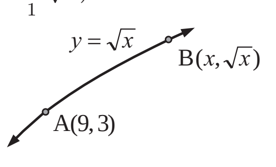
$\lim\limits_{x\to 2}(\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent $= 12$

3 **a** $\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

$= \dfrac{\cancelto{1}{\sqrt{x}-\sqrt{a}}}{(\sqrt{x}+\sqrt{a})\cancelto{1}{(\sqrt{x}-\sqrt{a})}}$

$= \dfrac{1}{\sqrt{x}+\sqrt{a}}$



b Slope AB $= \dfrac{\sqrt{x}-3}{x-9} = \dfrac{1}{\sqrt{x}+3}$

{from **a**, provided $x \neq 9$}

$\therefore \lim\limits_{x\to 9}(\text{slope AB}) = \frac{1}{6}$

But, as B $\to$ A, i.e., $x \to 9$,

$\lim\limits_{x\to 9}(\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent $= \frac{1}{6}$

EXERCISE 21B

1 **a**
$$f(x) = 1 - x^2$$
$$\therefore \quad f(2) = 1 - 2^2 = -3$$
$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{(1-x^2)-(-3)}{x-2} \\ &= \lim_{x\to 2} \frac{4-x^2}{x-2} \\ &= \lim_{x\to 2} \frac{-(x+2)\cancel{(x-2)}}{\cancel{x-2}} \\ &= \lim_{x\to 2} -(x+2) \qquad \{\text{as } x \neq 2\} \\ &= -4 \end{aligned}$$

b
$$f(x) = 2x^2 + 5x \quad \text{at} \quad x = -1$$
$$f(-1) = 2(-1)^2 + 5(-1) = -3$$
$$\begin{aligned} f'(-1) &= \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} \\ &= \lim_{x\to -1} \frac{(2x^2+5x)-(-3)}{x+1} \\ &= \lim_{x\to -1} \frac{2x^2+5x+3}{x+1} \\ &= \lim_{x\to -1} \frac{(2x+3)\cancel{(x+1)}}{\cancel{x+1}} \\ &= \lim_{x\to -1} 2x+3 \qquad \{\text{as } x \neq -1\} \\ &= 1 \end{aligned}$$

c
$$f(x) = 5 - 2x^2 \quad \text{at} \quad x = 3$$
$$\therefore \quad f(3) = 5 - 2(3)^2 = -13$$
$$\begin{aligned} f'(3) &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3} \frac{(5-2x^2)-(-13)}{x-3} \\ &= \lim_{x\to 3} \frac{18-2x^2}{x-3} \\ &= \lim_{x\to 3} \frac{-2(x^2-9)}{x-3} \\ &= \lim_{x\to 3} \frac{-2(x+3)\cancel{(x-3)}}{\cancel{x-3}} \\ &= \lim_{x\to 3} -2(x+3) \qquad \{\text{as } x \neq 3\} \\ &= -2(6) \\ &= -12 \end{aligned}$$

d
$$f(x) = 3x + 5 \quad \text{at} \quad x = -2$$
$$\therefore \quad f(-2) = 3(-2) + 5 = -1$$
$$\begin{aligned} f'(-2) &= \lim_{x\to -2} \frac{f(x)-f(-2)}{x+2} \quad \text{where} \\ &= \lim_{x\to -2} \frac{(3x+5)-(-1)}{x+2} \\ &= \lim_{x\to -2} \frac{3x+6}{x+2} \\ &= \lim_{x\to -2} \frac{3\cancel{(x+2)}}{\cancel{x+2}} \\ &= \lim_{x\to -2} 3 \qquad \{\text{as } x \neq -2\} \\ &= 3 \end{aligned}$$

2 **a**
$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{4}{x}-\frac{4}{2}}{x-2} \\ &= \lim_{x\to 2} \frac{8-4x}{2x(x-2)} \\ &= \lim_{x\to 2} \frac{-4\cancel{(x-2)}}{2x\cancel{(x-2)}} \\ &= \lim_{x\to 2} \frac{-4}{2x} \qquad \{\text{as } x \neq 2\} \\ &= -\tfrac{4}{4} \\ &= -1 \end{aligned}$$

b
$$\begin{aligned} f'(-2) &= \lim_{x\to -2} \frac{f(x)-f(-2)}{x-(-2)} \\ &= \lim_{x\to -2} \frac{-\frac{3}{x}-\frac{3}{2}}{(x+2)} \\ &= \lim_{x\to -2} \frac{-6-3x}{2x(x+2)} \\ &= \lim_{x\to -2} \frac{-3\cancel{(x+2)}}{2x\cancel{(x+2)}} \\ &= \lim_{x\to -2} \frac{-3}{2x} \qquad \{\text{as } x \neq -2\} \\ &= \tfrac{-3}{-4} \\ &= \tfrac{3}{4} \end{aligned}$$

c
$$\begin{aligned} f'(4) &= \lim_{x\to4} \frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to4} \frac{\frac{1}{x^2}-\frac{1}{16}}{x-4} \\ &= \lim_{x\to4} \frac{16-x^2}{16x^2(x-4)} \\ &= \lim_{x\to4} \frac{-(x+4)\cancel{(x-4)}}{16x^2\cancel{(x-4)}} \\ &= \lim_{x\to4} \frac{-(x+4)}{16x^2} \qquad \{\text{as } x\neq4\} \\ &= \tfrac{-8}{256} \\ &= -\tfrac{1}{32} \end{aligned}$$

d
$$\begin{aligned} f'(2) &= \lim_{x\to2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to2} \frac{\frac{4x}{x-3}-\left(-\frac{8}{1}\right)}{x-2} \\ &= \lim_{x\to2} \frac{4x+8(x-3)}{1(x-2)(x-3)} \\ &= \lim_{x\to2} \frac{12\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} \\ &= \lim_{x\to2} \frac{12}{x-3} \qquad \{\text{as } x\neq2\} \\ &= \tfrac{12}{-1} \\ &= -12 \end{aligned}$$

e
$$\begin{aligned} f'(5) &= \lim_{x\to5} \frac{f(x)-f(5)}{(x-5)} \\ &= \lim_{x\to5} \frac{\frac{4x+1}{x-2}-\frac{7}{1}}{x-5} \\ &= \lim_{x\to5} \frac{4x+1-7(x-2)}{(x-2)(x-5)} \\ &= \lim_{x\to5} \frac{4x+1-7x+14}{(x-2)(x-5)} \\ &= \lim_{x\to5} \frac{-3x+15}{(x-2)(x-5)} \\ &= \lim_{x\to5} \frac{-3\cancel{(x-5)}}{(x-2)\cancel{(x-5)}} \\ &= \lim_{x\to5} \frac{-3}{x-2} \qquad \{\text{as } x\neq5\} \\ &= -\tfrac{3}{3} \\ &= -1 \end{aligned}$$

f
$$\begin{aligned} f'(-4) &= \lim_{x\to-4} \frac{f(x)-f(-4)}{x-(-4)} \\ &= \lim_{x\to-4} \frac{\frac{3x}{x^2+1}-\left(-\frac{12}{17}\right)}{x+4} \\ &= \lim_{x\to-4} \frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\ &= \lim_{x\to-4} \frac{12x^2+51x+12}{17(x+4)(x^2+1)} \\ &= \lim_{x\to-4} \frac{\cancel{(x+4)}(12x+3)}{17(x^2+1)\cancel{(x+4)}} \\ &= \lim_{x\to-4} \frac{12x+3}{17(x^2+1)} \qquad \{x\neq-4\} \\ &= -\frac{45}{17\times17} \\ &= -\tfrac{45}{289} \end{aligned}$$

3 **a** $f(x)=\sqrt{x}$ and $f(4)=\sqrt{4}=2$

$$\begin{aligned} f'(4) &= \lim_{x\to4} \frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to4} \frac{\sqrt{x}-2}{x-4} \\ &= \lim_{x\to4} \frac{\cancel{\sqrt{x}-2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}} \\ &= \lim_{x\to4} \frac{1}{(\sqrt{x}+2)} \qquad \{\text{as } x\neq4\} \\ &= \frac{1}{2+2} \\ &= \tfrac{1}{4} \end{aligned}$$

b $f(x)=\sqrt{x}$ and $f\left(\frac{1}{4}\right)=\sqrt{\frac{1}{4}}=\frac{1}{2}$

$$\begin{aligned} f'(\tfrac{1}{4}) &= \lim_{x\to\frac{1}{4}} \frac{f(x)-f(\frac{1}{4})}{x-\frac{1}{4}} \\ &= \lim_{x\to\frac{1}{4}} \frac{\sqrt{x}-\frac{1}{2}}{x-\frac{1}{4}} \\ &= \lim_{x\to\frac{1}{4}} \frac{\cancel{(\sqrt{x}-\frac{1}{2})}}{(\sqrt{x}+\frac{1}{2})\cancel{(\sqrt{x}-\frac{1}{2})}} \\ &= \lim_{x\to\frac{1}{4}} \frac{1}{\sqrt{x}+\frac{1}{2}} \qquad \{\text{as } x\neq\tfrac{1}{4}\} \\ &= \frac{1}{\frac{1}{2}+\frac{1}{2}} \\ &= 1 \end{aligned}$$

c $f(x)=\dfrac{2}{\sqrt{x}}$ and $f(9)=\frac{2}{3}$

$$\begin{aligned} f'(9) &= \lim_{x\to 9}\frac{f(x)-f(9)}{x-9} \\ &= \lim_{x\to 9}\frac{\frac{2}{\sqrt{x}}-\frac{2}{3}}{x-9} \\ &= \lim_{x\to 9}\frac{2(3-\sqrt{x})}{3\sqrt{x}(x-9)} \\ &= \lim_{x\to 9}\frac{-2\cancel{(\sqrt{x}-3)}}{3\sqrt{x}(\sqrt{x}+3)\cancel{(\sqrt{x}-3)}} \\ &= \lim_{x\to 9}\frac{-2}{3\sqrt{x}(\sqrt{x}+3)} \quad \{x\neq 9\} \\ &= \frac{-2}{3(3)(6)} \\ &= -\tfrac{1}{27} \end{aligned}$$

d $f(x)=\sqrt{x-6}$ and $f(10)=2$

$$\begin{aligned} f'(10) &= \lim_{x\to 10}\frac{f(x)-f(10)}{x-10} \\ &= \lim_{x\to 10}\frac{\sqrt{x-6}-2}{x-10} \\ &= \lim_{x\to 10}\frac{(\sqrt{x-6}-2)(\sqrt{x-6}+2)}{(x-10)(\sqrt{x-6}+2)} \\ &= \lim_{x\to 10}\frac{x-6-4}{(x-10)(\sqrt{x-6}+2)} \\ &= \lim_{x\to 10}\frac{{}^{1}\cancel{x-10}}{\cancel{(x-10)}(\sqrt{x-6}+2)} \\ &= \lim_{x\to 10}\frac{1}{\sqrt{x-6}+2} \quad \{\text{as } x\neq 10\} \\ &= \frac{1}{2+2} \\ &= \tfrac{1}{4} \end{aligned}$$

4 **a** $f(x)=x^2+3x-4$ at $x=3$

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \quad \text{where } f(3)=3^2+3(3)-4=14 \\ &= \lim_{h\to 0}\frac{\left[(3+h)^2+3(3+h)-4\right]-14}{h} \\ &= \lim_{h\to 0}\frac{9+6h+h^2+9+3h-4-14}{h} \\ &= \lim_{h\to 0}\frac{9h+h^2}{h} \\ &= \lim_{h\to 0} 9+h \quad \{\text{as } h\neq 0\} \\ &= 9 \end{aligned}$$

b $f(x)=5-2x-3x^2$ at $x=-2$

$$\begin{aligned} f'(-2) &= \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} \quad \text{where } f(-2)=5-2(-2)-3(4)=-3 \\ &= \lim_{h\to 0}\frac{\left[5-2(-2+h)-3(-2+h)^2\right]-(-3)}{h} \\ &= \lim_{h\to 0}\frac{5+4-2h-12+12h-3h^2+3}{h} \\ &= \lim_{h\to 0}\frac{10h-3h^2}{h} \\ &= \lim_{h\to 0} 10-3h \quad \{\text{as } h\neq 0\} \\ &= 10 \end{aligned}$$

c $f(x)=\dfrac{1}{2x-1}$

$\therefore\ f(-2)=\dfrac{1}{2(-2)-1}=-\dfrac{1}{5}$

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2(-2+h)-5}-\left(-\frac{1}{5}\right)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2h-5}+\frac{1}{5}}{h}\\
&= \lim_{h\to 0}\frac{5+1\cancel{(2h-5)}}{5h\cancel{(2h-5)}}\\
&= \lim_{h\to 0}\frac{2h}{5h(2h-5)}\\
&= \lim_{h\to 0}\frac{2}{5(2h-5)}\quad \{\text{as}\quad h\neq 0\}\\
&= -\tfrac{2}{25}
\end{aligned}$$

d $f(x)=\dfrac{1}{x^2}$

$\therefore\ f(3)=\dfrac{1}{3^2}=\dfrac{1}{9}$

$$\begin{aligned}
f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{(3+h)^2}-\frac{1}{9}}{h}\\
&= \lim_{h\to 0}\frac{9-(3+h)^2}{9h(3+h)^2}\\
&= \lim_{h\to 0}\frac{9-9-6h-h^2}{9h(3+h)^2}\\
&= \lim_{h\to 0}\frac{-\cancel{h}(6+h)}{9\cancel{h}(3+h)^2}\\
&= \lim_{h\to 0}\frac{-(6+h)}{9(3+h)^2}\quad \{\text{as}\quad h\neq 0\}\\
&= \tfrac{-6}{81}\\
&= -\tfrac{2}{27}
\end{aligned}$$

e $f(x)=\sqrt{x}$

$\therefore\ f(4)=\sqrt{4}=2$

$$\begin{aligned}
f'(4) &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)\\
&= \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)}\\
&= \lim_{h\to 0}\frac{{}^{1}\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\quad \{\text{as}\quad h\neq 0\}\\
&= \tfrac{1}{4}
\end{aligned}$$

f $f(x)=\dfrac{1}{\sqrt{x}}$

$\therefore\ f(1)=\frac{1}{\sqrt{1}}=1$

$$\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{\sqrt{1+h}}-\frac{1}{1}}{h}\\
&= \lim_{h\to 0}\frac{1-\sqrt{1+h}}{h\sqrt{1+h}}\\
&= \lim_{h\to 0}\frac{(1-\sqrt{1+h})}{h\sqrt{1+h}}\left(\frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}\right)\\
&= \lim_{h\to 0}\frac{1-(1+h)}{h(\sqrt{1+h})(1+\sqrt{1+h})}\\
&= \lim_{h\to 0}\frac{-\cancel{h}^{\,1}}{\cancel{h}\sqrt{1+h}(1+\sqrt{1+h})}\\
&= \lim_{h\to 0}\frac{-1}{\sqrt{1+h}(1+\sqrt{1+h})}\quad \{h\neq 0\}\\
&= \frac{-1}{1(1+1)}\\
&= -\tfrac{1}{2}
\end{aligned}$$

5 **a** $f(x) = x^3, \quad \therefore \quad f'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ where $f(2) = 2^3 = 8$

$$\begin{aligned}
&= \lim_{h \to 0} \frac{(2+h)^3 - 8}{h} \\
&= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} \\
&= \lim_{h \to 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\} \\
&= 12
\end{aligned}$$

b $f(x) = x^4, \quad \therefore \quad f'(3) = \lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^4 = 81$

$$\begin{aligned}
&= \lim_{h \to 0} \frac{(3+h)^4 - 3^4}{h} \\
&= \lim_{h \to 0} \frac{81 + 108h + 54h^2 + 12h^3 + h^4 - 81}{h} \\
&= \lim_{h \to 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h} \\
&= \lim_{h \to 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\} \\
&= 108
\end{aligned}$$

EXERCISE 21C

1 **a**

$$\begin{aligned}
f(x) &= x \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h} \\
&= \lim_{h \to 0} \frac{h}{h} \\
&= \lim_{h \to 0} 1 \quad \{\text{as } h \neq 0\} \\
&= 1
\end{aligned}$$

b

$$\begin{aligned}
f(x) &= 5 \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{5 - 5}{h} \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= \lim_{h \to 0} 0 \quad \{\text{as } h \neq 0\} \\
&= 0
\end{aligned}$$

c $f(x) = x^3, \quad \therefore \quad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\begin{aligned}
&= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 \quad \{\text{as } h \neq 0\} \\
&= 3x^2
\end{aligned}$$

d $$\begin{aligned} f(x) &= x^4 \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^4-x^4}{h} \\ &= \lim_{h\to 0} \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} \\ &= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} \\ &= \lim_{h\to 0} 4x^3+6x^2h+4xh^2+h^3 \quad (\text{as } h\neq 0) \\ &= 4x^3 \end{aligned}$$

2 **a** $f(x)=2x+5$

$$\begin{aligned} &\therefore f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(2(x+h)+5)-(2x+5)}{h} \\ &= \lim_{h\to 0} \frac{2x+2h+5-2x-5}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \qquad \{\text{as } h\neq 0\} \\ &= 2 \end{aligned}$$

b $f(x)=x^2-3x$

$$\begin{aligned} &\therefore f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2-3(x+h)]-[x^2-3x]}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} \\ &= \lim_{h\to 0} 2x+h-3 \quad \{\text{as } h\neq 0\} \\ &= 2x-3 \end{aligned}$$

c $$\begin{aligned} f(x) &= x^3-2x^2+3 \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^3-2(x+h)^2+3]-[x^3-2x^2+3]}{h} \\ &= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2+3)-(x^3-2x^2+3)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-4xh-2h^2}{h} \\ &= \lim_{h\to 0} 3x^2+3xh+h^2-4x-2h \quad \{\text{as } h\neq 0\} \\ &= 3x^2-4x \end{aligned}$$

3 **a** $f(x) = \dfrac{1}{x+2}$,

$$\begin{aligned}
\therefore \quad & f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)+2} - \dfrac{1}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{-\cancel{h}^{\,1}}{\cancel{h}(x+2)(x+h+2)} \\
&= \lim_{h\to 0} \frac{-1}{(x+2)(x+h+2)} \quad \{\text{as } h \neq 0\} \\
&= \frac{-1}{(x+2)^2}
\end{aligned}$$

b $f(x) = \dfrac{1}{2x-1}$,

$$\begin{aligned}
\therefore \quad & f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)-1} - \dfrac{1}{2x-1}}{h} \\
&= \lim_{h\to 0} \frac{(2x-1)-[2(x+h)-1]}{h[2(x+h)-1](2x-1)} \\
&= \lim_{h\to 0} \frac{2x-1-2x-2h+1}{h[2(x+h)-1](2x-1)} \\
&= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}[2(x+h)-1](2x-1)} \\
&= \lim_{h\to 0} \frac{-2}{[2(x+h)-1](2x-1)} \quad \{\text{as } h \neq 0\} \\
&= \frac{-2}{(2x-1)^2}
\end{aligned}$$

c $f(x) = \dfrac{1}{x^2}$,

$$\begin{aligned}
\therefore \quad & f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{[x+(x+h)][x-(x+h)]}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{(2x+h)(-\overset{1}{\cancel{h}})}{\cancel{h}x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2} \quad \{\text{as } h \neq 0\} \\
&= \frac{-2x}{x^4} \\
&= -\frac{2}{x^3}
\end{aligned}$$

d $f(x) = \dfrac{1}{x^3}$,

$$\begin{aligned}
\therefore \quad & f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^3} - \dfrac{1}{x^3}}{h} \\
&= \lim_{h\to 0} \frac{x^3-(x+h)^3}{hx^3(x+h)^3} \\
&= \lim_{h\to 0} \frac{x^3-x^3-3x^2h-3xh^2-h^3}{hx^3(x+h)^3} \\
&= \lim_{h\to 0} \frac{-3x^2h-3xh^2-h^3}{hx^3(x+h)^3} \\
&= \lim_{h\to 0} \frac{-3x^2-3xh-h^2}{x^3(x+h)^3} \quad \{\text{as } h \neq 0\} \\
&= \frac{-3x^2}{x^3 \times x^3} \\
&= -\frac{3}{x^4}
\end{aligned}$$

4 **a** $f(x)=\sqrt{x+2}$, $\therefore$
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{x+2+h}-\sqrt{x+2})(\sqrt{x+2+h}+\sqrt{x+2})}{h(\sqrt{x+2+h}+\sqrt{x+2})} \\ &= \lim_{h\to 0}\frac{(x+2+h)-(x+2)}{h(\sqrt{x+2+h}+\sqrt{x+2})} \\ &= \lim_{h\to 0}\frac{\cancel{h}^{\,1}}{\cancel{h}(\sqrt{x+2+h}+\sqrt{x+2})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+2+h}+\sqrt{x+2}} \qquad \{\text{as } h\neq 0\} \\ &= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} \\ &= \frac{1}{2\sqrt{x+2}} \end{aligned}$$

b $f(x)=\dfrac{1}{\sqrt{x}}$, $\therefore$
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\times\sqrt{x}} \\ &= \lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x+h}\times\sqrt{x}}\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) \\ &= \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\ &= \lim_{h\to 0}\frac{-\cancel{h}^{\,1}}{\cancel{h}\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\ &= \lim_{h\to 0}\frac{-1}{\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \qquad \{\text{as } h\neq 0\} \\ &= \frac{-1}{\sqrt{x}\times\sqrt{x}\times 2\sqrt{x}} \\ &= \frac{-1}{2x\sqrt{x}} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \sqrt{2x+1} \\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\left(\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}\right) \\ &= \lim_{h\to 0} \frac{[2(x+h)+1]-[2x+1]}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\ &= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\ &= \lim_{h\to 0} \frac{2}{(\sqrt{2(x+h)+1}+\sqrt{2x+1}} \qquad \{\text{as } h \neq 0\} \\ &= \frac{2}{2\sqrt{2x+1}} \\ &= \frac{1}{\sqrt{2x+1}} \end{aligned}$$

5

Function	*Derivative*
x^1	$1x^0 = 1$
x^2	$2x^1 = 2x$
x^3	$3x^2$
x^4	$4x^3$

Function	*Derivative*
x^{-1}	$-1x^{-2}$
x^{-2}	$-2x^{-3}$
x^{-3}	$-3x^{-4}$

Function	*Derivative*
$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
$x^{-\frac{1}{2}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$

If $f(x) = x^n$, $f'(x) = nx^{n-1}$

EXERCISE 21D

1 **a** $f(x) = x^3$,

$\therefore \quad f'(x) = 3x^2$

b $f(x) = 2x^3$,

$$\therefore \quad f'(x) = 3 \times 2x^2 = 6x^2$$

c $f(x) = 7x^2$,

$$\therefore \quad f'(x) = 2 \times 7x = 14x$$

d $f(x) = x^2 + x$,

$\therefore \quad f'(x) = 2x + 1$

e $f(x) = 4 - 2x^2$,

$$\therefore \quad f'(x) = 0 - 2 \times 2x = -4x$$

f $f(x) = x^2 + 3x - 5$,

$$\therefore \quad f'(x) = 2x + 3 - 0 = 2x + 3$$

g
$$\begin{aligned} f(x) &= x^3 + 3x^2 + 4x - 1 \\ \therefore \quad f'(x) &= 3x^2 + 2(3x) + 4 - 0 \\ &= 3x^2 + 6x + 4 \end{aligned}$$

h
$$\begin{aligned} f(x) &= 5x^4 - 6x^2 \\ \therefore \quad f'(x) &= 4(5x^3) - 2(6x) \\ &= 20x^3 - 12x \end{aligned}$$

i
$$\begin{aligned} f(x) &= \frac{3x-6}{x} = 3 - 6x^{-1} \\ \therefore \quad f'(x) &= 0 - (-1) \times 6x^{-2} \\ &= \frac{6}{x^2} \end{aligned}$$

j
$$\begin{aligned} f(x) &= \frac{2x-3}{x^2} = \frac{2x}{x^2} - \frac{3}{x^2} \\ &= 2x^{-1} - 3x^2 \\ \therefore \quad f'(x) &= -2x^{-2} + 6x^{-3} \\ &= \frac{-2}{x^2} + \frac{6}{x^3} \end{aligned}$$

k $f(x) = \dfrac{x^3+5}{x} = x^2 + 5x^{-1}$

$\therefore\ f'(x) = 2x - 5x^{-2}$

$= 2x - \dfrac{5}{x^2}$

l $f(x) = \dfrac{x^3+x-3}{x} = x^2 + 1 - 3x^{-1}$

$\therefore\ f'(x) = 2x + 0 + 3x^{-2}$

$= 2x + \dfrac{3}{x^2}$

2 **a** $f(x) = \frac{1}{4}x^4$

$\therefore\ f'(x) = \frac{1}{4} \times 4x^3$

$= x^3$

b $f(x) = x + \dfrac{1}{x} = x + x^{-1}$

$\therefore\ f'(x) = 1 - x^{-2}$

$= 1 - \dfrac{1}{x^2}$

c $f(x) = \dfrac{x+1}{x} = 1 + x^{-1}$

$\therefore\ f'(x) = 0 - x^{-2}$

$= -\dfrac{1}{x^2}$

d $f(x) = \dfrac{x^2+5}{x^3} = \dfrac{x^2}{x^3} + \dfrac{5}{x^3} = x^{-1} + 5x^{-3}$

$\therefore\ f'(x) = -x^{-2} - 15x^{-4}$

$= \dfrac{-1}{x^2} - \dfrac{15}{x^4}$

e $f(x) = (x+1)(x-2)$

$= x^2 - x - 2$

$\therefore\ f'(x) = 2x - 1$

f $f(x) = \dfrac{1}{x^2} + 6\sqrt{x} = x^{-2} + 6x^{\frac{1}{2}}$

$\therefore\ f'(x) = -2x^{-3} + 3x^{-\frac{1}{2}}$

$= \dfrac{-2}{x^3} + \dfrac{3}{\sqrt{x}}$

g $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\therefore\ f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = \dfrac{-1}{2x\sqrt{x}}$

h $f(x) = (2x-1)^2 = 4x^2 - 4x + 1$

$\therefore\ f'(x) = 8x - 4$

i $f(x) = (x+2)^3$

$= x^3 + 3x^2(2) + 3x(2^2) + 2^3$

$= x^3 + 6x^2 + 12x + 8$

$\therefore\ f'(x) = 3x^2 + 12x + 12$

3 **a** $y = 2x^3 - 7x^2 - 1$

$\therefore\ \dfrac{dy}{dx} = 6x^2 - 14x$

b $y = \pi x^2$

$\therefore\ \dfrac{dy}{dx} = 2\pi x$

c $y = \dfrac{1}{5x^2} = \frac{1}{5}x^{-2}$

$\therefore\ \dfrac{dy}{dx} = -\frac{2}{5}x^{-3} = \dfrac{-2}{5x^3}$

d $y = 100x$

$\therefore\ \dfrac{dy}{dx} = 100$

e $y = 10(x+1)$

$= 10x + 10$

$\therefore\ \dfrac{dy}{dx} = 10$

f $y = 4\pi x^3$

$\therefore\ \dfrac{dy}{dx} = 12\pi x^2$

4 **a** $\dfrac{d}{dx}(6x+2)$

$= 6$

b $\dfrac{d}{dx}(x\sqrt{x})$

$= \dfrac{d}{dx}(x^{\frac{3}{2}})$

$= \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

c $\dfrac{d}{dx}(5-x)^2$

$= \dfrac{d}{dx}(25 - 10x + x^2)$

$= -10 + 2x$

$= 2x - 10$

d

$$\begin{aligned}&\frac{d}{dx}\left(\frac{6x^2-9x^4}{3x}\right)\\&=\frac{d}{dx}(2x-3x^3)\\&=2-9x^2\end{aligned}$$

e

$$\begin{aligned}&\frac{d}{dx}\left(4x-\frac{1}{4x}\right)\\&=\frac{d}{dx}\left(4x-\tfrac{1}{4}x^{-1}\right)\\&=4+\tfrac{1}{4}x^{-2}\\&=4+\frac{1}{4x^2}\end{aligned}$$

f

$$\begin{aligned}&\frac{d}{dx}\left(x(x+1)(2x-5)\right)\\&=\frac{d}{dx}\left(x(2x^2-3x-5)\right)\\&=\frac{d}{dx}\left(2x^3-3x^2-5x\right)\\&=6x^2-6x-5\end{aligned}$$

5 **a** $y=x^2$ at $x=2$

$$\begin{aligned}f(x)&=x^2\\ \therefore\quad f'(x)&=2x\\ \text{and so}\quad f'(2)&=2(2)\end{aligned}$$

$\therefore$ tangent has slope of 4

b $y=\dfrac{8}{x^2}$ at $x=9$

$$\begin{aligned}f(x)&=8x^{-2}\\ \therefore\quad f'(x)&=-16x^3\\&=-\frac{16}{x^3}\\ \text{and so}\quad f'(9)&=-\tfrac{16}{729}\end{aligned}$$

$\therefore$ tangent has slope of $-\frac{16}{729}$

c $y=2x^2-3x+7$ at $x=-1$

$$\begin{aligned}f(x)&=2x^2-3x+7\\ \therefore\quad f'(x)&=4x-3+0\\ \text{and so}\quad f'(-1)&=4(-1)-3\\&=-7\end{aligned}$$

$\therefore$ tangent has slope of -7

d $y=\dfrac{2x^2-5}{x}$ at $x=2$

$$\begin{aligned}f(x)&=2x-5x^{-1}\\ \therefore\quad f'(x)&=2+5x^{-2}\\&=2+\frac{5}{x^2}\\ \text{and so}\quad f'(2)&=2+\tfrac{5}{4}=\tfrac{13}{4}\end{aligned}$$

$\therefore$ tangent has slope of $\frac{13}{4}$

e $y=\dfrac{x^2-4}{x^2}$ at $x=4$

$$\begin{aligned}f(x)&=1-4x^{-2}\\ \therefore\quad f'(x)&=0+8x^{-3}\\&=\frac{8}{x^3}\\ \text{and so}\quad f'(4)&=\frac{8}{4^3}=\tfrac{1}{8}\end{aligned}$$

$\therefore$ tangent has slope of $\frac{1}{8}$

f $y=\dfrac{x^3-4x-8}{x^2}$ at $x=-1$

$$\begin{aligned}f(x)&=x-4x^{-1}-8x^{-2}\\ \therefore\quad f'(x)&=1+4x^{-2}+16x^{-3}\\&=1+\frac{4}{x^2}+\frac{16}{x^3}\\ \text{and so}\quad f'(-1)&=1+4-16\\&=-11\end{aligned}$$

$\therefore$ tangent has slope of -11

6 **a**

$$\begin{aligned}f(x)&=4\sqrt{x}+x=4x^{\frac{1}{2}}+x\\ \therefore\quad f'(x)&=4\left(\tfrac{1}{2}\right)x^{-\frac{1}{2}}+1\\&=\frac{2}{\sqrt{x}}+1\end{aligned}$$

b

$$\begin{aligned}f(x)&=\sqrt[3]{x}=x^{\frac{1}{3}}\\ \therefore\quad f'(x)&=\tfrac{1}{3}x^{-\frac{2}{3}}\\&=\frac{1}{3\sqrt[3]{x^2}}\end{aligned}$$

c

$$\begin{aligned}f(x)&=-\frac{2}{\sqrt{x}}=-2x^{-\frac{1}{2}}\\ \therefore\quad f'(x)&=-2(-\tfrac{1}{2})\,x^{-\frac{3}{2}}\\&=x^{-\frac{3}{2}}\\&=\frac{1}{x\sqrt{x}}\end{aligned}$$

d

$$\begin{aligned}f(x)&=2x-\sqrt{x}=2x-x^{\frac{1}{2}}\\ \therefore\quad f'(x)&=2-\tfrac{1}{2}x^{-\frac{1}{2}}\\&=2-\frac{1}{2\sqrt{x}}\end{aligned}$$

e $f(x) = \dfrac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5$

$\therefore \quad f'(x) = 4(-\tfrac{1}{2})\,x^{-\frac{3}{2}} - 5$

$= -2x^{-\frac{3}{2}}$ or $\dfrac{-2}{x\sqrt{x}}$

f $f(x) = 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}}$

$\therefore \quad f'(x) = 6x - \tfrac{3}{2}x^{\frac{1}{2}}$

$= 6x - \tfrac{3}{2}\sqrt{x}$

g $f(x) = \dfrac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}}$

$\therefore \quad f'(x) = 5(-\tfrac{5}{2})x^{-\frac{7}{2}}$

$= -\tfrac{25}{2}x^{-\frac{7}{2}}$

$= \dfrac{-25}{2x^3\sqrt{x}}$

h $f(x) = 2x - \dfrac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}}$

$\therefore \quad f'(x) = 2 - 3(-\tfrac{3}{2})x^{-\frac{5}{2}}$

$= 2 + \tfrac{9}{2}x^{-\frac{5}{2}}$

$= 2 + \dfrac{9}{2x^2\sqrt{x}}$

7 **a** $y = 4x - \dfrac{3}{x} = 4x - 3x^{-1}$ $\quad\therefore\quad \dfrac{dy}{dx} = 4 + 3x^{-2} = 4 + \dfrac{3}{x^2}$

$\dfrac{dy}{dx}$ is the slope function of $y = 4x - \dfrac{3}{x}$ from which the slope at any point can be found.

b $S = 2t^2 + 4t$ m $\quad\therefore\quad \dfrac{dS}{dt} = 4t + 4$ ms^{-1}

$\dfrac{dS}{dt}$ is the instantaneous rate of change in position at time t, i.e., it is the velocity function.

c $C = 1785 + 3x + 0.002x^2$ dollars.

$\dfrac{dC}{dx} = 3 + 2(0.002)x = 3 + 0.004x$ dollars/toaster

$\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 21E.1

1 **a** $f(x) = x^2, \quad g(x) = 2x + 7,$

$\therefore \quad f(g(x)) = f(2x+7)$

$= (2x+7)^2$

b $f(x) = 2x + 7, \quad g(x) = x^2,$

$f(g(x)) = f(x^2)$

$= 2x^2 + 7$

c $f(x) = \sqrt{x}, \quad g(x) = 3 - 4x,$

$f(g(x)) = f(3-4x) = \sqrt{3-4x}$

d $f(x) = 3 - 4x, \quad g(x) = \sqrt{x},$

$f(g(x)) = f(\sqrt{x}) = 3 - 4\sqrt{x}$

e $f(x) = \dfrac{2}{x}, \quad g(x) = x^2 + 3,$

$f(g(x)) = f(x^2 + 3)$

$= \dfrac{2}{x^2+3}$

f $f(x) = x^2 + 3, \quad g(x) = \dfrac{2}{x},$

$f(g(x)) = f(\tfrac{2}{x}) = \left(\dfrac{2}{x}\right)^2 + 3$

$= \dfrac{4}{x^2} + 3$

g $f(x) = 2^x, \quad g(x) = 3x + 4,$

$f(g(x)) = f(3x+4) = 2^{3x+4}$

h $f(x) = 3x + 4, \quad g(x) = 2^x,$

$f(g(x)) = f(2^x) = 3(2^x) + 4$

2 **a** $f(g(x)) = (3x+10)^3 \quad\therefore\quad f(x) = x^3, \quad g(x) = 3x + 10$

b $f(g(x)) = \dfrac{1}{2x+4} \quad\therefore\quad f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 4$

c $f(g(x)) = \sqrt{x^2 - 3x} \quad\therefore\quad f(x) = \sqrt{x}, \quad g(x) = x^2 - 3x$

d $f(g(x)) = \dfrac{1}{\sqrt{5-2x}}$ $\quad\therefore\quad f(x) = \dfrac{1}{\sqrt{x}},\quad g(x) = 5-2x$

e $f(g(x)) = (x^2+5x-1)^4$ $\quad\therefore\quad f(x) = x^4,\quad g(x) = x^2+5x-1$

f $f(g(x)) = \dfrac{10}{(3x-x^2)^3}$ $\quad\therefore\quad f(x) = \dfrac{10}{x^3},\quad g(x) = 3x-x^2$

EXERCISE 21E.2

1 **a** $\dfrac{1}{(2x-1)^2}$
$= (2x-1)^{-2}$
$= u^{-2},\quad \text{where}\quad u = 2x-1$

b $\sqrt{x^2-3x}$
$= (x^2-3x)^{\frac{1}{2}}$
$= u^{\frac{1}{2}},\quad \text{where}\quad u = x^2-3x$

c $\dfrac{2}{\sqrt{2-x^2}}$
$= 2(2-x^2)^{-\frac{1}{2}}$
$= 2u^{-\frac{1}{2}},\quad \text{where}\quad u = 2-x^2$

d $\sqrt[3]{x^3-x^2}$
$= (x^3-x^2)^{\frac{1}{3}}$
$= u^{\frac{1}{3}},\quad \text{where}\quad u = x^3-x^2$

e $\dfrac{4}{(3-x)^3}$
$= 4(3-x)^{-3}$
$= 4u^{-3},\quad \text{where}\quad u = 3-x$

f $\dfrac{10}{x^2-3}$
$= 10(x^2-3)^{-1}$
$= 10u^{-1},\quad \text{where}\quad u = x^2-3$

2 **a**
$$\begin{aligned} y &= (4x-5)^2 \\ \therefore\quad y &= u^2 \quad \text{where } u = 4x-5 \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 2u(4) \\ &= 8u \\ &= 8(4x-5) \end{aligned}$$

b
$$\begin{aligned} y &= \frac{1}{5-2x} = u^{-1} \quad \text{where } u = 5-2x \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = -u^{-2}(-2) \\ &= \frac{2}{u^2} \\ &= \frac{2}{(5-2x)^2} \end{aligned}$$

c
$$\begin{aligned} y &= \sqrt{3x-x^2} \\ \therefore\quad y &= u^{\frac{1}{2}} \quad \text{where } u = 3x-x^2 \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \tfrac{1}{2}u^{-\frac{1}{2}}(3-2x) \\ &= \frac{(3-2x)}{2\sqrt{u}} \\ &= \frac{3-2x}{2\sqrt{3x-x^2}} \end{aligned}$$

d
$$\begin{aligned} y &= (1-3x)^4 \\ \therefore\quad y &= u^4 \quad \text{where } u = 1-3x \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 4u^3(-3) \\ &= -12u^3 \\ &= -12(1-3x)^3 \end{aligned}$$

e
$$\begin{aligned} y &= 6(5-x)^3 \\ \therefore\quad y &= 6u^3 \quad \text{where } u = 5-x \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 18u^2(-1) \\ &= -18u^2 \\ &= -18(5-x)^2 \end{aligned}$$

f
$$\begin{aligned} y &= \sqrt[3]{2x^3-x^2} \\ \therefore\quad y &= u^{\frac{1}{3}} \quad \text{where } u = 2x^3-x^2 \\ \text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \tfrac{1}{3}u^{-\frac{2}{3}}(6x^2-3x) \\ &= \frac{6x^2-2x}{3\sqrt[3]{(2x^3-x^2)^2}} \end{aligned}$$

g $y = \dfrac{6}{(5x-4)^2}$

$\therefore \ y = 6u^{-2}$ where $u = 5x - 4$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= -12u^{-3}(5)$

$= -\dfrac{60}{u^3}$

$= \dfrac{-60}{(5x-4)^3}$

h $y = \dfrac{4}{3x - x^2}$

$\therefore \ y = 4u^{-1}$ where $u = 3x - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= -4u^{-2}(3-2x)$

$= \dfrac{-4(3-2x)}{u^2}$

$= \dfrac{-4(3-2x)}{(3x-x^2)^2}$

i $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

$\therefore \ y = 2u^3$ where $u = x^2 - 2x^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 6u^2(2x + 2x^{-2})$

$= 6\left(2x + \dfrac{2}{x^2}\right)\left(x^2 - \dfrac{2}{x}\right)^2$

3 **a** $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

$\therefore \ y = \sqrt{u}$ where $u = 1 - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x)$

$= \dfrac{-x}{\sqrt{u}}$

$= \dfrac{-x}{\sqrt{1-x^2}}$

at $x = \frac{1}{2}$, $\dfrac{dy}{dx} = \dfrac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$

$= -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)$

$\therefore$ slope of tangent $= -\frac{1}{\sqrt{3}}$

b $y = (3x+2)^6$ at $x = -1$

$\therefore \ y = u^6$ where $u = 3x + 2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= 6u^5(3)$

$= 18u^5$

$= 18(3x+2)^5$

at $x = -1$, $\dfrac{dy}{dx} = 18(-1)^5$

$\therefore$ slope of tangent $= -18$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

$\therefore \ y = u^{-4}$ where $u = 2x - 1$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= -4u^{-5}(2)$

$= \dfrac{-8}{u^5}$

$= \dfrac{-8}{(2x-1)^5}$

at $x = 1$, $\dfrac{dy}{dx} = \dfrac{-8}{1^5}$

$\therefore$ slope of tangent $= -8$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

$\therefore \ y = 6u^{\frac{1}{3}}$ where $u = 1 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= 6(\frac{1}{3})u^{-\frac{2}{3}}(-2)$

$= 2u^{-\frac{2}{3}}(-2)$

$= \dfrac{-4}{\sqrt[3]{u^2}}$

$= \dfrac{-4}{\sqrt[3]{(1-2x)^2}}$

at $x = 0$, $\dfrac{dy}{dx} = \dfrac{-4}{\sqrt[3]{1^2}}$

$\therefore$ slope of tangent $= -4$

e $y = \dfrac{4}{x + 2\sqrt{x}}$ at $x = 4$

$\therefore \; y = 4u^{-1}$ where $u = x + 2x^{\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= -4u^{-2}(1 + x^{-\frac{1}{2}})$

$= -\dfrac{4}{u^2}\left(1 + \dfrac{1}{\sqrt{x}}\right)$

$= \dfrac{-4}{(x + 2\sqrt{x})^2}\left(1 + \dfrac{1}{\sqrt{x}}\right)$

at $x = 4$, $\dfrac{dy}{dx} = \dfrac{-4}{(4+4)^2}(1 + \frac{1}{2})$

$= -\frac{6}{64}$

$\therefore$ slope of tangent $= -\frac{3}{32}$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$

$\therefore \; y = u^3$ where $u = x + x^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= 3u^2(1 - u^{-2})$

$= 3\left(x + \dfrac{1}{x}\right)^2\left(1 - \dfrac{1}{x^2}\right)$

at $x = 1$, $\dfrac{dy}{dx} = 3(1+1)^2(1-1)$

$\therefore$ slope of tangent $= 0$

4 **a** $y = x^3 \quad \therefore \; \dfrac{dy}{dx} = 3x^2$

$x = y^{\frac{1}{3}} \quad \therefore \; \dfrac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$\dfrac{dy}{dx}\dfrac{dx}{dy} = 3x^2\left(\frac{1}{3}\right)y^{-\frac{2}{3}}$

$= x^2(y)^{-\frac{2}{3}}$

$= x^2(x^3)^{-\frac{2}{3}}$ {substituting $y = x^3$}

$= x^2(x^{-2})$

$= x^0$

$= 1$ as required.

b We know that $\dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{dy}{dx}$ {chain rule}

$\therefore \; \dfrac{dy}{dx}\dfrac{dx}{dy} = \dfrac{dy}{dy}$

$= 1$

EXERCISE 21F.1

1 **a** $y = x^2(2x-1)$ is the product of $u = x^2$ and $v = 2x - 1$

$\therefore \; u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \; \dfrac{dy}{dx} = 2x(2x-1) + x^2(2)$

$= 4x^2 - 2x + 2x^2$

$= 6x^2 - 2x$

b $y = 4x(2x+1)^3$ is the product of $u = 4x$ and $v = (2x+1)^3$

$\therefore \; u' = 4$ and $v' = 6(2x+1)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore \; \dfrac{dy}{dx} = 4(2x+1)^3 + 24x(2x+1)^2$

$= [4(2x+1) + 24x](2x+1)^2$

$= [32x + 4](2x+1)^2$

c $y = x^2\sqrt{3-x}$ is the product of $u = x^2$ and $v = (3-x)^{\frac{1}{2}}$

$\therefore\ u' = 2x$ and $v' = -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} + x^2\left[-\tfrac{1}{2}(3-x)^{-\frac{1}{2}}\right]$$

$$= 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}}$$

d $y = \sqrt{x}(x-3)^2$ is the product of $y = x^{\frac{1}{2}}$ and $v = (x-3)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)^1$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$

$\therefore\ u' = 10x$ and $v' = 12x(3x^2-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1)$$
$$= 10x(3x^2-1)^2 + 60x^3(3x^2-1)$$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(1-2x)(x-x^2)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(1-2x)(x-x^2)^2$$

2 **a** $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$

$u' = 4x^3$ and $v' = 2(1-2x)^1(-2)$

$= -4(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

at $x = -1$, $\dfrac{dy}{dx} = -4(3^2) - 4(3) = -48$ $\therefore$ slope of tangent $= -48$

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(2x-1)(x^2-x+1)$$

at $x = 4$, $\dfrac{dy}{dx} = \frac{1}{4}(13)^2 + 4(7)(13) = 406\frac{1}{4}$

$\therefore$ slope of tangent $= 406\frac{1}{4}$

c $y = x\sqrt{1-2x}$ is the product of $u = x$ and $v = (1-2x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)$
$= -(1-2x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore\ \dfrac{dy}{dx} = \sqrt{1-2x} - \dfrac{x}{\sqrt{1-2x}}$

at $x = -4$, $\dfrac{dy}{dx} = \sqrt{9} - \dfrac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$

$\therefore$ slope of tangent $= \frac{13}{3}$

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$\therefore\ u' = 3x^2$ and $v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x)$
$= -x(5-x^2)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore\ \dfrac{dy}{dx} = 3x^2\sqrt{5-x^2} - \dfrac{x^4}{\sqrt{5-x^2}}$

at $x = 1$, $\dfrac{dy}{dx} = 3(1)^2\sqrt{4} - \dfrac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2}$ $\therefore$ slope of tangent $= \frac{11}{2}$

3 $y = \sqrt{x}(3-x)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(3-x)^1(-1) = -2(3-x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x)$$
$$= \frac{(3-x)^2 - 4x(3-x)}{2\sqrt{x}}$$
$$= \frac{(3-x)\left[(3-x) - 4x\right]}{2\sqrt{x}}$$
$$= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}$$

Tangents are horizontal when their slopes are 0.

$\therefore\ \dfrac{dy}{dx} = 0$ when $(3-x)(3-5x) = 0$
i.e., $3 - x = 0$ or $3 - 5x = 0$
i.e., $x = 3$ or $x = \frac{3}{5}$

EXERCISE 21F.2

1 **a** $y = \dfrac{1+3x}{2-x}$ is a quotient where $u = 1 + 3x$ and $v = 2 - x$

$\therefore\ u' = 3$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{3(2-x) - (-1)(1+3x)}{(2-x)^2} = \dfrac{7}{(2-x)^2}$

b $y = \dfrac{x^2}{2x+1}$ is a quotient where $u = x^2$ and $v = 2x+1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{2x(2x+1) - x^2(2)}{(2x+1)^2} = \dfrac{2x^2+2x}{(2x+1)^2}$

c $y = \dfrac{x}{x^2-3}$ is a quotient where $u = x$ and $v = x^2 - 3$

$\therefore\ u' = 1$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1(x^2-3) - x(2x)}{(x^2-3)^2} = \dfrac{-3-x^2}{(x^2-3)^2}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 1-2x$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{\dfrac{1-2x}{2\sqrt{x}} - (-2)\sqrt{x}}{(1-2x)^2} = \dfrac{1-2x+4x}{2\sqrt{x}(1-2x)^2} = \dfrac{1+2x}{2\sqrt{x}(1-2x)^2}$

e $y = \dfrac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2-3$ and $v = 3x - x^2$

$\therefore\ u' = 2x$ and $v' = 3-2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2} = \frac{6x^2-2x^3-3x^2+2x^3+9-6x}{(3x-x^2)^2}$$

$$= \frac{3x^2-6x+9}{(3x-x^2)^2}$$

f $y = \dfrac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\sqrt{1-3x} - x\left(\dfrac{-3}{2\sqrt{1-3x}}\right)}{(1-3x)} = \frac{2(1-3x)+3x}{2(1-3x)^{\frac{3}{2}}} = \frac{2-3x}{2(1-3x)^{\frac{3}{2}}}$$

2 **a** $y = \dfrac{x}{1-2x}$ is a quotient where $u = x$ and $v = 1-2x$

$\therefore\ u' = 1$ and $v' = -2$ Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1(1-2x) - x(-2)}{(1-2x)^2} = \dfrac{1}{(1-2x)^2}$

at $x = 1$ $\dfrac{dy}{dx} = \dfrac{1}{(1-2)^2} = \dfrac{1}{(-1)^2} = 1$ $\therefore$ slope of tangent $= 1$

b $y = \dfrac{x^3}{x^2+1}$ is a quotient where $u = x^3$ and $v = x^2 + 1$

$\therefore \quad u' = 3x^2$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2} = \dfrac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \dfrac{x^4 + 3x^2}{(x^2+1)^2}$

$\therefore$ at $x = -1$ $\dfrac{dy}{dx} = \dfrac{1+3}{(1+1)^2} = \dfrac{4}{4} = 1$

and so, the slope of tangent $= 1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 2x + 1$

$u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2} = \dfrac{\dfrac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}$

at $x = 4$ $\dfrac{dy}{dx} = \dfrac{\frac{9}{4} - 4}{81} = \dfrac{\left(\frac{9}{4} - 4\right)}{81} \times \dfrac{4}{4} = \dfrac{9 - 16}{324}$

$\therefore$ slope of tangent $= -\frac{7}{324}$

d $y = \dfrac{x^2}{\sqrt{x^2+5}}$ is a quotient where $u = x^2$ and $v = (x^2+5)^{\frac{1}{2}}$

$\therefore \quad u' = 2x$ and $v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x)$

$= x(x^2+5)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2} = \dfrac{2x\sqrt{x^2+5} - x^2\left(\dfrac{x}{\sqrt{x^2+5}}\right)}{(x^2+5)}$

at $x = -2$ $\dfrac{dy}{dx} = \dfrac{-4(3) - \left(\frac{-8}{3}\right)}{9} = \dfrac{\left(-12 + \frac{8}{3}\right)}{9} \times \dfrac{3}{3} = \dfrac{-36+8}{27}$

$\therefore$ slope of tangent $= -\frac{28}{27}$

3 **a** $y = \dfrac{2\sqrt{x}}{1-x}$ is a quotient where $u = 2x^{\frac{1}{2}}$ and $v = 1 - x$

$\therefore \quad u' = 2x^{\frac{1}{2}}$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \quad \dfrac{dy}{dx} = \left(\dfrac{\dfrac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2}\right)\dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{(1-x)+2x}{\sqrt{x}(1-x)^2} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$ as required.

i $\dfrac{dy}{dx} = 0$ when $x + 1 = 0$ i.e., $x = -1$. However $\dfrac{dy}{dx}$ is not defined for $x \leqslant 0$ because of the $\sqrt{x}$ term. Hence $\dfrac{dy}{dx}$ never equals 0.

ii $\dfrac{dy}{dx}$ is undefined when $x \leqslant 0$ and when $x = 1$

b $y = \dfrac{x^2 - 3x + 1}{x + 2}$ is a quotient where $y = x^2 - 3x + 1$ and $v = x + 2$

$\therefore$ $u' = 2x - 3$ and $v' = 1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore \quad \frac{dy}{dx} = \frac{(2x-3)(x+2) - (x^2 - 3x + 1)(1)}{(x+2)^2}$$

$$= \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x+2)^2}$$

$$= \frac{x^2 + 4x - 7}{(x+2)^2} \quad \text{as required.}$$

i $\dfrac{dy}{dx} = 0$ when $x^2 + 4x - 7 = 0$

i.e., $x = \dfrac{-4 \pm \sqrt{44}}{2} = -2 \pm \sqrt{11}$

ii $\dfrac{dy}{dx}$ is undefined when $(x+2)^2 = 0$ i.e., $x = -2$

EXERCISE 21G

1 **a** $y = x - 2x^2 + 3$ at $x = 2$

Since when $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$, the point of contact is $(2, -3)$.

Now $\dfrac{dy}{dx} = 1 - 4x$

$\therefore$ at $x = 2$, $\dfrac{dy}{dx} = 1 - 8 = -7$

$\therefore$ the tangent has equation $\dfrac{y - (-3)}{x - 2} = -7$ i.e., $y + 3 = -7(x - 2)$

i.e., $y = -7x + 14 - 3$

i.e., $y = -7x + 11$

b $y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$

Since when $x = 4$, $y = \sqrt{4} + 1 = 3$, the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

at $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore$ the tangent has equation $\dfrac{y - 3}{x - 4} = \frac{1}{4}$ i.e., $4y - 12 = x - 4$

i.e., $4y = x + 8$

c $y = x^3 - 5x$ at $x = 1$

Since when $x = 1$, $y = 1^3 - 5(1) = -4$ the point of contact is $(1, -4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$

at $x = 1$, $\dfrac{dy}{dx} = 3 - 5 = -2$

$\therefore$ tangent has equation $\dfrac{y - (-4)}{x - 1} = -2$ i.e., $y + 4 = -2x + 2$

i.e., $y = -2x - 2$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$. Now $y = \dfrac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = -2x^{-\frac{3}{2}}$$

$$\text{at} \quad x = 1, \quad \frac{dy}{dx} = -2\left(1^{-\frac{3}{2}}\right) = -2$$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-1} = -2$ i.e., $y - 4 = -2x + 2$
i.e., $y = -2x + 6$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$. Now $y = \dfrac{3}{x} - \dfrac{1}{x^2} = 3x^{-1} - x^{-2}$

$$\frac{dy}{dx} = -3x^{-2} + 2x^{-3}$$

$$\text{at} \quad x = -1, \quad \frac{dy}{dx} = \frac{-3}{(-1)^2} + \frac{2}{(-1)^3} = -5$$

$\therefore$ tangent has equation $\dfrac{y-(-4)}{x-(-1)} = -5$ i.e., $y + 4 = -5(x+1)$
i.e., $y = -5x - 5 - 4$
i.e., $y = -5x - 9$

f $y = 3x^2 - \dfrac{1}{x} = 3x^2 - x^{-1}$ at $x = -1$

Since when $x = -1$, $y = 3(-1)^2 - \dfrac{1}{(-1)} = 4$ the point of contact is $(1, 4)$.

$$\text{Now} \quad \frac{dy}{dx} = 6x - (-1x^{-2}) = 6x + \frac{1}{x^2}$$

$$\text{at} \quad x = -1, \quad \frac{dy}{dx} = 6(-1) + \frac{1}{(-1)^2} = -5$$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-(-1)} = -5$ i.e., $y - 4 = -5x - 5$
i.e., $y = -5x - 1$

2 **a** $y = x^2$ at $(3, 9)$

$$\frac{dy}{dx} = 2x$$

at $x = 3$, $\dfrac{dy}{dx} = 2(3) = 6 = \frac{6}{1}$

$\therefore$ the normal at (3, 9) has slope $-\frac{1}{6}$, so the equation of the normal is

$\dfrac{y-9}{x-3} = -\frac{1}{6}$ i.e., $6y - 54 = -x + 3$
i.e., $6y = -x + 57$

b $y = x^3 - 5x + 2$ at $x = -2$

Since when $x = -2$, $y = (-2)^3 - 5(-2) + 2 = 4$

and so the point of contact is $(-2, 4)$

$$\text{Now} \quad \frac{dy}{dx} = 3x^2 - 5$$

$\therefore$ at $x = -2$, $\dfrac{dy}{dx} = 3(-2)^2 - 5 = 7$

$\therefore$ the normal at $(-2, 4)$ has slope $-\frac{1}{7}$, so the equation of the normal is

$\dfrac{y-4}{x-(-2)} = -\frac{1}{7}$ i.e., $7y - 28 = -(x+2)$
i.e., $7y = -x + 26$

c $y = 2\sqrt{x} + 3$ at $x = 1$

Since when $x = 1$, $y = 2\sqrt{1} + 3 = 5$ the point of contact is $(1, 5)$

$$\text{Now} \quad y = 2\sqrt{x} + 3 = 2x^{\frac{1}{2}} + 3$$

$$\therefore \quad \frac{dy}{dx} = x^{-\frac{1}{2}}$$

and so, at $x = 1$, $\dfrac{dy}{dx} = \frac{1}{\sqrt{1}} = 1$

$\therefore$ the normal at (1, 5) has slope -1, and so the equation of the normal is

$$\frac{y-5}{x-1} = -1 \quad \text{i.e.,} \quad y - 5 = -x + 1$$
$$\text{i.e.,} \quad y = -x + 6$$

d $y = \dfrac{3}{\sqrt{x}}$ and so, when $x = 9$, $y = \frac{3}{\sqrt{9}} = 1$ $\therefore$ the point of contact is $(9, 1)$

$$y = \frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\tfrac{3}{2}x^{-\frac{3}{2}}$$

$\therefore$ at $x = 9$, $\dfrac{dy}{dx} = -\frac{3}{2}\left(9^{-\frac{3}{2}}\right) = -\frac{3}{2}\left(3^{-3}\right) = -\frac{1}{18}$

$\therefore$ the normal at (9, 1) has slope 18, so the equation of the normal is

$$\frac{y-1}{x-9} = 18 \quad \text{i.e.,} \quad y - 1 = 18x - 162$$
$$\text{i.e.,} \quad y = 18x - 161$$

e $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$

$$\text{Now} \quad y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\tfrac{5}{2}x^{-\frac{3}{2}} - \tfrac{1}{2}x^{-\frac{1}{2}}$$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -\frac{5}{2}\left(1^{-\frac{3}{2}}\right) - \frac{1}{2}\left(1^{-\frac{1}{2}}\right) = -\frac{5}{2} - \frac{1}{2} = -3$

$\therefore$ the normal at (1, 4) has slope $\frac{1}{3}$, so the equation of the normal is

$$\frac{y-4}{x-1} = \tfrac{1}{3} \quad \text{i.e.,} \quad 3y - 12 = x - 1$$
$$\text{i.e.,} \quad 3y = x + 11$$

f $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$

Since when $x = 1$, $y = 8\sqrt{1} - \dfrac{1}{1^2} = 7$ the point of contact is $(1, 7)$

$$\text{Now} \quad y = 8\sqrt{x} - \frac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$$

$$\therefore \quad \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 4 + 2 = 6$

$\therefore$ the normal at (1, 7) has slope $-\frac{1}{6}$, so the equation of the normal is

$$\frac{y-7}{x-1} = -\tfrac{1}{6} \quad \text{i.e.,} \quad 6y - 42 = -x + 1$$
$$\text{i.e.,} \quad 6y = -x + 43$$

3 **a** $y = 2x^3 + 3x^2 - 12x + 1 \quad \therefore \quad \dfrac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have slope $= 0$ so

$$\begin{aligned} 6x^2 + 6x - 12 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ \therefore \quad x = -2 \quad \text{or} \quad x &= 1 \end{aligned}$$

Now at $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$

and at $x = 1$, $y = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$

i.e., points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ tangents are $y = -6$ and $y = 21$

b Now $y = \dfrac{2x+1}{\sqrt{x}} = 2\sqrt{x} + \dfrac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}$

Horizontal tangents have slope $= 0$

$$\begin{aligned} \therefore \quad \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}} &= 0 \\ \therefore \quad \frac{2x-1}{2x\sqrt{x}} &= 0 \\ \therefore \quad 2x - 1 &= 0 \\ \therefore \quad x &= \tfrac{1}{2} \end{aligned}$$

Now at $x = \frac{1}{2}$, $y = \dfrac{2\left(\frac{1}{2}\right) + 1}{\sqrt{\frac{1}{2}}} = \dfrac{2}{\frac{1}{2}\sqrt{2}} = 2\sqrt{2}$

$\therefore$ the only horizontal tangent touches at $(\frac{1}{2}, 2\sqrt{2})$

c

$$\begin{aligned} \text{Now} \quad y &= 2x^3 + kx^2 - 3 \\ \therefore \quad \frac{dy}{dx} &= 6x^2 + 2kx \end{aligned}$$

and so when $x = 2$, $\dfrac{dy}{dx} = 4$

$$\begin{aligned} \therefore \quad 6(2)^2 + 2k(2) &= 4 \\ \therefore \quad 24 + 4k &= 4 \\ \therefore \quad 4k &= -20 \\ \therefore \quad k &= -5 \end{aligned}$$

d Now $y = 1 - 3x + 12x^2 - 8x^3 \quad \therefore \quad \dfrac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$, $\dfrac{dy}{dx} = -3 + 24 - 24 = -3$

i.e., the tangent at $(1, 2)$ has slope -3

$\therefore \quad -3 + 24x - 24x^2 = -3$ for all points x where the tangent has slope $= -3$

$\therefore \quad 24x^2 - 24x = 0$

$\therefore \quad 24x(x-1) = 0$

i.e., when $x = 0$ or $x = 1$

When $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

and when $x = 1$, $y = 1 - 3 + 12 - 8 = 2$

$\therefore$ the tangents are $\dfrac{y-1}{x-0} = -3$ and $\dfrac{y-2}{x-1} = -3$

i.e., $y = -3x + 1$ and $y = -3x + 5$

4 **a** Now $y = x^2 + ax + b \quad \therefore \ \dfrac{dy}{dx} = 2x + a$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 2 + a$

$\therefore$ the slope of the tangent to the curve at $x = 1$ will be $2 + a$

However the equation of the tangent is $y + 2x = 6$ i.e., $y = -2x + 6$

and so the slope of the tangent is -2. $\quad \therefore \ 2 + a = -2$

$$a = -4$$

$$\therefore \ y = x^2 - 4x + b$$

Also, at $x = 1$, the tangent line contacts the curve

i.e., $1^2 - 4(1) + b = -2(1) + 6$

$$1 - 4 + b = 4$$

$$b = 7 \qquad \therefore \ a = -4, \quad b = 7$$

b Now $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \ \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{a}{2}\left(4^{-\frac{1}{2}}\right) - \dfrac{b}{2}\left(4^{-\frac{3}{2}}\right) = \dfrac{a}{2}\left(\dfrac{1}{2}\right) - \dfrac{b}{2}\left(\dfrac{1}{8}\right) = \dfrac{a}{4} - \dfrac{b}{16}$

$\therefore$ the slope of the tangent to the curve at $x = 4$ will be $\dfrac{a}{4} - \dfrac{b}{16} = \dfrac{4a - b}{16}$

However the equation of the *normal* is $4x + y = 22$, i.e., $y = -4x + 22$

$\therefore$ the normal has slope -4

$\therefore$ the tangent has slope $\frac{1}{4}$ and so, $\dfrac{4a - b}{16} = \frac{1}{4}$

$$4a - b = 4$$

$$b = 4a - 4 \quad \text{...... (1)}$$

Also, at $x = 4$ the normal line intersects the curve.

i.e., $a\sqrt{4} + \dfrac{b}{\sqrt{4}} = -4(4) + 22$

$$\therefore \ 2a + \frac{b}{2} = 6$$

Consequently, $2a + \dfrac{4a - 4}{2} = 6$ using (1)

$$\therefore \ 2a + 2a - 2 = 6$$

$$\therefore \ 4a = 8$$

$\therefore \ a = 2$ and so $b = 4(2) - 4 = 4$ from (1)

5 **a** $y = \sqrt{2x + 1}$ when $x = 4$, $y = \sqrt{2(4) + 1} = 3 \quad \therefore$ the point of contact is $(4, 3)$

Since $y = (2x + 1)^{\frac{1}{2}}$ then $\dfrac{dy}{dx} = \frac{1}{2}(2x + 1)^{-\frac{1}{2}}(2) = \dfrac{1}{\sqrt{2x + 1}}$

$\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2(4) + 1}} = \frac{1}{3}$

so the tangent has equation $\dfrac{y - 3}{x - 4} = \frac{1}{3}$ i.e., $3y = x + 5$

b $y = \dfrac{1}{2-x} = (2-x)^{-1}$ at $x=-1$, $y = \dfrac{1}{2-(-1)} = \frac{1}{3}$

$\therefore$ the point of contact is $(-1, \frac{1}{3})$

Now $\dfrac{dy}{dx} = -1(2-x)^{-2}(-1) = \dfrac{1}{(2-x)^2}$

$\therefore$ at $x=-1$, $\dfrac{dy}{dx} = \dfrac{1}{(2-(-1))^2} = \frac{1}{9}$

$\therefore$ the tangent has equation $\dfrac{y-\frac{1}{3}}{x-(-1)} = \frac{1}{9}$ i.e., $9y - 3 = x+1$

i.e., $9y = x+4$

c $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$

as $y = (x^2+1)^{-2}$

$\dfrac{dy}{dx} = -2(x^2+1)^{-3}(2x) = \dfrac{-4x}{(x^2+1)^3}$

$\therefore$ at $x=1$, $\dfrac{dy}{dx} = \dfrac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$

$\therefore$ normal at $(1, \frac{1}{4})$ has slope 2.

So the equation of the normal is $\dfrac{y-\frac{1}{4}}{x-1} = 2$ i.e., $y - \frac{1}{4} = 2x-2$

$2x - y = \frac{7}{4}$

i.e., $y = 2x - \frac{7}{4}$

d $y = \dfrac{1}{\sqrt{3-2x}}$ at $x=-3$, $y = \dfrac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

Now $y = (3-2x)^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = -\frac{1}{2}(3-2x)^{-\frac{3}{2}}(-2) = (3-2x)^{-\frac{3}{2}}$

$\therefore$ at $x=-3$, $\dfrac{dy}{dx} = (3-2(-3))^{-\frac{3}{2}} = 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$

$\therefore$ normal at $(-3, \frac{1}{3})$ has slope -27,

So the equation of the normal is $\dfrac{y-\frac{1}{3}}{x-(-3)} = -27$ i.e., $y - \frac{1}{3} = -27(x+3)$

i.e., $y = -27x - \frac{242}{3}$

6 The tangent has equation $3x+y=5$, i.e., $y=-3x+5$

$\therefore$ tangent has slope -3 ...... (1)

Also, at $x=-1$, $y=-3(-1)+5=8$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ ...... (2)

Now $y = a(1-bx)^{\frac{1}{2}}$, $\therefore$ $\dfrac{dy}{dx} = \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$

$\therefore$ $-3 = \frac{1}{2}a(1+b)^{-\frac{1}{2}}(-b)$ using (1)

$6 = \dfrac{ab}{\sqrt{1+b}}$ ...... (3)

and $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

and so $8 = a\sqrt{1+b}$ ...... (4)

$\therefore \quad \dfrac{6\sqrt{1+b}}{b} = \dfrac{8}{\sqrt{1+b}}$ {equating a's in (3) and (4)}

$\therefore \quad 6(1+b) = 8b$

$\therefore \quad 6+6b = 8b$

$\therefore \quad 6 = 2b$

$\therefore \quad b = 3$ and $a = \dfrac{8}{\sqrt{4}} = 4$

7 **a** $f(x) = \dfrac{x}{1-3x}$ at $(-1, -\frac{1}{4})$

$f(x)$ is a quotient where $u = x$ and $v = 1-3x$

$\therefore \quad u' = 1$ and $v' = -3$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \quad f'(x) = \dfrac{1(1-3x) - x(-3)}{(1-3x)^2} = \dfrac{1}{(1-3x)^2}$

$\therefore \quad f(-1) = \dfrac{1}{(1-3(-1))^2} = \frac{1}{16}$

So the tangent has equation $\dfrac{y - \left(-\frac{1}{4}\right)}{x-(-1)} = \frac{1}{16}$ i.e., $16y + 4 = x+1$

i.e., $16y = x - 3$

b $f(x) = \sqrt{x}(1-x)^2$

since $f(4) = \sqrt{4}(1-4)^2 = 18$, the point of contact is (4, 18)

Now $f(x)$ is a product where $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$\therefore \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-x)(-1) = -2(1-x)$

Now $f'(x) = u'v + uv'$ {product rule}

$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$

$\therefore \quad f'(4) = \frac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4) = \frac{1}{4}(9) - 2(2)(-3) = \frac{57}{4}$

$\therefore$ the normal at (4, 18) has slope $-\frac{4}{57}$.

So the equation of the normal is $\dfrac{y-18}{x-4} = -\frac{4}{57}$ i.e., $57(y-18) = -4(x-4)$

i.e., $57y = -4x + 1042$

c $f(x) = \dfrac{x^2}{1-x}$ at $(2, -4)$

$f(x)$ is a quotient where $u = x^2$ and $v = 1-x$

$u' = 2x$ and $v' = -1$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \quad f'(x) = \dfrac{2x(1-x) - x^2(-1)}{(1-x)^2} = \dfrac{2x - 2x^2 + x^2}{(1-x)^2} = \dfrac{2x - x^2}{(1-x)^2}$

$\therefore \quad f'(2) = \dfrac{2(2) - 2^2}{(1-2)^2} = \dfrac{4-4}{1} = 0$

As the tangent has slope 0, i.e., is horizontal, it has equation $y = c$ and as the contact point is $(2, -4)$, the tangent has equation $y = -4$

d $\quad f(x) = \dfrac{x^2-1}{2x+3}$

Since $\quad f(-1) = \dfrac{(-1)^2-1}{2(-1)+3} = \dfrac{0}{1} = 0 \quad$ the point of contact is $(-1,\ 0)$

Now $f(x)$ is a quotient where $\quad u = x^2-1 \quad$ and $\quad v = 2x+3$

$\therefore \quad u' = 2x \quad$ and $\quad v' = 2$

Now $\quad f'(x) = \dfrac{u'v-uv'}{v^2} = \dfrac{2x(2x+3)-(x^2-1)(2)}{(2x+3)^2}$

$\therefore \quad f'(-1) = \dfrac{2(-1)(-2+3)-((-1)^2-1)(2)}{(2(-1)+3)^2} = \dfrac{-2(-1)-(0)(2)}{(1)^2} = -2$

$\therefore \quad$ the normal at $(-1,\ 0)$ has slope $\frac{1}{2}$.

So the equation of the normal is $\quad \dfrac{y-0}{x-(-1)} = \frac{1}{2} \quad$ i.e., $\quad 2y = x+1$

8 **a** $y = x^3 \quad$ at $\quad x = 2$

Since when $\ x = 2, \quad y = 2^3 = 8 \quad$ the point of contact is $(2,\ 8)$

Now $\quad \dfrac{dy}{dx} = 3x^2 \quad$ and so at $\quad x = 2, \quad \dfrac{dy}{dx} = 3(2)^2 = 12$

$\therefore$ tangent at (2, 8) has slope 12 and its equation is $\dfrac{y-8}{x-2} = 12 \quad$ i.e., $\ y-8 = 12x-24$

i.e., $\quad y = 12x-16$

$\therefore \quad$ the tangent meets the curve where $\quad 12x-16 = x^3$

i.e., $\quad x^3-12x+16 = 0$

Because the tangent touches the curve at $\ x = 2,\ $ there must be a repeated solution at this point. $\quad \therefore \quad (x-2)^2$ must be a factor of this cubic

$\therefore \quad (x-2)^2(x+4) = 0$

$\therefore \quad$ tangent meets curve again when $\ x = -4$

and when $\quad x = -4, \quad y = (-4)^3 = -64$

$\therefore \quad$ tangent meets curve again at $(-4,\ -64)$

b $y = -x^3+2x^2+1 \quad$ at $\quad x = -1$

Since when $\quad x = -1, \quad y = -(-1)^3+2(-1)^2+1 = 4$

and so the point of contact is $(-1,\ 4)$

Now $\quad \dfrac{dy}{dx} = -3x^2+4x \qquad \therefore \quad$ at $\quad x = -1, \quad \dfrac{dy}{dx} = -3(-1)^2+4(-1) = -7$

$\therefore \quad$ tangent at $(-1,\ 4)$ has slope -7

$\therefore \quad$ its equation is $\quad \dfrac{y-4}{x-(-1)} = -7 \qquad \therefore \quad y-4 = -7(x+1)$

i.e., $\quad y = -7x-3$

Now the tangent meets the curve where $\quad -7x-3 = -x^3+2x^2+1$

i.e., $\quad x^3-2x^2-7x-4 = 0$

Because the tangent touches the curve at $\ x = -1,\ $ there must be a repeated solution at this point. $\quad \therefore \quad (x+1)^2$ must be a factor of this cubic

$\therefore \quad (x+1)^2(x-4) = 0$

$\therefore \quad$ tangent meets curve again when $\ x = 4$

and when $\quad x = 4, \quad y = -(4)^3+2(4)^2+1 = -64+32+1 = -31$

$\therefore \quad$ tangent meets curve again at $(4,\ -31)$

c $y = x^2 - \dfrac{3}{x}$ at $x = 3$

Since when $x = 3$, $y = 3^2 - \frac{3}{3} = 8$, the point of contact is (3, 8)

$$\text{Now } \frac{dy}{dx} = 2x + \frac{3}{x^2}$$

$$\therefore \text{ at } x = 3, \quad \frac{dy}{dx} = 2(3) + \frac{3}{3^2} = 6 + \tfrac{1}{3} = \tfrac{19}{3}$$

$\therefore$ tangent at (3, 8) has slope $\frac{19}{3}$ and therefore its equation is

$$\frac{y-8}{x-3} = \tfrac{19}{3} \quad \text{i.e.,} \quad 3(y-8) = 19(x-3)$$
$$3y = 19x - 33$$
$$\text{i.e.,} \quad y = \tfrac{19}{3}x - 11$$

Now the tangent meets the curve where $\frac{19}{3}x - 11 = x^2 - \dfrac{3}{x}$

$$\therefore \quad 19x^2 - 33x = 3x^3 - 9$$
$$\text{i.e.,} \quad 3x^3 - 19x^2 + 33x - 9 = 0$$

Because the tangent touches the curve at $x = 3$, there must be a repeated solution at this point.

Hence, $3x^3 - 19x^2 + 33x - 9 = (x-3)^2(3x-1) = 0$

{since coefficient of x^3 is 3 and constant term is -9}

$\therefore$ the tangent meets the curve again at $x = \frac{1}{3}$ where $y = \left(\frac{1}{3}\right)^2 - \dfrac{3}{\left(\frac{1}{3}\right)} = \frac{1}{9} - 9 = -\frac{80}{9}$

$\therefore$ tangent meets curve again at $\left(\frac{1}{3}, -\frac{80}{9}\right)$

d $y = x^3 + \dfrac{4}{x}$ at $x = 1$

Since when $x = 1$, $y = 1^3 + \frac{4}{1} = 5$ and so the point of contact is (1, 5)

Now $\dfrac{dy}{dx} = 3x^2 - \dfrac{4}{x^2}$ and at $x = 1$, $\dfrac{dy}{dx} = 3 - 4 = -1$

$\therefore$ tangent at (1, 5) has slope -1 and therefore its equation is

$$\frac{y-5}{x-1} = -1 \quad \text{i.e.,} \quad y - 5 = -x + 1$$
$$\text{i.e.,} \quad y = -x + 6$$

Now the tangent meets the curve where $-x + 6 = x^3 + \dfrac{4}{x}$

$$\therefore \quad x^3 + x - 6 + \frac{4}{x} = 0$$
$$\therefore \quad x^4 + x^2 - 6x + 4 = 0$$

Using a graphics calculator, this quartic has a graph which touches the x-axis at $x = 1$ and has no other x-intercepts. i.e., the tangent *never* meets the curve again.

9 **a** $y = x^2 - x + 9$ at $x = a$

Since when $x = a$, $y = a^2 - a + 9$ the point of contact is $(a, a^2 - a + 9)$

$$\text{Now } \frac{dy}{dx} = 2x - 1 \quad \therefore \text{ at } x = a, \quad \frac{dy}{dx} = 2a - 1$$

$\therefore$ the slope of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\dfrac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

$$\text{i.e.,} \quad y - (a^2 - a + 9) = (2a-1)(x-a)$$
$$\text{i.e.,} \quad y = (2a-1)x - 2a^2 + a + a^2 - a + 9$$
$$\text{i.e.,} \quad y = (2a-1)x - a^2 - 9$$

But this tangent passes through (0, 0) $\therefore\ 0 = a^2 - 9$

$\therefore\ (a+3)(a-3) = 0$

$\therefore\ a = \pm 3$

$\therefore$ tangents are: at $a = 3$: $y - (9 - 3 + 9) = 5(x - 3)$

$\therefore\ y = 5x$, with contact at (3, 15)

at $a = -3$: $y - (9 + 3 + 9) = -7(x + 3)$

$\therefore\ y = -7x$, with contact at (−3, 21)

points of contact are (3, 15) and (−3, 21)

b Let (a, a^3) lie on $y = x^3$

Now $\dfrac{dy}{dx} = 3x^2$, $\therefore$ at $x = a$, $\dfrac{dy}{dx} = 3a^2$

$\therefore$ the slope of the tangent at (a, a^3) is $3a^2$

$\therefore$ the equation of the tangent is $\dfrac{y - a^3}{x - a} = 3a^2$ i.e., $y - a^3 = (3a^2)(x - a)$

But this tangent passes through (−2, 0) $\therefore\ a - a^3 = 3a^2(-2 - a)$

$\therefore\ -a^3 = -6a^2 - 3a^3$

$\therefore\ 2a^3 + 6a^2 = 0$

$\therefore\ 2a^2(a + 3) = 0$

$\therefore\ a = 0$ or -3

If $a = 0$, tangent equation is $y = 0$, with contact point (0, 0)

If $a = -3$, tangent equation is $y - (-27) = 27(x + 3)$

i.e., $y = 27x + 54$, with contact point (−3, −27)

c Let $(a, \sqrt{a})$ lie on $y = \sqrt{x}$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ $\therefore$ at $x = a$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a}}$

$\therefore$ the slope of the tangent at $(a, \sqrt{a})$ is $\dfrac{1}{2\sqrt{a}}$

$\therefore$ the slope of the normal at $(a, \sqrt{a})$ is $-2\sqrt{a}$

$\therefore$ the normal has equation $\dfrac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$ i.e., $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through (4, 0) $\therefore\ 0 - \sqrt{a} = -2\sqrt{a}(4 - a)$

$\therefore\ 2\sqrt{a}(4 - a) - \sqrt{a} = 0$

$\therefore\ \sqrt{a}(8 - 2a - 1) = 0$

$\therefore\ \sqrt{a}(7 - 2a) = 0$

$\therefore\ a = 0$ or $\frac{7}{2}$

when $a = 0$, normal equation is $y = 0$, and contact point is (0, 0)

when $a = \frac{7}{2}$, $y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}\left(x - \frac{7}{2}\right)$

i.e., $\sqrt{2}y - \sqrt{7} = -2\sqrt{7}\left(x - \frac{7}{2}\right)$

i.e., $\sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7}$

i.e., $\sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$

i.e., $y = -\sqrt{14}x + 4\sqrt{14}$ with contact point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$

EXERCISE 21H

1 **a**
$$\begin{aligned} f(x) &= 3x^2 - 6x + 2 \\ \therefore\ f'(x) &= 6x - 6 \\ \therefore\ f''(x) &= 6 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - x + 5 \\ \therefore\ f'(x) &= 6x^2 - 6x - 1 + 0 \\ \therefore\ f''(x) &= 12x - 6 \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1 \\ \therefore\ f'(x) &= -x^{-\frac{3}{2}} \\ \therefore\ f''(x) &= \tfrac{3}{2}x^{-\frac{5}{2}} \end{aligned}$$

d
$$\begin{aligned} f(x) &= \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1} \\ \therefore\ f'(x) &= -4x^{-3} + 3x^{-2} \\ \therefore\ f''(x) &= 12x^{-4} - 6x^{-3} \\ &= \frac{12-6x}{x^4} \end{aligned}$$

e
$$\begin{aligned} f(x) &= (1-2x)^3 \\ \therefore\ f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2 \\ \therefore\ f''(x) &= -12(1-2x)^1(-2) = 24(1-2x) \end{aligned}$$

f $f(x) = \dfrac{x+2}{2x-1}$ is a quotient where $u = x+2$ and $v = 2x-1$

$\therefore\ u' = 1$ and $v' = 2$

$$\begin{aligned} \therefore\ f'(x) &= \frac{1(2x-1) - 2(x+2)}{(2x-1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-5}{(2x-1)^2} \\ &= -5(2x-1)^{-2} \end{aligned}$$

and
$$\begin{aligned} f''(x) &= 10(2x-1)^{-3}(2) \\ &= \frac{20}{(2x-1)^3} \end{aligned}$$

2 **a**
$$\begin{aligned} y &= x - x^3 \\ \therefore\ \frac{dy}{dx} &= 1 - 3x^2 \\ \therefore\ \frac{d^2y}{dx^2} &= -6x \end{aligned}$$

b
$$\begin{aligned} y &= x^2 - \frac{5}{x^2} \\ &= x^2 - 5x^{-2} \\ \therefore\ \frac{dy}{dx} &= 2x + 10x^{-3} \\ \therefore\ \frac{d^2y}{dx^2} &= 2 - 30x^{-4} = 2 - \frac{30}{x^4} \end{aligned}$$

c
$$\begin{aligned} y &= 2 - \frac{3}{\sqrt{x}} \\ &= 2 - 3x^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{3}{2}x^{-\frac{3}{2}} \\ \therefore\ \frac{d^2y}{dx^2} &= -\tfrac{9}{4}x^{-\frac{5}{2}} \end{aligned}$$

d
$$\begin{aligned} y &= \frac{4-x}{x} = 4x^{-1} - 1 \\ \therefore\ \frac{dy}{dx} &= -4x^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 8x^{-3} = \frac{8}{x^3} \end{aligned}$$

e
$$\begin{aligned} y &= (x^2-3x)^3 \\ \therefore\ \frac{dy}{dx} &= 3(x^2-3x)^2(2x-3) = (6x-9)(x^2-3x)^2 \end{aligned}$$

which is a product where $u = 6x-9$ and $v = (x^2-3x)^2$

$\therefore\ u' = 6$ and $v' = 2(x^2-3x)^1(2x-3)$

$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= 6(x^2-3x)^2 + (6x-9)(2)(x^2-3x)(2x-3) \\ &= 6(x^2-3x)\left[(x^2-3x) + (2x-3)^2\right] \\ &= 6(x^2-3x)(x^2-3x+4x^2-12x+9) \\ &= 6(x^2-3x)(5x^2-15x+9) \end{aligned}$$

f
$$\begin{aligned} y &= x^2 - x + \frac{1}{1-x} = x^2 - x + (1-x)^{-1} \\ \therefore\ \frac{dy}{dx} &= 2x - 1 + (-1)(1-x)^{-2}(-1) = 2x - 1 + (1-x)^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 2 - 2(1-x)^{-3}(-1) = 2 + \frac{2}{(1-x)^3} \end{aligned}$$

3 **a**

$$\begin{aligned} f(x) &= 2x^3 - 6x^2 + 5x + 1 \\ \therefore\quad f'(x) &= 6x^2 - 12x + 5 \\ \therefore\quad f''(x) &= 12x - 12 \end{aligned}$$

and so $f''(x) = 0$ when

$$\begin{aligned} 12x - 12 &= 0 \\ \therefore\quad 12x &= 12 \\ \text{i.e., when}\quad x &= 1 \end{aligned}$$

b $f(x) = \dfrac{x}{x^2+2}$ is a quotient where $u = x$ and $v = x^2 + 2$

$\therefore\quad u' = 1,\quad v' = 2x$

$$\begin{aligned} \therefore\quad f'(x) &= \frac{1(x^2+2) - 2x^2}{(x^2+2)^2} \quad \{\text{using the quotient rule}\} \\ &= \frac{2 - x^2}{(x^2+2)^2} \quad \text{which is another quotient} \end{aligned}$$

with $u = 2 - x^2$ and $v = (x^2+2)^2$

$\therefore\quad u' = -2x,\quad v' = 2(x^2+2)(2x)$

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4} \\ &= \frac{-2x(x^2+2)[x^2+2+2(2-x^2)]}{(x^2+2)^4} \\ &= \frac{-2x[-x^2+6]}{(x^2+2)^3} \\ &= \frac{2x[x^2-6]}{(x^2+2)^3} \end{aligned}$$

and so $f''(x) = 0$ when

$$\begin{aligned} 2x[x^2-6] &= 0 \\ \text{i.e.,}\quad x &= 0 \quad \text{or} \quad x^2 - 6 = 0 \\ \text{i.e., when}\quad x &= 0 \quad \text{or} \quad x = \pm\sqrt{6} \end{aligned}$$

REVIEW SET 21A

1 $y = -2x^2$. Since when $x = -1$, $y = -2(-1)^2 = -2$, the point of contact is $(-1, -2)$.

$$\text{Now}\quad \frac{dy}{dx} = -4x$$

$$\therefore\quad \text{at}\quad x = -1,\quad \frac{dy}{dx} = -4(-1) = 4$$

$\therefore$ tangent has equation $\dfrac{y - (-2)}{x + 1} = 4$ i.e., $y + 2 = 4x + 4$

i.e., $y = 4x + 2$

2 **a**

$$\begin{aligned} y &= 3x^2 - x^4 \\ \therefore\quad \frac{dy}{dx} &= 6x - 4x^3 \end{aligned}$$

b

$$\begin{aligned} y &= \frac{x^3 - x}{x^2} = x - x^{-1} \\ \therefore\quad \frac{dy}{dx} &= 1 + x^{-2} = 1 + \frac{1}{x^2} \end{aligned}$$

3 $f(x) = x^2 + 2x$, $\therefore$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h \to 0} 2x + 2 + h \quad \{\text{as}\quad h \neq 0\} \\ &= 2x + 2 \end{aligned}$$

Checking: $f(x) = x^2 + 2x$, $\therefore\quad f'(x) = 2x + 2$ ✓

4 $y = \dfrac{1-2x}{x^2}$. Since when $x=1$, $y = \dfrac{1-2(1)}{1^2} = -1$ the point of contact is $(1, -1)$

Since $y = \dfrac{1}{x^2} - \dfrac{2}{x}$ then $\dfrac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\dfrac{2}{x^3} + \dfrac{2}{x^2}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -2 + 2 = 0$

i.e., the tangent is a horizontal line. $\therefore$ the normal is a vertical line, of the form $x = k$.

As the normal passes through $(1, -1)$ its equation must be $x = 1$

5 $y = 2x^3 + 4x - 1$ at $(1, 5)$

Now $\dfrac{dy}{dx} = 6x^2 + 4$ $\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 6(1)^2 + 4 = 10$

$\therefore$ the tangent has equation $\dfrac{y-5}{x-1} = 10$ i.e., $y = 10x - 5$

Now the tangent meets the curve again where
$$10x - 5 = 2x^3 + 4x - 1$$
$$2x^3 - 6x + 4 = 0$$
$$x^3 - 3x + 2 = 0$$

We know that $(x-1)^2$ is a factor since the line is tangent to the curve (i.e., touches) at $x = 1$

Consequently, $x^3 - 3x + 2 = (x-1)^2(x+2) = 0$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ tangent meets curve again at $(-2, -25)$

6 $y = \dfrac{ax+b}{\sqrt{x}} = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{a}{2}x^{-\frac{1}{2}} - \dfrac{b}{2}x^{-\frac{3}{2}} = \dfrac{a}{2\sqrt{x}} - \dfrac{b}{2x\sqrt{x}}$

The equation of the tangent at $x = 1$ is $2x - y = 1$

i.e., $y = 2x - 1$ $\therefore$ the slope is 2

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{a}{2} - \dfrac{b}{2} = 2$

$\therefore$ $a - b = 4$

$\therefore$ $a = b + 4$ ...... (1)

Also, at $x = 1$ the tangent touches the curve i.e., $\dfrac{a(1)+b}{\sqrt{1}} = 2(1) - 1$

i.e., $a + b = 1$

$\therefore$ $b + 4 + b = 1$ {using (1)}

$\therefore$ $2b = -3$

$\therefore$ $b = -\frac{3}{2}$

$\therefore$ $a = 4 - \frac{3}{2} = \frac{5}{2}$

i.e., $a = \frac{5}{2}$, $b = -\frac{3}{2}$

7 $y = 4(ax+1)^{-2}$

Since when $x = 0$, $y = 4(0+1)^{-2} = 4$, the point of contact is $(0, 4)$

Now $\dfrac{dy}{dx} = -8(ax+1)^{-3}(a) = \dfrac{-8a}{(ax+1)^3}$ $\therefore$ at $x = 0$, $\dfrac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-0} = -8a$ i.e., $y - 4 = -8ax$

and this passes through $(1, 0)$ $\therefore$ $0 - 4 = -8a(1)$ $\therefore$ $a = \frac{1}{2}$

8 $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$

Since when $x = 4$ $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ the point of contact is $(4, \frac{1}{2})$

Now $\dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $\therefore$ at $x = 4$, $\dfrac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has slope 16.

So the equation is $\dfrac{y - \frac{1}{2}}{x - 4} = 16$ i.e., $y - \frac{1}{2} = 16x - 64$ i.e., $y = 16x - \frac{127}{2}$

9 **a**

$$\begin{aligned} M &= (t^2+3)^4 \\ \frac{dM}{dt} &= 4(t^2+3)^3(2t) \\ \frac{dM}{dt} &= 8t(t^2+3)^3 \end{aligned}$$

b $A = \dfrac{\sqrt{t+5}}{t^2}$ is a quotient with

$u = (t+5)^{\frac{1}{2}}$ and $v = t^2$

$\therefore$ $u' = \frac{1}{2}(t+5)^{-\frac{1}{2}}$, $v' = 2t$

$$\begin{aligned} \frac{dA}{dt} &= \frac{\left(\dfrac{1}{2\sqrt{t+5}}\right)t^2 - \sqrt{t+5}(2t)}{t^4} \\ &= \frac{t^2 - 2(2t)(t+5)}{2t^4\sqrt{t+5}} \\ &= \frac{t - 4(t+5)}{2t^3\sqrt{t+5}} \\ &= \frac{-3t-20}{2t^3\sqrt{t+5}} \end{aligned}$$

10 **a**

$$\begin{aligned} y &= \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x \\ \therefore \quad \frac{dy}{dt} &= -2x^{-\frac{3}{2}} - 3 = \frac{-2}{x\sqrt{x}} - 3 \end{aligned}$$

b

$$\begin{aligned} y &= \left(x - \frac{1}{x}\right)^4 = (x - x^{-1})^4 \\ \therefore \quad \frac{dy}{dx} &= 4(x - x^{-1})^3(1 + x^{-2}) \\ &= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right) \end{aligned}$$

c

$$\begin{aligned} y &= \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}} \\ \therefore \quad \frac{dy}{dx} &= \tfrac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3) \\ &= \frac{2x-3}{2\sqrt{x^2-3x}} \end{aligned}$$

REVIEW SET 21B

1 **a**

$$\begin{aligned} y &= 5x - 3x^{-1} \\ \frac{dy}{dx} &= 5 + 3x^{-2} = 5 + \frac{3}{x^2} \end{aligned}$$

b

$$\begin{aligned} y &= (3x^2 + x)^4 \\ \frac{dy}{dx} &= 4(3x^2+x)^3(6x+1) \end{aligned}$$

c $y = (x^2+1)(1-x^2)^3$ is a product with $u = x^2 + 1$ and $v = (1-x^2)^3$

$\therefore$ $u' = 2x$ and $v' = 3(1-x^2)^2(-2x)$

$\dfrac{dy}{dx} = 2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$ {using the product rule}

2 $y = x^3 - 3x^2 - 9x + 2$, $\therefore$ $\dfrac{dy}{dx} = 3x^2 - 6x - 9$

horizontal tangents occur when $\dfrac{dy}{dx} = 0$ i.e., when

$$\begin{aligned} 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ \therefore \quad x &= 3 \quad \text{or} \quad x = -1 \end{aligned}$$

when $x = 3$, horizontal tangent has equation $y = -25$

when $x = -1$, horizontal tangent has equation $y = 7$

3 $y = \dfrac{x+1}{x^2-2}$ and since when $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

Now $y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2 - 2$

$\therefore$ $u' = 1$ and $v' = 2x$

$$\frac{dy}{dx} = \frac{1(x^2-2)-(x+1)(2x)}{(x^2-2)^2} \quad \{\text{using the quotient rule}\}$$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{(1^2-2)-(1+1)2(1)}{(1^2-2)^2} = \dfrac{(-1)-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$

$\therefore$ the equation of the normal is $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$ i.e., $5y = x - 11$

4 **a** $f(x) = \dfrac{(x+3)^3}{\sqrt{x}} = (x+3)^3 x^{-\frac{1}{2}}$ which is a product with $u = (x+3)^3$ and $v = x^{-\frac{1}{2}}$

$\therefore$ $u' = 3(x+3)^2$ and $v' = -\frac{1}{2}x^{-\frac{3}{2}}$

$\therefore$ $f'(x) = \dfrac{3(x+3)^2}{\sqrt{x}} - \dfrac{(x+3)^3}{2x\sqrt{x}}$ {using the product rule}

b $f(x) = x^4\sqrt{x^2+3}$ which is a product with $u = x^4$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore$ $u' = 4x^3$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$

$\therefore$ $f'(x) = 4x^3\sqrt{x^2+3} + \dfrac{x^5}{\sqrt{x^2+3}}$ {using the product rule}

5 **a** $f(x) = \sqrt{x}(1-x)^2$ is a product with $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-x)(-1)$

$\therefore$ $f'(x) = \dfrac{(1-x)^2}{2\sqrt{x}} - 2\sqrt{x}(1-x)$ {product rule}

b $f(x) = \sqrt{3x-x^2} = (3x-x^2)^{\frac{1}{2}}$

$\therefore$ $f'(x) = \frac{1}{2}(3x-x^2)^{-\frac{1}{2}}(3-2x)$

c $f(x) = \dfrac{1}{2-x} = (2-x)^{-1}$

$\therefore$ $f'(x) = -1(2-x)^{-2}(-1)$

$= \dfrac{1}{(2-x)^2}$

6 **a** $f(x) = 3x^2 - \dfrac{1}{x} = 3x^2 - x^{-1}$

$\therefore$ $f'(x) = 6x + x^{-2}$

and $f''(x) = 6 - 2x^{-3} = 6 - \dfrac{2}{x^3}$

b $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\therefore$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore$ $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

7 $y = x^2\sqrt{1-x}$ at $x = -3$

Since when $x = -3$, $y = (-3)^2\sqrt{1-(-3)} = 9\sqrt{4} = 18$,

the point of contact is $(-3, 18)$

Also, $y = x^2\sqrt{1-x}$ is a product, with $u = x^2$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore$ $u' = 2x$ and $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$\therefore \quad \dfrac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2\left(\frac{1}{2}(1-x)^{-\frac{1}{2}}\right)$

$\therefore$ at $x=-3$, $\dfrac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2(\frac{1}{2})(1-(-3))^{-\frac{1}{2}} = -6(2) - 9(\frac{1}{2})^2 = -\frac{57}{4}$

$\therefore$ the tangent at (−3, 18) has equation $\dfrac{y-18}{x-(-3)} = -\frac{57}{4}$ i.e., $4y - 72 = -57x - 171$
i.e., $4y = -57x - 99$

Now when $x=0$, $y = -\frac{99}{4}$ and when $y=0$, $x=-\frac{99}{57}$

$\therefore$ area of ΔOAB $= \frac{1}{2}\left(\frac{99}{4}\right)\left(\frac{99}{57}\right) \doteqdot 21.5$ units^2

8 $y = x^3 + ax + b \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + a$ and so at $x=1$, $\dfrac{dy}{dx} = 3+a$

Now the equation of the tangent at $x=1$ is $y=2x$

$\therefore$ the slope is 2 $\quad \therefore \quad 3+a = 2$ and so $a=-1$

Also, at $x=1$ the tangent touches the curve.

i.e., $x^3 + ax + b = 2x$ when $x=1$

$\therefore \quad (1)^3 + (-1)(1) + b = 2(1)$

$\therefore \quad 1 - 1 + b = 2$

$\therefore \quad b = 2$ and so $a = -1$.

9 $y = x^3 + ax^2 - 4x + 3 \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x=1$ is parallel to $y=3x$, so at $x=1$, $\dfrac{dy}{dx} = 3$

i.e., $3 = 3(1)^2 + 2a(1) - 4$

$\therefore \quad 2a = 4$

$\therefore \quad a = 2$

Since when $x=1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

The contact point is (1, 2) and since the slope is 3, the tangent at (1, 2) has equation

$\dfrac{y-2}{x-1} = 3$ i.e., $y - 2 = 3x - 3$
i.e., $y = 3x - 1$

The tangent meets the curve where $x^3 + 2x^2 - 4x + 3 = 3x - 1$
i.e., $x^3 + 2x^2 - 7x + 4 = 0$

Since the line touches the curve (i.e., is tangent to it) at $x=1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ {since the constant term is 4}

$\therefore$ the curve cuts the tangent at $x=-4$, $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

i.e., at $(-4, -13)$

10 $f(x) = 2x^3 + Ax + B \quad \therefore \quad f'(x) = 6x^2 + A$

Now as the slope at (−2, 33) is 10,

$\therefore \quad f'(-2) = 10$

$\therefore \quad 10 = 6(-2)^2 + A$

$\therefore \quad A = -14$

$\therefore \quad f(x) = 2x^3 - 14x + B$

and as (−2, 33) lies on the curve

$f(-2) = 33$

$\therefore \quad 2(-2)^3 - 14(-2) + B = 33$

$\therefore \quad -16 + 28 + B = 33$

$\therefore \quad B = 49 - 28$

$\therefore \quad B = 21$

REVIEW SET 21C

1 **a** $y = x^3\sqrt{1-x^2}$ is a product where $y = x^3$ and $v = (1-x^2)^{\frac{1}{2}}$

$\therefore \quad u' = 3x^2$ and $v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -x(1-x^2)^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = 3x^2\sqrt{1-x^2} - \dfrac{x^4}{\sqrt{1-x^2}}$ {using the product rule}

b $y = \dfrac{x^2-3x}{\sqrt{x+1}}$ is a quotient where $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$

$u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$ {using the quotient rule}

2 $y = \dfrac{x+1}{x^2-2}$ at $x = 1$

Since when $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

$y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2 - 2$

$\therefore \quad u' = 1$ and $v' = 2x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1(x^2-2)-(x+1)2x}{(x^2-2)^2}$ {using the quotient rule}

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{(1-2)-2(1+1)}{(1-2)^2} = \dfrac{-1-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$.

So the normal has equation: $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$ i.e., $5y + 10 = x - 1$

i.e., $5y = x - 11$

3
$$\begin{aligned} f(x) &= 2x^4 - 4x^3 - 9x^2 + 4x + 7 \\ \therefore \quad f'(x) &= 8x^3 - 12x^2 - 18x + 4 \\ \therefore \quad f''(x) &= 24x^2 - 24x - 18, \quad \therefore \quad f''(x) = 0 \text{ where } 24x^2 - 24x - 18 = 0 \end{aligned}$$
$$\begin{aligned} 4x^2 - 4x - 3 &= 0 \\ (2x+1)(2x-3) &= 0 \\ \therefore \quad x = -\tfrac{1}{2} &\text{ or } x = \tfrac{3}{2} \end{aligned}$$

4 $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$

$f(x)$ is the product of u and v where $u = 3x$ and $v = 1 + x$ $\therefore$ $u' = 3$ and $v' = 1$

$\therefore$ by the quotient rule $f'(x) = \dfrac{3(1+x)-1(3x)}{(1+x)^2} = \dfrac{3}{(1+x)^2}$ $\therefore$ $f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore$ the normal at $(2, 2)$ has slope -3

So the equation of the normal is $\dfrac{y-2}{x-2} = -3$ $\therefore$ $y - 2 = -3(x-2)$

i.e., $y - 2 = -3x + 6$

i.e., $y = -3x + 8$

$\therefore$ when $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$\therefore$ B and C are at $(0, 8)$ and $(\frac{8}{3}, 0)$, and the distance BC $= \sqrt{\left(0-\frac{8}{3}\right)^2 + (8-0)^2}$

$$= \sqrt{\tfrac{64}{9} + 64}$$
$$= \sqrt{\tfrac{640}{9}}$$
$$= \tfrac{8\sqrt{10}}{3} \text{ units}$$

5 **a** $y = \dfrac{x^2}{3-2x}$ is a quotient with $u = x^2$ and $v = 3 - 2x$

$\therefore$ $u' = 2x$ and $v' = -2$

$$\frac{dy}{dx} = \frac{2x(3-2x) - x^2(-2)}{(3-2x)^2} = \frac{6x - 2x^2}{(3-2x)^2} \quad \{\text{using the quotient rule}\}$$

b $y = \sqrt{x}(x^2 - x)^3$ is a product with $u = x^{\frac{1}{2}}$ and $v = (x^2 - x)^3$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(x^2 - x)^2(2x - 1)$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2-x)^3}{2\sqrt{x}} + 3\sqrt{x}(2x-1)(x^2-x)^2 \quad \{\text{using the product rule}\}$$

6 **a**
$$y = 3x^4 - \frac{2}{x} = 3x^4 - 2x^{-1}$$
$$\therefore \quad \frac{dy}{dx} = 12x^3 + 2x^{-2}$$
$$\therefore \quad \frac{d^2y}{dx^2} = 36x^2 - 4x^{-3} = 36x^2 - \frac{4}{x^3}$$

b
$$y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$$
$$\therefore \quad \frac{dy}{dx} = 3x^2 - 1 - \tfrac{1}{2}x^{-\frac{3}{2}}$$
$$\therefore \quad \frac{d^2y}{dx^2} = 6x + \tfrac{3}{4}x^{-\frac{5}{2}}$$

7 $y = \dfrac{x}{\sqrt{1-x}}$ at $x = -3$.

Since when $x = -3$, $y = \dfrac{-3}{\sqrt{1-(-3)}} = -\frac{3}{2}$ the point of contact is $(-3, -\frac{3}{2})$

Now $y = \dfrac{x}{\sqrt{1-x}}$ is a quotient with $u = x$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore$ $u' = 1$, $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$$\therefore \quad \frac{dy}{dx} = \frac{(1)\sqrt{1-x} - x(\frac{1}{2})(1-x)^{-\frac{1}{2}}(-1)}{1-x} \quad \{\text{quotient rule}\}$$

at $x = -3$, $$\frac{dy}{dx} = \frac{\sqrt{4} - (-3)(\frac{1}{2})(4)^{-\frac{1}{2}}(-1)}{4} = \frac{2 - 3(\frac{1}{2})(\frac{1}{2})}{4} = \frac{2 - \frac{3}{4}}{4} = \frac{5}{16}$$

$\therefore$ the tangent at $(-3, -\frac{3}{2})$ has equation $\dfrac{y - (-\frac{3}{2})}{x - (-3)} = \frac{5}{16}$ i.e., $16(y + \frac{3}{2}) = 5(x+3)$

$$16y + 24 = 5x + 15$$

i.e., $5x - 16y = 9$

$\therefore$ $b = -16$ and $a = 9$

8 $f(x) = 3x^3 + Ax^2 + B$

As the point $(-2, 14)$ lies on the curve, $14 = -24 + 4A + B$

$$4A + B = 38 \quad \ldots\ldots (1)$$

Now $f'(x) = 9x^2 + 2Ax$ and as $f'(-2) = 0$

$36 - 4A = 0$

$\therefore$ $A = 9$

$\therefore$ $4(9) + B = 38$ {using (1)}

$\therefore$ $B = 38 - 36 = 2$

That is, $A = 9$ and $B = 2$

$\therefore \quad f'(x) = 9x^2 + 18x$

$\therefore \quad f''(x) = 18x + 18$ and so $f''(-2) = -36 + 18 = -18$

9 $y = \dfrac{a}{(x+2)^2} = a(x+2)^{-2}$

The slope of the line AB is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-4}{0-2} = \dfrac{4}{-2} = -2$

$\therefore$ the equation of the tangent is $\dfrac{y-8}{x-0} = -2$ i.e., $y = -2x + 8$

$\dfrac{dy}{dx} = -2(a)(x+2)^{-3} = -2$

$\therefore \quad \dfrac{a}{(x+2)^3} = 1 \quad \therefore \quad a = (x+2)^3$ ...... (1)

The line AB meets the curve where
$$-2x + 8 = \frac{a}{(x+2)^2}$$
$$\therefore \quad -2x + 8 = \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using (1)}\}$$
$$\therefore \quad -2x + 8 = x + 2$$
$$\therefore \quad -3x = -6$$
$$\therefore \quad x = 2$$
and so $a = (2+2)^3 = 64$

10 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ meet when $\sqrt{(3x+1)} = \sqrt{5x - x^2}$

Squaring both sides, $3x + 1 = 5x - x^2$
$$\therefore \quad x^2 - 2x + 1 = 0$$
$$\therefore \quad (x-1)^2 = 0$$
$$\therefore \quad x = 1$$

and when $x = 1$, $y = \sqrt{3+1} = 2$ $\quad\therefore$ the curves meet at (1, 2)

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$
$$\frac{dy}{dx} = \tfrac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$
$\therefore$ at (1, 2) $\dfrac{dy}{dx} = \dfrac{3}{2(3+1)^{\frac{1}{2}}} = \frac{3}{4}$

and for $y = \sqrt{5x - x^2} = (5x - x^2)^{\frac{1}{2}}$
$$\frac{dy}{dx} = \tfrac{1}{2}(5x - x^2)^{-\frac{1}{2}}(5 - 2x) = \frac{5 - 2x}{2\sqrt{5x - x^2}}$$
$\therefore$ at (1, 2) $\dfrac{dy}{dx} = \dfrac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$

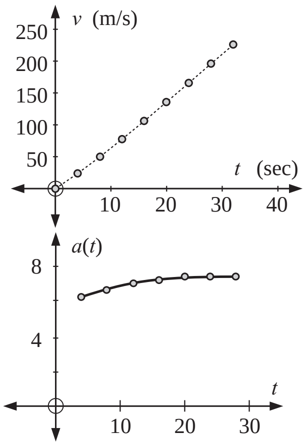
i.e., the curves have the same slope of $\frac{3}{4}$ at their point of intersection.

Now the equation of the common tangent at (1, 2) is $\dfrac{y-2}{x-1} = \frac{3}{4}$ i.e., $4(y-2) = 3(x-1)$
$$4y - 8 = 3x - 3$$
i.e., $4y = 3x + 5$

Chapter 22
APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 22A

1 **a**



b
$$v'(4) \doteqdot \frac{v(8)-v(0)}{8} \doteqdot \tfrac{50}{8} \doteqdot 6.25$$
$$v'(8) \doteqdot \frac{v(12)-v(4)}{8} \doteqdot \tfrac{54}{8} \doteqdot 6.75$$
$$v'(12) \doteqdot \frac{v(16)-v(8)}{8} \doteqdot \tfrac{56}{8} \doteqdot 7$$
$$v'(16) \doteqdot \frac{v(20)-v(12)}{8} \doteqdot \tfrac{58}{8} \doteqdot 7.25$$
$$v'(20) \doteqdot \frac{v(24)-v(16)}{8} \doteqdot \tfrac{60}{8} \doteqdot 7.5$$
$$v'(24) \doteqdot \frac{v(28)-v(20)}{8} \doteqdot \tfrac{60}{8} \doteqdot 7.5$$
$$v'(28) \doteqdot \frac{v(32)-v(24)}{8} \doteqdot \tfrac{60}{8} \doteqdot 7.5$$

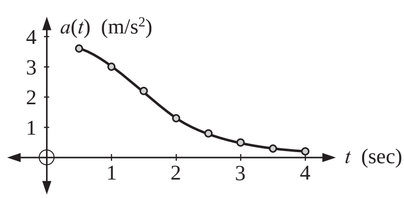
c For $v(t) = 5.242 \times t^{1.0865}$
$\therefore \quad v'(t) = 5.242 \times 1.0865t^{0.0865}$
$\therefore \quad v'(t) = 5.695t^{0.0865}$

$v'(4) \doteqdot 6.42$ $\quad v'(20) \doteqdot 7.38$
$v'(8) \doteqdot 6.82$ $\quad v'(24) \doteqdot 7.50$
$v'(12) \doteqdot 7.06$ $\quad v'(28) \doteqdot 7.60$
$v'(16) \doteqdot 7.24$ $\quad$ The fit is excellent.

2 **a** **i**
$$v'(1) \doteqdot \frac{v(1.5)-v(0.5)}{1.5-0.5} \doteqdot \frac{5.0-2.0}{1} \doteqdot 3 \text{ m/s}^2$$

ii
$$v'(2.5) \doteqdot \frac{v(3)-v(2)}{3-2} \doteqdot \frac{6.6-5.8}{1} \doteqdot 0.8 \text{ m/s}^2$$

b
$v'(0.5) \doteqdot 3.6 - 0.0 \doteqdot 3.6$
$v'(1.5) \doteqdot 5.8 - 3.6 \doteqdot 2.2$
$v'(2) \doteqdot 6.3 - 5 \doteqdot 1.3$
$v'(3) \doteqdot 6.8 - 6.3 \doteqdot 0.5$
$v'(3.5) \doteqdot 6.9 - 6.6 \doteqdot 0.3$



EXERCISE 22B

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geqslant 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\dfrac{dP}{dt} = 4t - 12$ thousand dollars/year

c $\dfrac{dP}{dt}$ is the rate of change in profit with time

d **i** Profit decreases when $\dfrac{dP}{dt} \leqslant 0$ i.e., $4t - 12 \leqslant 0$
$\therefore \quad 4t \leqslant 12$
i.e., $t \leqslant 3$

But $t \geqslant 0$ $\therefore$ $0 \leqslant t \leqslant 3$ years

ii Profit increases when $\dfrac{dP}{dt} \geqslant 0$ i.e., $t \geqslant 3$ years

e The profit function is a quadratic with $a > 0$ $\therefore$ shape is 

So, a minimum profit occurs when $\dfrac{dP}{dt} = 0$ i.e., at $t = 3$ years

and $P(3) = 18 - 36 + 118 = 100$ thousand dollars i.e., \$100 000.

f **i** When $t = 4$, $\dfrac{dP}{dt} = 4$ thousand dollars per year.

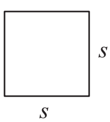
So, the profit is increasing at \$4000/year after 4 years.

ii When $t = 10$, $\dfrac{dP}{dt} = 28$ thousand dollars per year.

So, the profit is increasing at \$28 000/year after 10 years.

iii When $t = 25$, $\dfrac{dP}{dt} = 88$ thousand dollars per year.

So, the profit is increasing at \$88 000/year after 25 years.

2 

Area, $A = s^2$

a average rate of change in A for $4 \leqslant s \leqslant 4.5$

$$= \frac{A(4.5) - A(4)}{4.5 - 4}$$
$$= \frac{4.5^2 - 4^2}{0.5}$$
$$= 8.5 \text{ cm}^2 \text{ / cm}$$

b $\dfrac{dA}{dt} = 2s$ So, when $s = 4$, $\dfrac{dA}{dt} = 8 \text{ cm}^2 \text{ / cm}$

3 $V = 200(50 - t)^2 \text{ m}^3$

a average rate on $0 \leqslant t \leqslant 5$

$$= \frac{V(5) - V(0)}{5 - 0}$$
$$= \frac{200(45)^2 - 200(50)^2}{5}$$
$$= -19\,000 \text{ m}^3\text{/min}$$

i.e., leaving at 19 000 m^3/min

b
$$V'(t) = 400(50 - t)^1 \times (-1)$$
$$\therefore \quad V'(5) = 400 \times 45 \times -1$$
$$= -18\,000 \text{ m}^3\text{/min}$$

i.e., leaving at 18 000 m^3/min

4 **a** $\dfrac{dV}{dt} = 1.2 \text{ m}^3\text{/min}$

b
$$V = \tfrac{4}{3}\pi r^3$$
$$\therefore \quad \frac{dV}{dr} = \tfrac{4}{3}\pi \times 3r^2 = 4\pi r^2$$

c
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} \quad \{\text{chain rule}\}$$
$$\therefore \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

d when $r = 3.6$

$$1.2 = 4\pi(3.6)^2 \frac{dr}{dt}$$
$$\therefore \quad \frac{1.2}{4\pi(3.6)^2} = \frac{dr}{dt}$$
$$\therefore \quad \frac{dr}{dt} \doteqdot 0.007\,37 \text{ m/min}$$

5 $s(t) = 1.2 + 28.1t - 4.9t^2$ metres

a When released, $t = 0$ and $s(0) = 1.2$ m $\therefore$ it is released 1.2 m above the ground.

b $s'(t) = 28.1 - 9.8t$ m/s is the instantaneous velocity of the ball at the time t seconds after release.

c When $s'(t) = 0$, $28.1 - 9.8t = 0$ $\therefore$ $t = \frac{28.1}{9.8} \doteqdot 2.87$ sec.

So, after 2.87 sec the ball has reached its maximum height.

d $s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \doteqdot 41.5$ m
So, the maximum height reached is about 41.5 m.

e **i** $s'(0) = 28.1$ m/s **ii** $s'(2) = 28.1 - 19.6 = 8.5$ m/s **iii** $s'(5) = 28.1 - 49 = -20.9$ m/s

If $s'(t) \geqslant 0$, the ball is travelling upwards.
If $s'(t) \leqslant 0$, the ball is travelling downwards.

f $s(t) = 0$ when
$$1.2 + 28.1t - 4.9t^2 = 0$$
i.e., $$4.9t^2 - 28.1t - 1.2 = 0$$
$$\therefore \quad t = \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8}$$
$$\doteqdot -0.0424 \quad \text{or} \quad 5.777$$
$\therefore$ hits the ground after 5.78 sec.

g $\dfrac{d^2s}{dt^2} = -9.8$ m/s/s and is the constant rate of change in $\dfrac{ds}{dt}$
i.e., the instantaneous acceleration is constant at -9.8 m/s^2 for the entire motion.

6 **a**
$$s(t) = bt - 4.9t^2$$
$$s'(t) = b - 9.8t$$
$$\therefore \quad s'(0) = b$$
i.e., the initial velocity is b m/s.

b since $s(14.2) = 0$
$$b(14.2) - 4.9(14.2)^2 = 0$$
$$\therefore \quad 14.2\,[b - 4.9 \times 14.2] = 0$$
$$\therefore \quad b = 4.9 \times 14.2$$
$$\therefore \quad b \doteqdot 69.6$$
$\therefore$ the initial velocity is 69.6 m/s

EXERCISE 22C.1

1 $Q = 100 - 10\sqrt{t}$, $t \geqslant 0$

a **i** At $t = 0$, $Q = 100$ units
ii At $t = 25$, $Q = 50$ units
iii At $t = 100$, $Q = 0$ units

b $\dfrac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\dfrac{5}{\sqrt{t}}$

i At $t = 25$, $\dfrac{dQ}{dt} = -1$ units/year
i.e., decreasing at 1 unit/year

ii At $t = 50$, $\dfrac{dQ}{dt} = -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}}$ units/year
i.e., decreasing at $\frac{1}{\sqrt{2}}$ units/year

c $\dfrac{dQ}{dt} = -\dfrac{5}{\sqrt{t}}$

$\therefore$ $\dfrac{dQ}{dt} < 0$, for all $t > 0$, $\therefore$ $\dfrac{dQ}{dt}$ is decreasing for all $t > 0$

2 $H = 20 - \dfrac{9}{t+5}$ m, $t \geqslant 0$

a At planting, $t = 0$ $\therefore$ $H(0) = 20 - \dfrac{9}{0+5} = 18.2$ m

b **i** At $t = 4$,
$H(4) = 20 - \dfrac{9}{4+5} = 19$ m

ii At $t = 8$,
$H(8) = 20 - \dfrac{9}{8+5} \doteqdot 19.3$ m

iii At $t = 12$,
$H(12) = 20 - \dfrac{9}{12+5} \doteqdot 19.5$ m

c Now $\dfrac{dH}{dt} = 9(t+5)^{-2} = \dfrac{9}{(t+5)^2}$

i When $t = 0$,
$\dfrac{dH}{dt} = \frac{9}{25} = 0.36$ m/yr

ii When $t = 5$,
$\dfrac{dH}{dt} = \frac{9}{100} = 0.09$ m/yr

iii When $t = 10$,
$\dfrac{dH}{dt} = \frac{9}{225} = 0.04$ m/yr

d Now $\dfrac{dH}{dt} = \dfrac{9}{(t+5)^2}$, $\therefore$ as $(t+5)^2 > 0$ for all $t \geqslant 0$, $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$

i.e., the height of the tree is always increasing, which means that the tree is always growing.

3 **a** $R = 20 + \frac{1}{10}T + \frac{1}{200}T^2$

When $T = 0^o\text{C}$, $R = 20$ units
When $T = 20^o\text{C}$, $R = 24$ units
When $T = 40^o\text{C}$, $R = 32$ units

b Now $\dfrac{dR}{dT} = \frac{1}{10} + \dfrac{T}{100} = 0.1 + 0.01T$

c $\therefore$ $\dfrac{dR}{dT} = 0$ when $0.01T = -0.1$ i.e., $T = -10$

Sign diagram of $\dfrac{dR}{dT}$ is: $-$ | $+$ (at -10)

R increases when $\dfrac{dR}{dT} > 0$
i.e., when $T > -10^o\text{C}$

4 **a** $C(v) = 200v + 10\,000v^{-1}$ dollars

i At $v = 20$ kmph, $C = \$4500$
ii At $v = 40$ kmph, $C = \$8250$

b $\dfrac{dC}{dv} = 200 - 10\,000v^{-2} = 200 - \dfrac{10\,000}{v^2}$

i At $v = 10$ kmph, $\dfrac{dC}{dv} = \$100$ per kmph

ii At $v = 30$ kmph, $\dfrac{dC}{dv} = \$188.89$ per kmph

c Cost is a minimum when $\dfrac{dC}{dv} = 0$, i.e., $200 - \dfrac{10\,000}{v^2} = 0$

$$\therefore\ 200 = \frac{10\,000}{v^2}$$
$$\therefore\ v^2 = 50$$
$$\therefore\ v = \pm\sqrt{50}$$
i.e., $v = \sqrt{50}$ kmph

5 $y = \frac{1}{10}x(x-2)(x-3) = \frac{1}{10}(x^3 - 5x^2 + 6x)$

a When $y = 0$, $x = 0,\ 2$ or 3

$\therefore$ the lake is between 2 and 3 km from the shoreline.

b $\dfrac{dy}{dx} = \frac{1}{10}(3x^2 - 10x + 6)$

When $x = \frac{1}{2}$, $\dfrac{dy}{dx} = \frac{7}{40}$ $\therefore$ land is sloping upwards.

When $x = \frac{3}{2}$, $\dfrac{dy}{dx} = -\frac{9}{40}$ $\therefore$ land is sloping downwards.

c The deepest point of the lake occurs when the slope of the land is 0, i.e., $\dfrac{dy}{dx} = 0$

$\therefore$ $\frac{1}{10}(3x^2 - 10x + 6) = 0$

i.e., $3x^2 - 10x + 6 = 0$ $\therefore$ $x = \dfrac{10 \pm \sqrt{100-72}}{6} = \dfrac{5 \pm \sqrt{7}}{3}$

but it must be the value between 2 and 3 km, i.e., $x = \dfrac{5+\sqrt{7}}{3} \doteqdot 2.549$ km

$\therefore$ the depth at this point is $y(2.549) \doteqdot \frac{1}{10}(2.549)(0.549)(-0.451)$
$\doteqdot -0.06311$ km
i.e., $\doteqdot 63.11$ m below sea level.

6 **a** $V = x^3 \quad \therefore \quad \dfrac{dV}{dx} = 3x^2 \quad \text{mm}^3/\text{mm}$

This is the rate at which the volume increases as the length of side increases.

b When $x = 2$, $\dfrac{dV}{dx} = 3(2)^2 = 12\ \text{mm}^3/\text{mm}$

i.e., for every mm increase in length of side the volume increases by 12 mm^3.

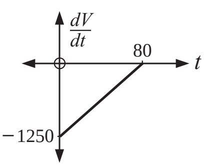
c When $V = x^3$, $\dfrac{dV}{dx} = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}A$, where A is the surface area of the cube

$$\therefore \quad \frac{dV}{dx} \propto A$$

As the salt crystal grows, its additional volume is related exactly to the outer surface area of the crystal because as it grows it deposits on the outer surface.

7 **a** $V = 50\,000\left(1 - \frac{t}{80}\right)^2, \quad 0 \leqslant t \leqslant 80$

$$\frac{dV}{dt} = 2 \times 50\,000\left(1 - \frac{t}{80}\right)^1 \times \left(-\tfrac{1}{80}\right) = -1250\left(1 - \frac{t}{80}\right)$$



b Outflow was fastest when $t = 0$ (when the tap was first opened).

c As $\dfrac{dV}{dt} = -1250 + \frac{1250}{80}t$, then $\dfrac{d^2V}{dt^2} = \frac{1250}{80} = \frac{125}{8}$

and since $\dfrac{d^2V}{dt^2}$ is constant and positive, it shows that $\dfrac{dV}{dt}$ is constantly increasing

i.e., the outflow is decreasing at a constant rate.

8 **a** $\dfrac{dP}{dt} = aP\left(1 - \dfrac{P}{b}\right) - \left(\dfrac{c}{100}\right)P$ and when $\dfrac{dP}{dt} = 0$, the rate of change of population is zero, $\therefore$ the population is not changing and is stable.

b If $a = 0.06$, $b = 24\,000$, $c = 5$ then

$$\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \tfrac{5}{100}P$$
$$= 0.06P - 0.05P - \frac{0.06P^2}{24\,000}$$
$$= P\left(0.01 - \frac{P}{400\,000}\right)$$

Now for a stable population

$$\frac{dP}{dt} = 0$$
$$\therefore \quad P = 0 \quad \text{or} \quad \frac{P}{400\,000} = 0.01$$

i.e., $P = 0$ or 4000

i.e, the stable population is 4000 fish

c If the harvest rate is 4%, then $\dfrac{dP}{dt} = 0.06P\left(1 - \dfrac{P}{24\,000}\right) - \tfrac{4}{100}P$

$$= P\left(0.02 - \frac{0.06P}{24\,000}\right)$$

For stable population $\dfrac{dP}{dt} = 0 \quad \therefore \quad 0 = P\left(0.02 - \dfrac{0.06P}{24\,000}\right)$

$$\therefore \quad P = 0 \quad \text{or} \quad \frac{0.06P}{24\,000} = 0.02$$
$$\therefore \quad P = 0 \quad \text{or} \quad \frac{0.02 \times 24\,000}{0.06}$$
$$\therefore \quad P = 0 \quad \text{or} \quad 8000$$

i.e., the stable population is 8000 fish.

EXERCISE 22C.2

1 **a** $$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250$$
$$\therefore \quad C'(x) = 0.0009x^2 + 0.04x + 4 \quad \text{(dollars/pair)}$$

b $$C'(220) = 0.0009(220)^2 + 0.04(220) + 4$$
$$= \$56.36 \text{ per pair}$$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $$C(221) - C(220) = \$7348.98 - \$7292.40$$
$$= \$56.58$$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

d $$C''(x) = 0.0018x + 0.04$$
$$C''(x) = 0 \quad \text{when} \quad 0.0018x + 0.04 = 0 \quad \text{i.e.,} \quad x = -\frac{0.04}{0.0018}$$
$$\doteqdot -22.2 \quad \text{but } x \geqslant 0$$

$\therefore$ when the rate of change is a minimum it is out of the bounds of our model (we cannot make < 0 jeans!)

2 **a** $$C(x) = 0.000\,072x^3 - 0.000\,61x^2 + 0.19x + 893 \text{ dollars}$$
$$\therefore \quad C'(x) = 0.000\,216x^2 - 0.001\,22x + 0.19 \quad \text{(dollars/item)}$$

$C'(x)$ represents the cost/item for the x items of production.

b $C'(300) = 0.000\,216(300)^2 - 0.001\,22(300) + 0.19 = \$19.26$

This estimates the cost of making the 301st item if 300 items are currently being made.

c $$\text{Actual cost} = C(301) - C(300)$$
$$= \$2858.43 - \$2839.10$$
$$= \$19.33 \quad \text{i.e., the actual cost of the 301st item is } \$19.33$$

d $$C''(x) = 0.000\,432x - 0.001\,22 \quad \text{which is 0} \quad \text{when} \quad 0.000\,432x = 0.001\,22$$
$$x = \frac{0.001\,22}{0.000\,432}$$
$$x \doteqdot 2.82$$

$\therefore$ the rate of change in price is a minimum when $x \doteqdot 2.8$

EXERCISE 22D.1

1 **a** $s(t) = t^2 + 3t - 2 \quad t \geqslant 0$

$$\text{Average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(3) - s(1)}{3 - 2} = \frac{16 - 2}{2} = 7 \text{ ms}^{-1}$$

b Average velocity
$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$
$$= \frac{s(1+h) - s(1)}{(1+h) - 1}$$
$$= \frac{(1+h)^2 + 3(1+h) - 2 - 2}{h}$$
$$= \frac{2h + h^2 + 3h}{h}$$
$$= (5 + h) \text{ ms}^{-1}$$

c $$\lim_{h \to 0} \frac{s(1+h) - s(1)}{h}$$
$$= \lim_{h \to 0} 5 + h$$
$$= 5 \text{ ms}^{-1}$$

This is the *instantaneous velocity* at $t = 1$ second.

d Average velocity $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$= \dfrac{s(t+h)-s(t)}{(t+h)-t}$

$= \dfrac{\left[(t+h)^2+3(t+h)-2\right]-\left[t^2+3t-2\right]}{h}$

$= \dfrac{2ht+h^2+3h}{h}$

$= (2t+3+h)\ \text{ms}^{-1}$

Now $\displaystyle\lim_{h\to 0}\frac{s(t+h)-s(t)}{h} = \lim_{h\to 0}(2t+3+h)$

$= (2t+3)\ \text{ms}^{-1}$

This is the *instantaneous velocity* at t seconds.

2 **a** $s(t) = 5-2t^2$ cm

Average velocity $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$= \dfrac{s(5)-s(2)}{5-2}$

$= \dfrac{(-45)-(-3)}{3}$

$= -\frac{42}{3}$

$= -14\ \text{cm s}^{-1}$

b Average velocity $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$= \dfrac{s(2+h)-s(2)}{(2+h)-2}$

$= \dfrac{5-2(2+h)^2+3}{h}$

$= \dfrac{-8h-2h^2}{h}$

$= (-8-2h)\ \text{cm s}^{-1}$

c $\displaystyle\lim_{h\to 0}\frac{s(2+h)-s(2)}{h} = \lim_{h\to 0}(-8-2h)$

$= -8\ \text{cm s}^{-1}$

This is the *instantaneous velocity* when $t = 2$ seconds.

d $\displaystyle\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$

$\displaystyle = \lim_{h\to 0}\frac{\left[5-2(t+h)^2\right]-\left[5-2t^2\right]}{h}$

$\displaystyle = \lim_{h\to 0}\frac{-4th-2h^2}{h}$

$\displaystyle = \lim_{h\to 0}(-4t-2h)$

$= -4t\ \text{cm s}^{-1}$

$-4t$ cm s^{-1} is the *instantaneous velocity* at t seconds.

3 $v(t) = 2\sqrt{t}+3\ \text{cm s}^{-1},\quad t \geqslant 0$

a Average acceleration

$= \dfrac{v(t_2)-v(t_1)}{t_2-t_1}$

$= \dfrac{v(4)-v(1)}{4-1}$

$= \dfrac{7-5}{3}$

$= \frac{2}{3}\ \text{cm s}^{-2}$

b Average acceleration

$= \dfrac{v(t_2)-v(t_1)}{t_2-t_1}$

$= \dfrac{v(1+h)-v(1)}{(1+h)-1}$

$= \dfrac{[2\sqrt{1+h}+3]-[2\sqrt{1}+3]}{h}$

$= \dfrac{2\sqrt{1+h}-2}{h}\ \text{cm s}^{-2}$

c $\lim\limits_{h\to 0} \dfrac{v(1+h)-v(1)}{(1+h)-1}$

$= \lim\limits_{h\to 0} \dfrac{[2\sqrt{1+h}+3]-[2+3]}{h}$

$= \lim\limits_{h\to 0} \dfrac{2[\sqrt{1+h}-1]}{h} \times \dfrac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$

$= \lim\limits_{h\to 0} \dfrac{2h}{h(\sqrt{1+h}+1)}$

$= \frac{2}{2}$

$= 1$ cm s^{-2} This is the *instantaneous acceleration* when $t = 1$ second.

d $\lim\limits_{h\to 0} \dfrac{v(t+h)-v(t)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2\sqrt{t+h}-2\sqrt{t}}{h}$

$= \lim\limits_{h\to 0} \dfrac{2(\sqrt{t+h}-\sqrt{t})}{h} \times \dfrac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+t}$

$= \lim\limits_{h\to 0} \dfrac{2h}{h(\sqrt{t+h}+\sqrt{t})}$

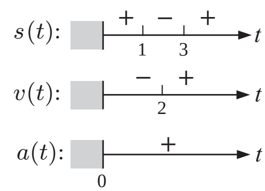
$= \dfrac{2}{2\sqrt{t}}$

$= \dfrac{1}{\sqrt{t}}$ cm s^{-2} This is the *instantaneous acceleration* at t seconds.

4 **a** This is the *instantaneous velocity* at $t = 3$ seconds.
b This is the *instantaneous acceleration* at $t = 5$ seconds.
c This is the *instantaneous velocity* at t seconds.
d This is the *instantaneous acceleration* at t seconds.

EXERCISE 22D.2

1 **a** $s(t) = t^2 - 4t + 3$ cm, $t \geqslant 0$ $\therefore$ $v(t) = 2t - 4$ cm s^{-1}
and $a(t) = 2$ cm s^{-2}

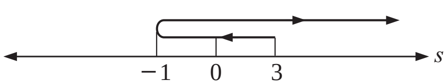


b When $t = 0$, $s(0) = 3$ cm
$v(0) = -4$ cm s^{-1}
$a(0) = 2$ cm s^{-2}
$\therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.

c When $t = 2$, $s(2) = -1$ cm
$v(2) = 0$ cm s^{-1}
$a(2) = 2$ cm s^{-2}
$\therefore$ the object is 1 cm left of O, momentarily at rest, but with acceleration 2 cm s^{-2} to the right.

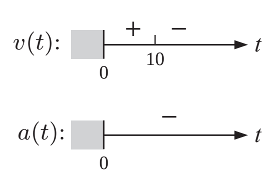
d The object reverses direction when $v(t) = 0$ i.e., at $t = 2$ seconds.
At $t = 2$, the particle is 1 cm left of O.

e



f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, i.e., when $0 \leqslant t \leqslant 2$.

2 $s(t) = 98t - 4.9t^2$ m $t \geqslant 0$



a $v(t) = 98 - 9.8t$ ms^{-1}
$a(t) = -9.8$ ms^{-2}

b When $t = 0$, $s(0) = 0$ m, $v(0) = 98$ ms^{-1}

c When $t = 5$, $s(5) = 367.5$ m
$v(5) = 49$ ms^{-1}
$a(5) = -9.8$ ms^{-2}
The stone is 367.5 m above the ground, travelling upwards at 49 ms^{-1} and slowing down.

When $t = 12$, $s(5) = 470.4$ m
$v(5) = -19.6$ ms^{-1}
$a(5) = -9.8$ ms^{-2}
The stone is 470.4 m above the ground and travelling downwards at 19.6 ms^{-1}, increasing in speed.

d Maximum height is reached when $v(t) = 0$ ms^{-1}

$\therefore\quad 98 - 9.8t = 0$

$9.8t = 98$

$t = 10$ seconds

$\therefore$ the maximum height is

$s(10) = 9.8(10) - 4.9\,(100)$
$= 980 - 490$
$= 490$ m

e The stone is at ground level when $s(t) = 0$ i.e., when $98t - 4.9t^2 = 0$

$\therefore\quad 4.9t(20 - t) = 0$

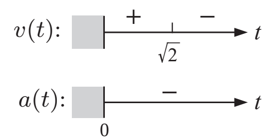
$\therefore\quad t = 0$ or 20 seconds

i.e., it hits the ground after 20 seconds.

3 **a** $s(t) = 12t - 2t^3 - 1$ cm, $t \geqslant 0$

$\therefore\quad v(t) = 12 - 6t^2$ cm s^{-1}

and $a(t) = -12t$ cm s^{-2}



b When $t = 0$, $s(0) = -1$ cm
$v(0) = 12$ cm s^{-1}
$a(0) = 0$ cm s^{-2}

The particle is 1 cm left of O, moving right at 12 cm s^{-1} with constant speed.

c The particle reverses direction when $v(t) = 0$ i.e., at $t = \sqrt{2}$ seconds.

When $t = \sqrt{2}$, $s(\sqrt{2}) = 12\sqrt{2} - 2(2\sqrt{2}) - 1$
$= 8\sqrt{2} - 1$ i.e., the particle is $(8\sqrt{2} - 1)$ cm to the right of O.

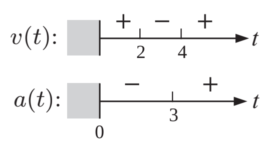
d **i** From the sign diagrams in **a**, the speed increases for $t \geqslant \sqrt{2}$ seconds.

ii The velocity of the particle never increases $\{a(t) \leqslant 0\}$.

4 **a** $x(t) = t^3 - 9t^2 + 24t$ m, $t \geqslant 0$

$v(t) = 3t^2 - 18t + 24$
$= 3(t^2 - 6t + 8)$
$= 3(t - 4)(t - 2)$ ms^{-1}

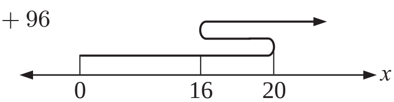
and $a(t) = 6t - 18$
$= 6(t - 3)$ ms^{-2}



b Reverses direction when $v(t) = 0$, i.e., at $t = 2$ seconds and $t = 4$ seconds.

$x(2) = 8 - 36 + 48$ m
$= 20$ m

and $x(4) = 64 - 144 + 96$
$= 16$ m



c **i** The speed decreases when $v(t)$ and $a(t)$ have the same sign, i.e., when $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$.

ii The velocity decreases when $a(t) < 0$, i.e., when $0 \leqslant t \leqslant 3$.

d When $t = 5$, $s(5) = 5^3 - 9.5^2 + 24.5$
$= 125 - 225 + 120$
$= 20$ m

$\therefore$ distance travelled $= 20 + 4 + 4$ m
$= 28$ m

5 **a** Let the equation be $s(t) = at^2 + bt + c$

$\therefore\quad v(t) = 2at + b$

and $a(t) = 2a = g$ {gravitational acceleration}

$\therefore\quad a = \frac{1}{2}g$

Also $v(t) = gt + b$

But, when $t = 0$, $v(0) = g \times 0 + b$

$\therefore\quad v(0) = b$ i.e., initial velocity is b

$\therefore\quad v(t) = v(0) + gt$ as required

b Now when $t = 0, \quad s(0) = 0$

$$\therefore \quad a \times 0^2 + b \times 0 + c = 0$$

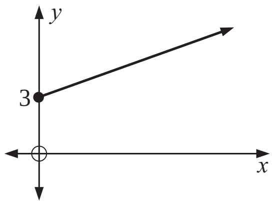
$$\therefore \quad c = 0$$

and so $s(t) = \left(\tfrac{1}{2}g\right) t^2 + v(0)t$

i.e., $s(t) = v(0) \times t + \tfrac{1}{2}gt^2$ as required

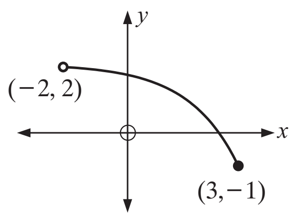
EXERCISE 22E.1

1 **a**



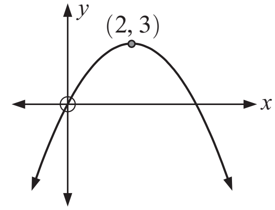
i $x \geqslant 0$
ii never

b



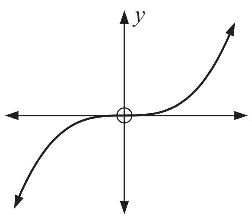
i never
ii $-2 < x \leqslant 3$

c



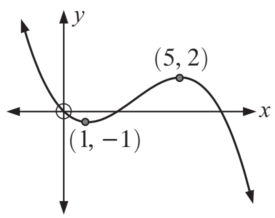
i $x \leqslant 2$
ii $x \geqslant 2$

d



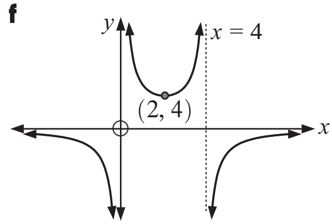
i all real x
ii never

e



i $1 \leqslant x \leqslant 5$
ii $x \leqslant 1, \quad x \geqslant 5$

f

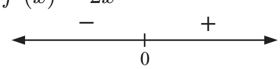


i $2 \leqslant x < 4, \quad x > 4$
ii $x < 0, \quad 0 < x \leqslant 2$

EXERCISE 22E.2

1 **a** $f(x) = x^2, \quad f'(x) = 2x$

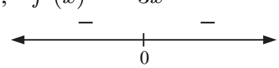
Sign $f'(x)$:



increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

b $f(x) = -x^3, \quad f'(x) = -3x^2$

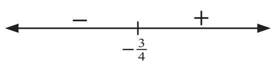
Sign $f'(x)$:



decreasing for all x

c $f(x) = 2x^2 + 3x - 4, \quad f'(x) = 4x + 3$

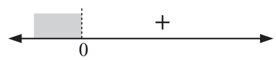
Sign $f'(x)$:



increasing when $x \geqslant -\frac{3}{4}$,
decreasing when $x \leqslant -\frac{3}{4}$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}, \;\; f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

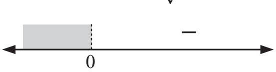
Sign $f'(x)$:



only defined when $x \geqslant 0$,
increasing when $x \geqslant 0$, never decreasing

e $f(x) = \dfrac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}, \qquad f'(x) = -x^{-\frac{3}{2}} = \dfrac{-1}{x\sqrt{x}}$

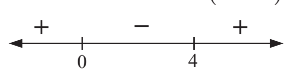
Sign $f'(x)$:



$f(x)$ is only defined for $x > 0$
decreasing when $x > 0$, never increasing

f $f(x) = x^3 - 6x^2, \quad f'(x) = 3x^2 - 12x = 3x(x-4)$

Sign $f'(x)$:



increasing when $x \leqslant 0$ or $x \geqslant 4$,
decreasing when $0 \leqslant x \leqslant 4$

g $f(x) = -2x^3 + 4x$

$$\begin{aligned} f'(x) &= -6x^2 + 4 \\ &= -2(3x^2 - 2) \end{aligned}$$

Sign $f'(x)$: $\quad - \;\big|_{-\sqrt{\frac{2}{3}}}\; + \;\big|_{\sqrt{\frac{2}{3}}}\; -$

increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$ or $x \geqslant \sqrt{\frac{2}{3}}$

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$\begin{aligned} f'(x) &= -12x^2 + 30x + 18 \\ &= -6(2x^2 - 5x - 3) \\ &= -6(2x + 1)(x - 3) \end{aligned}$$

Sign $f'(x)$: $\quad - \;\big|_{-\frac{1}{2}}\; + \;\big|_{3}\; -$

increasing when $-\frac{1}{2} \leqslant x \leqslant 3$,
decreasing when $x \leqslant -\frac{1}{2}$ and $x \geqslant 3$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,

$$\begin{aligned} f'(x) &= 12x^3 + 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x - 2)^2 \end{aligned}$$

Sign $f'(x)$: $\quad - \;\big|_{0}\; + \;\big|_{2}\; +$

increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$,

$$\begin{aligned} f'(x) &= 6x^2 + 18x + 6 \\ &= 6(x^2 + 3x + 1) \end{aligned}$$

Sign $f'(x)$: $\quad + \;\big|_{\frac{-3-\sqrt{5}}{2}}\; - \;\big|_{\frac{-3+\sqrt{5}}{2}}\; +$

$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

increasing for $x \leqslant \frac{-3-\sqrt{5}}{2}$ or $x \geqslant \frac{-3+\sqrt{5}}{2}$,
decreasing for $\frac{-3-\sqrt{5}}{2} \leqslant x \leqslant \frac{-3+\sqrt{5}}{2}$

k $f(x) = x^3 - 6x^2 + 3x - 1$,

$$\begin{aligned} f(x) &= 3x^2 - 12x + 3 \\ &= 3(x^2 - 4x + 1) \end{aligned}$$

$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

Sign $f'(x)$: $\quad + \;\big|_{2-\sqrt{3}}\; - \;\big|_{2+\sqrt{3}}\; +$

increasing when $x \leqslant 2 - \sqrt{3}$
or $x \geqslant 2 + \sqrt{3}$,
decreasing when $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

l $$\begin{aligned} f(x) &= x - 2\sqrt{x}, \\ &= x - 2x^{\frac{1}{2}} \end{aligned} \qquad \begin{aligned} f'(x) &= 1 - x^{-\frac{1}{2}} \\ &= 1 - \frac{1}{\sqrt{x}} \\ &= \frac{\sqrt{x} - 1}{\sqrt{x}} \end{aligned}$$

Sign $f'(x)$: $\quad \big|_{0}\; - \;\big|_{1}\; +$

increasing when $x \geqslant 1$,
decreasing when $0 \leqslant x \leqslant 1$

m $y = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 - 24x^2 - 12x + 24 \\ &= 12(x^3 - 2x^2 - x + 2) \end{aligned}$$

Using technology, a root is -1,

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 12(x + 1)(x^2 - 3x + 2) \\ &= 12(x + 1)(x - 1)(x - 2) \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:

$- \;\big|_{-1}\; + \;\big|_{1}\; - \;\big|_{2}\; +$

So $f(x)$ is increasing for $-1 \leqslant x \leqslant 1$ and $x \geqslant 2$, and decreasing for $x \leqslant -1$ and $1 \leqslant x \leqslant 2$.

n $y = x^4 - 4x^3 + 2x^2 + 4x + 1$,

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 12x^2 + 4x + 4 \\ &= 4(x^3 - 3x^2 + x + 1) \end{aligned}$$

Using technology, a root is 1,

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 4(x - 1)(x^2 - 2x - 1) \\ &= 0 \end{aligned}$$

when $x = 1$ or $x = \dfrac{2 \pm \sqrt{8}}{2}$

i.e., $x = 1$ or $x = 1 \pm \sqrt{2}$

Sign diagram of $\frac{dy}{dx}$: $\quad - \;\big|_{1-\sqrt{2}}\; + \;\big|_{1}\; - \;\big|_{1+\sqrt{2}}\; +$

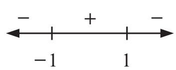
$\therefore$ increasing for $1 - \sqrt{2} \leqslant x \leqslant 1$ and $x \geqslant 1 + \sqrt{2}$, decreasing for $x \leqslant 1 - \sqrt{2}$ and $1 \leqslant x \leqslant 1 + \sqrt{2}$

2 **a** **i** $f(x) = \dfrac{4x}{x^2+1}$,

let $u = 4x, \quad u' = 4,$
$v = x^2+1, \quad v' = 2x$

$$\therefore \quad f'(x) = \frac{4(x^2+1) - 4x \times 2x}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2} = \frac{-4(x^2-1)}{(x^2+1)^2} = \frac{-4(x+1)(x-1)}{(x^2+1)^2}$$

Sign diagram of $\dfrac{dy}{dx}$:



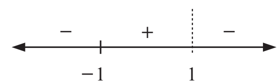
ii $\therefore$ increasing for $-1 \leqslant x \leqslant 1$,
decreasing for $x \leqslant -1$ and $x \geqslant 1$

b **i** $f(x) = \dfrac{4x}{(x-1)^2}$,

let $u = 4x, \quad u' = 4,$
$v = (x-1)^2, \quad v' = 2(x-1)^1$

$$\therefore \quad f'(x) = \frac{4(x-1)^2 - 8x(x-1)}{(x-1)^4} = \frac{4(x-1)\left((x-1)-2x\right)}{(x-1)^4} = \frac{4(-1-x)}{(x-1)^3} = \frac{-4(x+1)}{(x-1)^3}$$

Sign diagram of $\dfrac{dy}{dx}$:

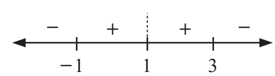


ii $\therefore$ increasing for $-1 \leqslant x < 1$,
decreasing for $x \leqslant -1$ and $x > 1$

c **i** $f(x) = \dfrac{-x^2+4x-7}{x-1}$, let $u = -x^2+4x-7, \ u' = -2x+4, \ v = x-1, \ v' = 1$

$$\therefore \quad f'(x) = \frac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2} = \frac{-2x^2+6x-4+x^2-4x+7}{(x-1)^2} = \frac{-x^2+2x+3}{(x-1)^2} = \frac{-(x^2-2x-3)}{(x-1)^2} = \frac{-(x-3)(x+1)}{(x-1)^2}$$

Sign diagram of $\dfrac{dy}{dx}$:

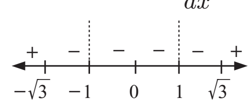


ii $\therefore$ $f(x)$ is increasing for $-1 \leqslant x < 1$ and $1 < x \leqslant 3$, and
decreasing for $x \leqslant -1$ and $x \geqslant 3$

3 **a** $f(x) = \dfrac{x^3}{x^2-1}$, let $u = x^3, \quad u' = 3x^2, \quad v = x^2-1, \quad v' = 2x$

$$\therefore \quad f'(x) = \frac{3x^2(x^2-1) - x^3 \times 2x}{(x^2-1)^2} = \frac{3x^4-3x^2-2x^4}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2} = \frac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x^2-1)^2}$$

Sign diagram of $\dfrac{dy}{dx}$:

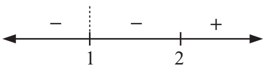


$\therefore$ $f(x)$ is increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$, and
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x \leqslant 0$, $0 \leqslant x < 1$ and $1 < x \leqslant \sqrt{3}$.

b $f(x) = x^2 + \dfrac{4}{x-1}$, $\quad = x^2 + 4(x-1)^{-1}$

$$\begin{aligned} f'(x) &= 2x - 4(x-1)^{-2} \times 1 \\ &= 2x - \frac{4}{(x-1)^2} \\ &= \frac{2x(x-1)^2 - 4}{(x-1)^2} \\ &= \frac{2x(x^2-2x+1) - 4}{(x-1)^2} \\ &= \frac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2} \\ &= \frac{(x-2)(2x^2+2)}{(x-1)^2} \end{aligned}$$

Sign of $\dfrac{dy}{dx}$:

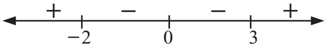


$\therefore$ $f(x)$ is increasing for $x \geqslant 2$, and decreasing for $1 < x \leqslant 2$ and $x < 1$.

EXERCISE 22E.3

1 **a** A is a *local minimum*, B is a *local maximum*, C is a *horizontal inflection*.

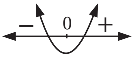
b $f'(x)$ has sign diagram:



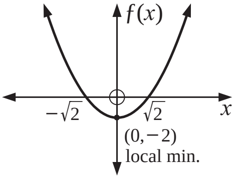
c **i** $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$ **ii** $f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$.

2 **a** $f(x) = x^2 - 2$ $\therefore$ $f'(x) = 2x$

with sign diagram:

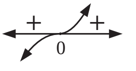


local minimum at $(0, -2)$

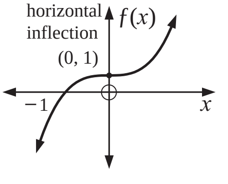


b $f(x) = x^3 + 1$ $\therefore$ $f'(x) = 3x^2$

with sign diagram:



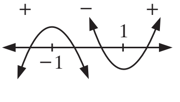
horizontal inflection at $(0, 1)$



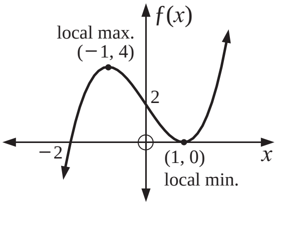
c $f(x) = x^3 - 3x + 2$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 3 \\ &= 3(x^2-1) \\ &= 3(x+1)(x-1) \end{aligned}$$

with sign diagram:



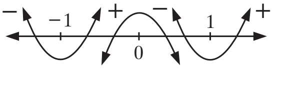
local maximum at $(-1, 4)$,
local minimum at $(1, 0)$



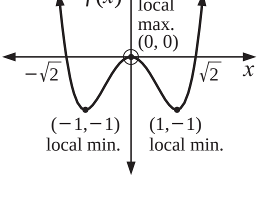
d $f(x) = x^4 - 2x^2$

$$\begin{aligned} \therefore \quad f'(x) &= 4x^3 - 4x \\ &= 4x(x^2-1) \\ &= 4x(x+1)(x-1) \end{aligned}$$

with sign diagram:

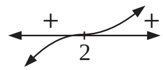


local minima at $(-1, -1)$ and $(1, -1)$,
local maximum at $(0, 0)$

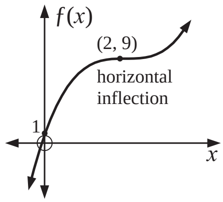


e $f(x) = x^3 - 6x^2 + 12x + 1$

$\therefore \quad f'(x) = 3x^2 - 12x + 12$
$= 3(x^2 - 4x + 4)$
$= 3(x - 2)^2$

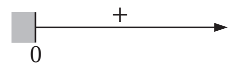
with sign diagram: 

$\therefore$ horizontal inflection at (2, 9)

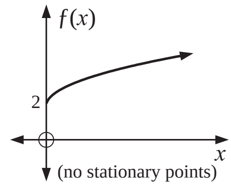


f $f(x) = \sqrt{x} + 2$

$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$= \dfrac{1}{2\sqrt{x}} \neq 0$

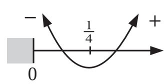
with sign diagram: 

$\therefore$ no stationary points.



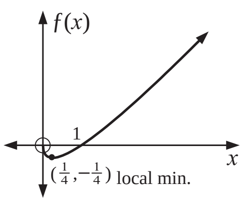
g $f(x) = x - \sqrt{x}$

$\therefore \quad f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$
$= 1 - \dfrac{1}{2\sqrt{x}}$
$= \dfrac{2\sqrt{x} - 1}{2\sqrt{x}}$

with sign diagram: 

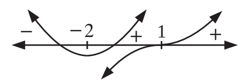
$f(x)$ is defined for all $x \geqslant 0$

local minimum at $(\frac{1}{4}, -\frac{1}{4})$



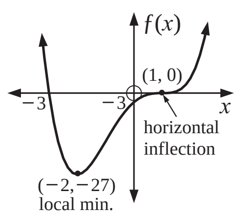
h $f(x) = x^4 - 6x^2 + 8x - 3$

$\therefore \quad f'(x) = 4x^3 - 12x + 8$
$= 4(x^3 - 3x + 2)$
$= 4(x - 1)(x^2 + x - 2)$
$= 4(x - 1)(x + 2)(x - 1)$

with sign diagram: 

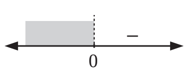
local minimum at (−2, −27),

horizontal inflection at (1, 0)



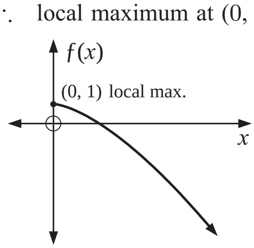
i $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$

$\therefore \quad f'(x) = -\frac{3}{2}x^{\frac{1}{2}}$
$= \dfrac{-3\sqrt{x}}{2}$

with sign diagram: 

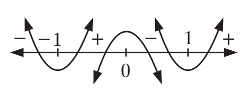
$f(x)$ is only defined when $x \geqslant 0$

$\therefore$ local maximum at (0, 1)



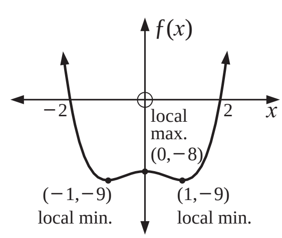
j $f(x) = x^4 - 2x^2 - 8$

$\therefore \quad f'(x) = 4x^3 - 4x$
$= 4x(x^2 - 1)$
$= 4x(x + 1)(x - 1)$

with sign diagram: 

local minima at (−1, −9) and (1, −9),

local maximum at (0, −8)



3 $f(x) = ax^2 + bx + c, \quad a \neq 0$

$f'(x) = 2ax + b$ and $f(x)$ has a stationary point when $f'(x) = 0$ i.e., $x = -\dfrac{b}{2a}$

There is a local maximum when $a < 0$ and there is a local minimum when $a > 0$.

4 $f(x) = 2x^3 + ax^2 - 24x + 1, \quad \therefore \quad f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0, \quad \therefore \quad 96 - 8a - 24 = 0$

$\therefore \quad 72 = 8a$ and so $a = 9$.

5 **a** $f(x) = x^3 + ax + b, \quad \therefore \quad f'(x) = 3x^2 + a$

Now $3x^2 + a = 0$ when $x = -2$

$\therefore \quad 12 + a = 0$

i.e., $a = -12$

Now $(-2, 3)$ is a point on the curve.

$\therefore \quad f(x) = x^3 - 12x + b$

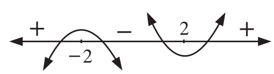
$\therefore \quad 3 = -8 + 24 + b$

$\therefore \quad b = -13$

b $\therefore \quad f(x) = x^3 - 12x - 13$ and $f'(x) = 3x^2 - 12$

$f'(x) = 3(x+2)(x-2)$

where $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $(-2, 3)$ and a local minimum at $(2, -29)$

6 Let the cubic polynomial be $P(x) = ax^3 + bx^2 + cx + d$ ...... (1)

$\therefore \quad P'(x) = 3ax^2 + 2bx + c$ ...... (2)

(0, 2) lies on $P(x)$ $\quad \therefore \quad P(0) = 2$ and so $a(0) + b(0) + c(0) + d = 2$

i.e., $d = 2$

Now the tangent is $y = 9x + 2$, $\therefore$ slope at (0, 2) is 9 $\quad \therefore \quad P'(0) = 9$

$\therefore \quad 3a(0) + 2b(0) + c = 9$

$\therefore \quad c = 9$

and a stationary point at $(-1, -7)$ means that $P'(-1) = 0$

$\therefore \quad 3a(-1)^2 + 2b(-1) + c = 0 \quad$ {using (2)}

$\therefore \quad 3a - 2b + c = 0$

but since $c = 9$

$3a - 2b = -9$ ...... (3)

Now $(-1, -7)$ also lies on $P(x)$ $\quad \therefore \quad a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore \quad -a + b - 9 + 2 = -7$

$\therefore \quad a - b = 0$ ...... (4)

$\therefore \quad a = b$

$\therefore \quad 3a - 2a = -9 \quad$ {in (3)}

$\therefore \quad a = -9$

$\therefore \quad a = b = -9$

$\therefore \quad P(x) = -9x^3 - 9x^2 + 9x + 2$

7 **a** $f(x) = x^3 - 12x - 2$, for $-3 \leqslant x \leqslant 5$

$\therefore \quad f'(x) = 3x^2 - 12$

$= 3(x+2)(x-2)$

which is 0 when $x = -2$ or 2

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$\therefore$ maximum value is 63 when $x = 5$, and minimum value is -18, when $x = 2$.

b $f(x) = 4 - 3x^2 + x^3$, for $-2 \leqslant x \leqslant 3$

$\therefore \quad f'(x) = -6x + 3x^2$

$= 3x(x-2)$

which is 0 when $x = 0$ or 2

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ maximum value is 4 when $x = 0$ or $x = 3$, minimum value is -16, when $x = -2$.

8 $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$

$C'(x) = 0.0021x^2 - 0.3592x + 14.663$

$C'(x) = 0$ when $0.0021x^2 - 0.3592x + 14.663 = 0$

Using technology, $x \doteqdot 103.74$ or $x \doteqdot 67.30$

x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made. The minimum hourly cost is \$529.80 when 104 hinges are made.

EXERCISE 22F.1

1 **a** $f(x) = \dfrac{3x-2}{x+1}$

$= \dfrac{3(x+1)-5}{x+1}$

$= 3 - \dfrac{5}{x+1}$

$\therefore$ VA is $x + 1 = 0$, i.e., $x = -1$

and the HA is $y = 3$

$\{$as $|x| \to \infty$, $f(x) \to 3\}$

b $f(x) = \dfrac{x-4}{2x-1}$

$= \dfrac{\frac{1}{2}(2x-1) - 3\frac{1}{2}}{2x-1}$

$= \frac{1}{2} + \dfrac{-3\frac{1}{2}}{2x-1}$

$= \frac{1}{2} - \dfrac{7}{2(2x-1)}$

$\therefore$ VA is $2x - 1 = 0$, i.e., $x = \frac{1}{2}$

and the HA is $y = \frac{1}{2}$

$\{$as $|x| \to \infty$, $f(x) \to \frac{1}{2}\}$

c $f(x) = \dfrac{4-2x}{x-1}$

$= \dfrac{-2(x-1)+2}{x-1}$

$= -2 + \dfrac{2}{x-1}$

$\therefore$ VA is $x - 1 = 0$, i.e., $x = 1$ and the HA is $y = -2$ $\{$as $|x| \to \infty$, $f(x) \to -2\}$

2 **a** **i** $f(x) = -3 + \dfrac{1}{4-x}$

$\therefore$ VA is $4 - x = 0$ i.e., $x = 4$

and HA is $y = -3$

$\{$as $|x| \to \infty$, $y \to -3\}$

ii $f(x) = -3 + (4-x)^{-1}$

$\therefore \quad f'(x) = -(4-x)^{-2} \times (-1)$

$= \dfrac{1}{(4-x)^2}$

Sign diagram of $f'(x)$ is

iii When $x = 0$, $y = -3 + \frac{1}{4} = -2\frac{3}{4}$

$\therefore$ y intercept is $-2\frac{3}{4}$

When $y = 0$, $-3 + \dfrac{1}{4-x} = 0$

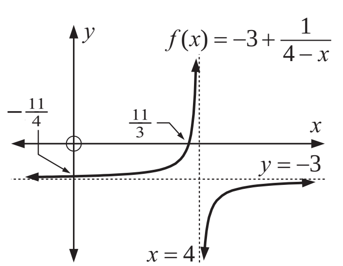
$\therefore \quad \dfrac{1}{4-x} = 3$

$\therefore \quad 4 - x = \frac{1}{3}$

$\therefore \quad x = 3\frac{2}{3}$

$\therefore$ x-intercept is $3\frac{2}{3}$

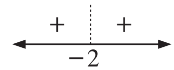
iv



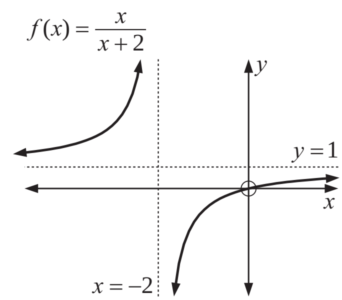
b **i** $f(x) = \dfrac{x}{x+2} = \dfrac{(x+2)-2}{x+2} = 1 - \dfrac{2}{x+2}$

$\therefore$ VA is $x = -2$ $\{x+2=0\}$
and HA is $y = 1$.
$\{\text{as } |x| \to \infty, \quad y \to 1\}$

ii $f'(x) = \dfrac{1(x+2) - x(1)}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

and has sign diagram 

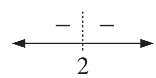
iii when $x = 0$, $f(0) = 0$
$\therefore$ y-intercept is 0
when $y = 0$, $\dfrac{x}{x+2} = 0$
$\therefore$ $x = 0$
$\therefore$ x-intercept is 0

iv 

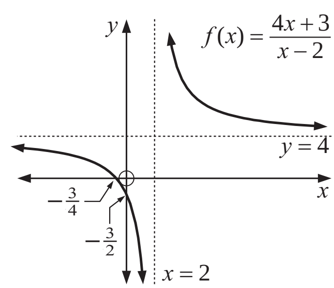
c **i** $f(x) = \dfrac{4x+3}{x-2} = \dfrac{4(x-2)+11}{x-2} = 4 + \dfrac{11}{x-2}$

$\therefore$ VA is $x - 2 = 0$ i.e., $x = 2$
and HA is $y = 4$
$\{\text{as } |x| \to \infty, \quad y \to 4\}$

ii $f'(x) = \dfrac{4(x-2) - (4x+3)1}{(x-2)^2} = \dfrac{-11}{(x-2)^2}$

and has sign diagram 

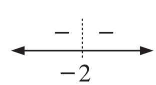
iii when $x = 0$, $f(0) = \frac{3}{-2}$
$\therefore$ y-intercept is $-1\frac{1}{2}$
when $y = 0$, $4x + 3 = 0$
$\therefore$ $x = -\frac{3}{4}$
$\therefore$ x-intercept is $-\frac{3}{4}$

iv 

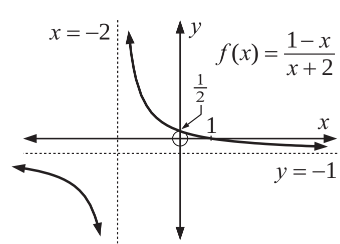
d **i** $f(x) = \dfrac{1-x}{x+2} = \dfrac{-(x+2)+3}{x+2} = -1 + \dfrac{3}{x+2}$

$\therefore$ VA is $x + 2 = 0$ i.e., $x = -2$
and HA is $y = -1$
$\{\text{as } |x| \to \infty, \quad y \to -1\}$

ii $f'(x) = \dfrac{-1(x+2) - (1-x)1}{(x+2)^2} = \dfrac{-3}{(x+2)^2}$

and has sign diagram 

iii when $x = 0$, $f(0) = \frac{1}{2}$
$\therefore$ y-intercept is $\frac{1}{2}$
when $y = 0$, $\dfrac{1-x}{x+2} = 0$
$\therefore$ $1 - x = 0$
$\therefore$ $x = 1$
$\therefore$ x-intercept is 1

iv 

EXERCISE 22F.2

1 **a** $y = \dfrac{2x}{x^2-4} = \dfrac{2x}{(x+2)(x-2)}$

VAs are $x+2=0$ and $x-2=0$

i.e., $x=-2$ and $x=2$

HA is $y=0$ $\{$as $|x| \to \infty,\ y \to 0\}$

b $y = \dfrac{1-x}{(x+2)^2}$

VA is $x+2=0$ i.e., $x=-2$

HA is $y=0$ $\{$as $|x| \to \infty,\ y \to 0\}$

c $y = \dfrac{3x+2}{x^2+1}$ has no VAs $\{$as $x^2+1=0$ has no real solutions$\}$ and a HA of $y=0$ $\{$as $|x| \to \infty,\ y \to 0\}$

2 **a** **i** $f(x) = \dfrac{4x}{x^2+1}$

has no VAs $\{$as $x^2+1=0$ has no real solutions$\}$

and a HA of $y=0$ $\{$as $|x| \to \infty,\ y \to 0\}$

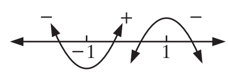
ii $f'(x) = \dfrac{4(x^2+1)-4x(2x)}{(x^2+1)^2}$

$= \dfrac{4x^2+4-8x^2}{(x^2+1)^2}$

$= \dfrac{4-4x^2}{(x^2+1)^2}$

$= \dfrac{4(1+x)(1-x)}{(x^2+1)^2}$

and has a sign diagram of



$\therefore$ local min. is $\left(-1, \dfrac{4(-1)}{1+1}\right)$

i.e., $(-1, -2)$

and local max is $\left(1, \dfrac{4(1)}{1+1}\right)$

i.e., $(1, 2)$

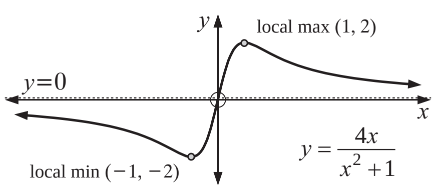
iii when $x=0$, $f(0)=0$

$\therefore$ y-intercept is 0

when $y=0$, $\dfrac{4x}{x^2+1}=0$

$\therefore$ $x=0$ and so the x-intercept is 0

iv



b $f(x) = \dfrac{4x}{x^2-4x-5} = \dfrac{4x}{(x-5)(x+1)}$

i Vertical asymptotes occur when $(x-5)(x+1)=0$, i.e., at $x=-1$ and $x=5$.

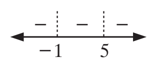
Horizontal asymptote is $y=0$ $\{$as $|x| \to \infty,\ y \to 0\}$

ii $f'(x) = \dfrac{4(x^2-4x-5)-4x(2x-4)}{(x^2-4x-5)^2}$

$= \dfrac{4x^2-16x-20-8x^2+16x}{(x^2-4x-5)^2}$

$= \dfrac{-4x^2-20}{(x-5)^2(x+1)^2}$

Since $-4x^2-20$ is always negative we have no turning points, and sign diagram is:



iii When $x=0$, $y = \frac{0}{-5} = 0$

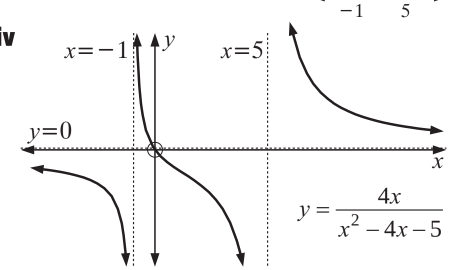
$\therefore$ y-intercept is 0

When $y=0$, $\dfrac{4x}{x^2-4x-5}=0$

$\therefore$ $x=0$

$\therefore$ x-intercept is 0

iv

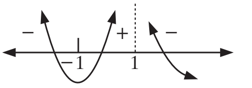


c **i** $f(x) = \dfrac{4x}{(x-1)^2}$

has VA $x - 1 = 0$ i.e., $x = 1$
and a HA of $y = 0$
$\{$as $|x| \to \infty, \quad y \to 0\}$

ii
$$\begin{aligned} f'(x) &= \frac{4(x-1)^2 - 4x \times 2(x-1)1}{(x-1)^4} \\ &= \frac{4(x-1)[x-1-2x]}{(x-1)^4} \\ &= \frac{4(-x-1)}{(x-1)^3} \\ &= \frac{-4(x+1)}{(x-1)^3} \end{aligned}$$

which has sign diagram



local min. at $\left(-1, \frac{4(-1)}{(-2)^2}\right)$ i.e., at $(-1, -1)$

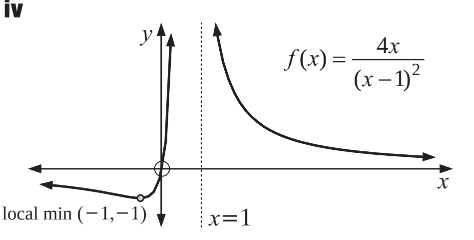
iii when $x = 0$, $y = \frac{0}{1} = 0$

$\therefore$ y-intercept is 0

when $y = 0$, $\dfrac{4x}{(x-1)^2} = 0$

$\therefore$ x-intercept is 0

iv



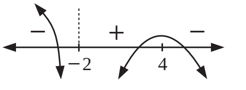
d $f(x) = \dfrac{3x-3}{(x+2)^2}$

i Vertical asymptotes occur when $(x+2)^2 = 0$, i.e., at $x = -2$.

HA is $y = 0$ $\{$as $|x| \to \infty, \quad y \to 0\}$

ii
$$\begin{aligned} f'(x) &= \frac{3(x+2)^2 - 2(x+2)(3x-3)}{(x+2)^4} \\ &= \frac{3(x+2) - 2(3x-3)}{(x+2)^3} \\ &= \frac{3x+6-6x+6}{(x+2)^3} \\ &= \frac{12-3x}{(x+2)^3} \\ &= \frac{-3(x-4)}{(x+2)^3} \end{aligned}$$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(4, \frac{1}{4})$.

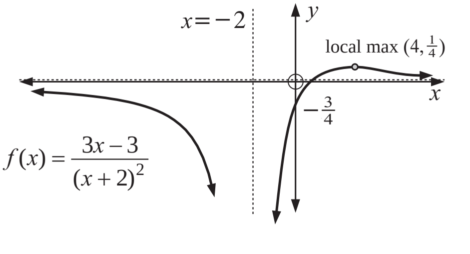
iii when $x = 0$, $y = \frac{-3}{2^2} = -\frac{3}{4}$

$\therefore$ y-intercept is $-\frac{3}{4}$

when $y = 0$, $3x - 3 = 0$
$\therefore$ $x = 1$

$\therefore$ x-intercept is 1

iv



EXERCISE 22F.3

1 **a** $y = \dfrac{2x^2 - x + 2}{x^2 - 1} = \dfrac{2x^2 - x + 2}{(x+1)(x-1)}$

has VA when $x + 1 = 0, \quad x - 1 = 0$

i.e., $x = -1, \quad x = 1$

and as $y = \dfrac{2 - \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$

HA is $y = 2$

$\{$as $|x| \to \infty, \quad y \to \frac{2}{1}\}$

b $y = \dfrac{-x^2 + 2x - 1}{x^2 + x + 1}$

has a VA when $x^2 + x + 1 = 0$

But $\Delta = 1^2 - 4(1)(1) < 0 \quad \therefore$ no real solutions $\therefore$ no VA's exist

and as $y = \dfrac{-1 + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$

HA is $y = -1 \quad \{$as $|x| \to \infty, \quad y = \dfrac{-1}{1}\}$

c $y = \dfrac{3x^2 - x + 2}{(x+2)^2} = \dfrac{3x^2 - x + 2}{x^2 + 4x + 4} = \dfrac{3 - \frac{1}{x} + \frac{2}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}}$

has VA $x + 2 = 0$ i.e., $x = -2$ had HA $y = 3$ $\{$as $|x| \to \infty, \quad y \to \frac{3}{1}\}$

2 **a** **i** $y = \dfrac{x^2 - x}{x^2 - x - 6} = \dfrac{x(x-1)}{(x-3)(x+2)}$

has VA when $x - 3 = 0, \quad x + 2 = 0$

i.e., $x = 3, \quad x = -2$

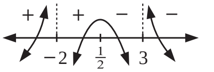
and since $y = \dfrac{1 - \frac{1}{x}}{1 - \frac{1}{x} - \frac{6}{x^2}}$

the HA is $y = 1$

$\{$as $|x| \to \infty, \quad y \to \frac{1}{1}\}$

ii
$$\begin{aligned} f'(x) &= \frac{(2x-1)(x^2 - x - 6) - (x^2 - x)(2x-1)}{(x^2 - x - 6)^2} \\ &= \frac{(2x-1)(x^2 - x - 6 - x^2 + x)}{(x^2 - x - 6)^2} \\ &= \frac{-6(2x-1)}{(x^2 - x - 6)^2} \end{aligned}$$

Turning points are when $f'(x) = 0$, i.e., when $x = \frac{1}{2}$.
Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(\frac{1}{2}, \frac{1}{25})$.

iii When $x = 0, \quad y = \frac{0}{-6} = 0$

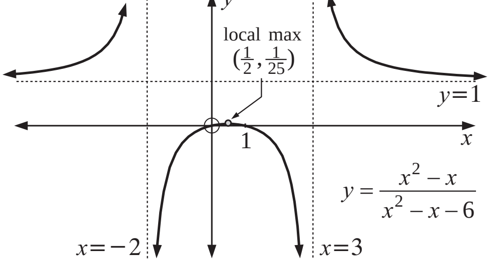
$\therefore$ y-intercept is 0

When $y = 0, \quad x(x-1) = 0$

$\therefore$ $x = 0$ or 1

$\therefore$ x-intercepts are 0 and 1

iv



b **i** $y = \dfrac{x^2-1}{x^2+1} = \dfrac{(x+1)(x-1)}{x^2+1}$

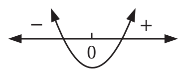
has no VA as $x^2+1=0$ has no real solutions and since

$y = \dfrac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}}$ the HA is $y=1$

$\{$as $|x| \to \infty$, $y \to \frac{1}{1}\}$

ii
$$\begin{aligned} f'(x) &= \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} \\ &= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

and has sign diagram



$\therefore$ a local minimum of $(0, -1)$.

iii When $x=0$, $y = \frac{-1}{1} = -1$

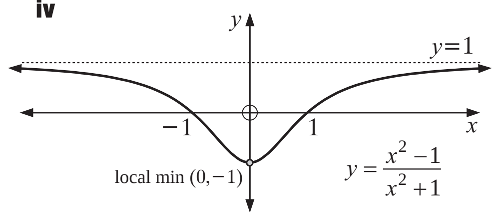
$\therefore$ y-intercept is -1

When $y=0$, $(x+1)(x-1)=0$

$\therefore$ $x = \pm 1$

$\therefore$ x-intercepts are ± 1

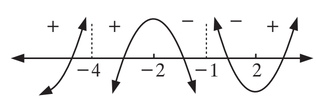
iv



c **i** $y = \dfrac{x^2-5x+4}{x^2+5x+4} = \dfrac{(x-1)(x-4)}{(x+1)(x+4)}$ has VAs of $x+1=0$ and $x+4=0$ i.e., $x=-1$, $x=-4$

and as $y = \dfrac{1-\dfrac{5}{x}+\dfrac{4}{x^2}}{1+\dfrac{5}{x}+\dfrac{4}{x^2}}$ the HA is $y=1$ $\{$as $|x| \to \infty$, $y \to \frac{1}{1}\}$

ii
$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x-5)(x^2+5x+4)-(x^2-5x+4)(2x+5)}{(x+1)^2(x+4)^2} \\ &= \frac{[2x^3+10x^2+8x-5x^2-25x-20]-[2x^3-10x^2+8x+5x^2-25x+20]}{(x+1)^2(x+4)^2} \\ &= \frac{10x^2-40}{(x+1)^2(x+4)^2} \\ &= \frac{10(x+2)(x-2)}{(x+1)^2(x+4)^2} \end{aligned}$$
which has sign diagram



$\therefore$ local max at $\left(-2, \dfrac{4+10+4}{4-10+4}\right)$ i.e., at $(-2, -9)$

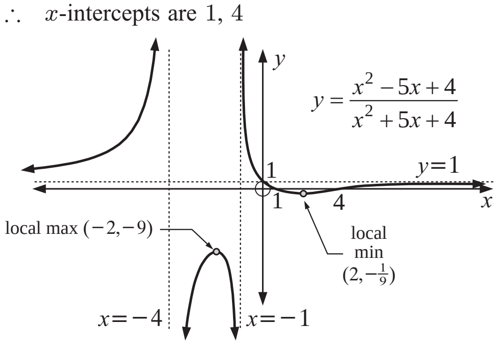
local min at $\left(2, \dfrac{4-10+4}{4+10+4}\right)$ i.e., at $(2, -\frac{1}{9})$

iii When $x=0$, $y = \frac{4}{4} = 1$ $\therefore$ the y-intercept is 1

When $y=0$, $(x-1)(x-4)=0$ $\therefore$ $x = 1$ or 4

$\therefore$ x-intercepts are 1, 4

iv



d **i** $y = \dfrac{x^2-6x+5}{(x+1)^2} = \dfrac{(x-1)(x-5)}{(x+1)^2}$

has VA $x+1=0$ i.e., $x=-1$ and HA $y=1$

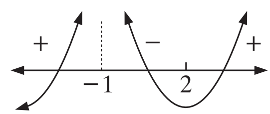
$\left\{\text{as } y = \dfrac{1-\frac{6}{x}+\frac{5}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}} \to \frac{1}{1} \text{ as } |x| \to \infty\right\}$

ii $\dfrac{dy}{dx} = \dfrac{(2x-6)(x+1)^2-(x^2-6x+5)2(x+1)^1 1}{(x+1)^4}$

$= \dfrac{(x+1)[2x^2-4x-6-2x^2+12x-10]}{(x+1)^4}$

$= \dfrac{8x-16}{(x+1)^3}$

$= \dfrac{8(x-2)}{(x+1)^3}$ and has sign diagram



$\therefore$ local minimum at $\left(2, \frac{4-12+5}{3^2}\right)$ i.e., $(2, -\frac{1}{3})$

iii When $x=0$, $y=\frac{5}{1}=5$

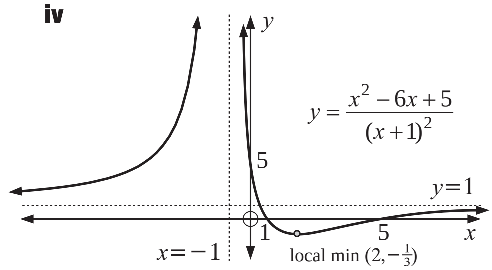
$\therefore$ y-intercept is 5

When $y=0$, $(x-1)(x-5)=0$

$\therefore$ $x=1$ or 5

$\therefore$ x-intercepts are 1, 5

iv



EXERCISE 22G

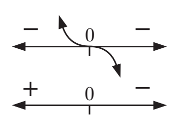
1 **a** $f(x)=x^2+3$

$\therefore$ $f'(x)=2x$ and $f''(x)=2$

Since $f''(x) \neq 0$,

no points of inflection exist.

b $f(x)=2-x^3$



$\therefore$ $f'(x)=-3x^2$

and $f''(x)=-6x$

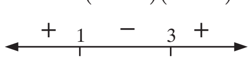
Now $f''(x)=0$ when $x=0$

and $f'(0)=0$

$\therefore$ there is a horizontal inflection at (0, 2)

c

$$\begin{aligned} f(x) &= x^3-6x^2+9x+1 \\ \therefore\quad f'(x) &= 3x^2-12x+9 \\ &= 3(x^2-4x+3) \\ &= 3(x-3)(x-1) \end{aligned}$$



$$\begin{aligned} \text{and}\quad f''(x) &= 6x-12 \\ &= 6(x-2) \end{aligned}$$

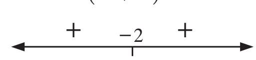
Now $f''(x) = 0$ when $x=2$

and $f'(2) \neq 0$

$\therefore$ there is a non-horizontal inflection at (2, 3)

d

$$\begin{aligned} f(x) &= x^3+6x^2+12x+5 \\ \therefore\quad f'(x) &= 3x^2+12x+12 \\ &= 3(x^2+4x+4) \\ &= 3(x+2)^2 \end{aligned}$$

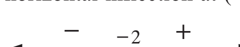


$$\begin{aligned} \text{and}\quad f''(x) &= 6x+12 \\ &= 6(x+2) \end{aligned}$$

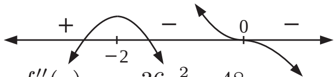
Now $f''(x) = 0$ when $x=-2$

and $f'(2) = 0$

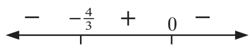
$\therefore$ there is a horizontal inflection at (–2, –3)



e
$$\begin{aligned} f(x) &= -3x^4 - 8x^3 + 2 \\ \therefore\ f'(x) &= -12x^3 - 24x^2 \\ &= -12x^2(x+2) \end{aligned}$$



$$\begin{aligned} \text{and}\quad f''(x) &= -36x^2 - 48x \\ &= -12x(3x+4) \end{aligned}$$



$\therefore$ horizontal inflection at (0, 2),

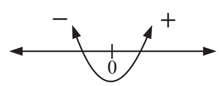
non-horizontal inflection at $\left(-\frac{4}{3}, \frac{310}{27}\right)$

f
$$\begin{aligned} f(x) &= 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}} \\ \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{3}{2}} \\ \text{and}\quad f''(x) &= -\tfrac{3}{4}x^{-\frac{5}{2}} \\ &= \frac{-3}{4x^2\sqrt{x}} \end{aligned}$$

$f''(x) \neq 0$ for all x

$\therefore$ no points of inflection.

2 **a**
$$\begin{aligned} f(x) &= x^2 \\ \therefore\ f'(x) &= 2x \end{aligned}$$
which has sign diagram:



and $f''(x) = 2$

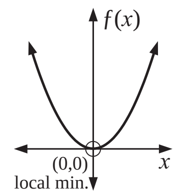
i There is a local minimum at (0, 0).

ii There are no points of inflection as $f''(x) \neq 0$.

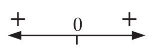
iii $f(x)$ is increasing when $x \geqslant 0$, and decreasing when $x \leqslant 0$.

iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x.

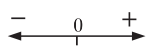
v



b
$$\begin{aligned} f(x) &= x^3 \\ \therefore\ f'(x) &= 3x^2 \end{aligned}$$
which has sign diagram:



and $f''(x) = 6x$ which has sign diagram:



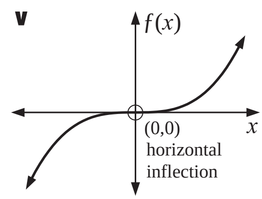
i A horizontal inflection at (0, 0). $\{f'(x) = 0\}$

ii see **i**

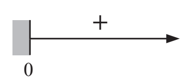
iii $f(x)$ is increasing for all x.

iv $f(x)$ is concave up $x \geqslant 0$, and concave down $x \leqslant 0$

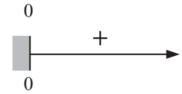
v



c
$$\begin{aligned} f(x) &= \sqrt{x} \\ \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \end{aligned}$$
which has sign diagram:



and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \dfrac{-1}{4x\sqrt{x}}$ which has sign diagram:



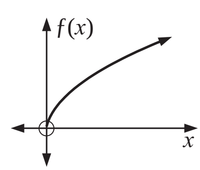
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

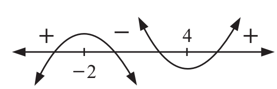
iii $f(x)$ is increasing for all $x \geqslant 0$.

iv $f(x)$ is concave down for all $x \geqslant 0$ as $f''(x) < 0$ for all $x > 0$.

v



d
$$\begin{aligned} f(x) &= x^3 - 3x^2 - 24x + 1 \\ \therefore\ f'(x) &= 3x^2 - 6x - 24 \\ &= 3(x^2 - 2x - 8) \\ &= 3(x-4)(x+2) \end{aligned}$$
which has sign diagram:



$$\begin{aligned} \text{and}\quad f''(x) &= 6x - 6 \\ &= 6(x-1) \end{aligned}$$
which has sign diagram:

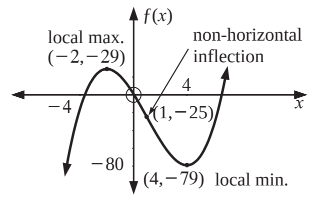
i There is a local maximum at (−2, 29), and a local minimum at (4, −79).

ii There is a non-horizontal inflection at $(1, -25)$.

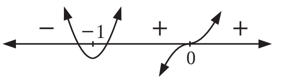
iii $f(x)$ is increasing for $x \leqslant -2$ or $x \geqslant 4$, and decreasing for $-2 \leqslant x \leqslant 4$.

iv $f(x)$ is concave down when $x \leqslant 1$, and concave up when $x \geqslant 1$.

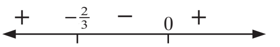
v



e
$$\begin{aligned} f(x) &= 3x^4 + 4x^3 - 2 \\ \therefore \quad f'(x) &= 12x^3 + 12x^2 \\ &= 12x^2(x+1) \quad \text{which has sign diagram:} \end{aligned}$$



$$\begin{aligned} \text{and} \quad f''(x) &= 36x^2 + 24x \\ &= 12x(3x+2) \quad \text{which has sign diagram:} \end{aligned}$$



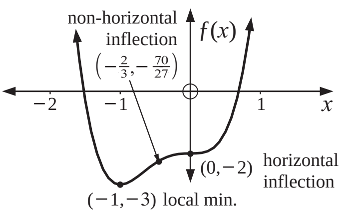
i There is a local minimum at $(-1, -3)$, and a horizontal inflection at $(0, -2)$

ii There is a non-horizontal inflection at $(-\frac{2}{3}, -\frac{70}{27})$ and a horizontal inflection at $(0, -2)$

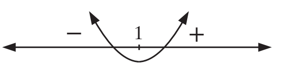
iii $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

iv $f(x)$ is concave down for $-\frac{2}{3} \leqslant x \leqslant 0$, and concave up for $x \leqslant -\frac{2}{3}$ or $x \geqslant 0$.

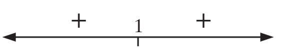
v



f
$$\begin{aligned} f(x) &= (x-1)^4 \\ \therefore \quad f'(x) &= 4(x-1)^3 \times 1 \\ &= 4(x-1)^3 \quad \text{which has sign diagram:} \end{aligned}$$



$$\begin{aligned} \text{and} \quad f''(x) &= 12(x-1)^2 \times 1 \\ &= 12(x-1)^2 \quad \text{which has sign diagram:} \end{aligned}$$



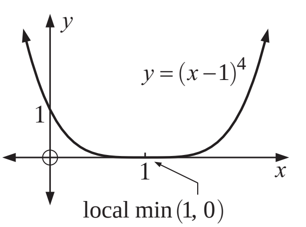
i There is a local minimum at $(1, 0)$.

ii There are no points of inflection.

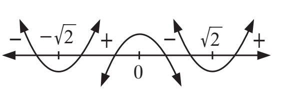
iii $f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

iv $f(x)$ is concave up for all x.

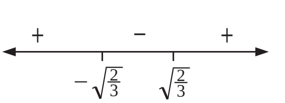
v



g
$$\begin{aligned} f(x) &= x^4 - 4x^2 + 3 \\ f'(x) &= 4x^3 - 8x \\ &= 4x(x^2 - 2) \\ &= 4x(x+\sqrt{2})(x-\sqrt{2}) \quad \text{which has sign diagram:} \end{aligned}$$



$$\begin{aligned} f''(x) &= 12x^2 - 8 \\ &= 4(3x^2 - 2) \\ &= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2}) \quad \text{which has sign diagram:} \end{aligned}$$

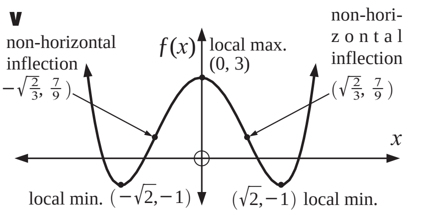


i There is a local maximum at $(0, 3)$, and local min. at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.

ii There are non-horizontal inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$. $\{f''(x) = 0\}$

iii $f(x)$ is increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, and decreasing for $0 \leqslant x \leqslant \sqrt{2}$ and $x \leqslant -\sqrt{2}$.

v

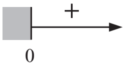


iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

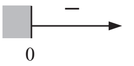
h $f(x) = 3 - \dfrac{4}{\sqrt{x}}, \quad x > 0$

$= 3 - 4x^{-\frac{1}{2}}$

$\therefore \quad f'(x) = 2x^{-\frac{3}{2}} = \dfrac{2}{x\sqrt{x}}$ with sign diag:



and $\quad f''(x) = -3x^{-\frac{5}{2}} = -\dfrac{3}{x^2\sqrt{x}}$ with sign diag:



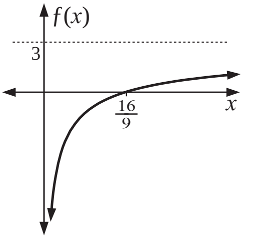
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x > 0$ as $f'(x) > 0$ for all x.

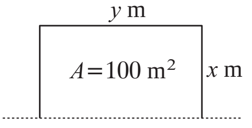
iv $f(x)$ is concave down for all $x > 0$ as $f''(x) < 0$ for all x.

v



EXERCISE 22H

1 **a**



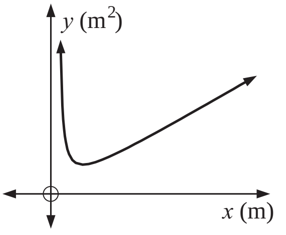
$L = 2x + y$

but $\quad xy = 100$

$\therefore \quad y = \dfrac{100}{x}$

$\therefore \quad L = 2x + \dfrac{100}{x}$

b



c $\dfrac{dL}{dx} = 2 - 100x^{-2} = 2 - \dfrac{100}{x^2}$

which is 0 when $\quad \dfrac{100}{x^2} = 2$

$\therefore \quad x^2 = 50$

$\therefore \quad x = \sqrt{50} \quad \{x > 0\}$

$\dfrac{d^2L}{dt^2} = 200x^{-3} = \dfrac{200}{x^3} > 0$ for $x > 0$

$\therefore$ min $L = 28.28$ m when $x = \sqrt{50}$ m

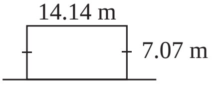
$\therefore \quad L_{\min} = 2\sqrt{50} + \dfrac{100}{\sqrt{50}}$

$= 2\sqrt{50} + 2\sqrt{50}$

$= 4\sqrt{50}$

$= 20\sqrt{2}$ m when $x = 5\sqrt{2}$ m

d



2 **a** Inner length of box $= 2x$ cm

b Volume $= 200$ cm^3

$\therefore \quad x \times 2x \times h = 200$

$2x^2h = 200$

$\therefore \quad x^2h = 100 \quad \text{......} \ (1)$

c From (1) $\quad h = \dfrac{100}{x^2}$.

Now area of inner surface is

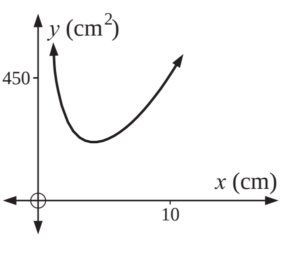
$A(x) = 2(2x \times x) + 2(2x \times h) + 2(x \times h)$

$= 4x^2 + 4xh + 2xh$

$= 4x^2 + 6xh$

i.e., $\quad A(x) = 4x^2 + \dfrac{600}{x}$ cm^2

d



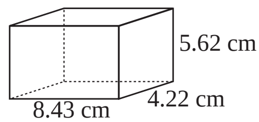
e
$$\begin{aligned} A(x) &= 4x^2 + 600x^{-1} \\ \therefore\ A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \\ \therefore\ A'(x) &= 0 \quad \text{when} \\ 8x &= \frac{600}{x^2} \\ 8x^3 &= 600 \\ x^3 &= 75 \\ x &= \sqrt[3]{75} \\ x &\doteqdot 4.127 \text{ cm} \end{aligned}$$

$$\begin{aligned} A''(x) &= 8 + 1200x^{-3} \\ &= 8 + \frac{1200}{x^3} \\ \therefore\ A''(x) &> 0 \qquad \{\text{as } x > 0\} \end{aligned}$$

$\therefore$ minimum when $x \doteqdot 4.22$ cm

$$\begin{aligned} \therefore\ A_{\min} &= 4(4.217)^2 + \frac{600}{(4.217)} \\ &\doteqdot 213.41 \text{ cm}^2 \end{aligned}$$

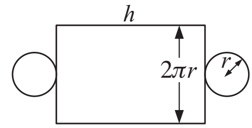
f



3 **a** Volume of can $= \pi r^2 h$

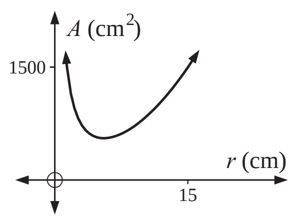
$$\begin{aligned} \therefore\ 1000 &= \pi r^2 h \quad \text{(in cm)} \\ \therefore\ h &= \frac{1000}{\pi r^2} \text{ cm} \end{aligned}$$

b Opening the can up we get



$$\begin{aligned} \therefore\ A(r) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c



d
$$\begin{aligned} A(r) &= 2\pi r^2 + 2000r^{-1} \\ A'(r) &= 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2} \end{aligned}$$

So, $A'(r) = 0$ when
$$\begin{aligned} 4\pi r &= \frac{2000}{r^2} \\ r^3 &= \frac{2000}{4\pi} \\ r &= \sqrt[3]{\frac{500}{\pi}} \\ \therefore\ r &\doteqdot 5.419 \text{ cm} \end{aligned}$$

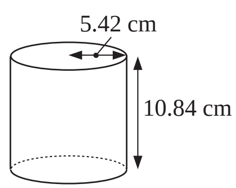
$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

and as $r > 0$, $A''(r) > 0$

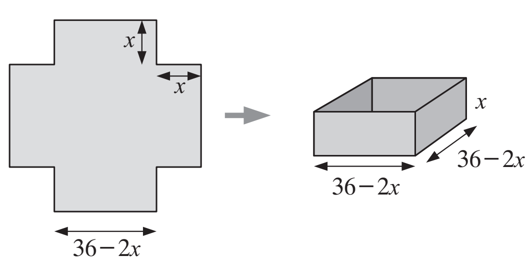
$\therefore$ area is a minimum when $r \doteqdot 5.42$ cm

and $h = \dfrac{1000}{\pi r^2} \doteqdot 10.84$ cm

e



4



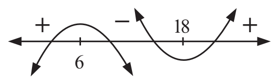
Now volume of container is
$$\begin{aligned} V &= lbd \\ &= x(36-2x)(36-2x) \\ \therefore\quad V &= x(36-2x)^2 \text{ cm}^3 \end{aligned}$$

let $u = x,\quad u' = 1,\quad v = (36-2x)^2,\quad v' = 2(36-2x)(-2)$

$$\begin{aligned} \therefore\quad V'(x) &= (36-2x)^2 - 4x(36-2x) \\ &= (36-2x)[(36-2x)-4x] \\ &= (36-2x)(36-6x) \end{aligned}$$

$\therefore\quad V'(x) = 0$ when $x = 6$ or $x = 18$

Sign diagram of $V'(x)$ is:



$\therefore$ volume is maximised when $x = 6$ cm

$\therefore$ cut out 6 cm × 6 cm squares.

5 **a**
$$\begin{aligned} P &= 2\pi r + 2l \\ \therefore\quad 400 &= 2\pi(x) + 2l \\ \therefore\quad 200 &= \pi x + l \\ \therefore\quad l &= 200 - \pi x \end{aligned}$$

Now $0 \leqslant l \leqslant 200$

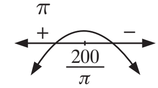
and $l > 0$ means that
$$\begin{aligned} \pi x &< 200 \\ \therefore\quad x &< \frac{200}{\pi} \end{aligned}$$
and $x > 0$
$$\therefore\quad 0 \leqslant x \leqslant \frac{200}{\pi}$$

b Area,
$$\begin{aligned} A &= \pi r^2 + (2x) \times l \\ &= \pi x^2 + 2xl \\ &= \pi x^2 + 2x(200 - \pi x) \\ &= \pi x^2 + 400x - 2\pi x^2 \\ \therefore\quad A &= 400x - \pi x^2 \end{aligned}$$

c Now $\dfrac{dA}{dx} = 400 - 2\pi x$

which is 0 when $2\pi x = 400$
$$x = \frac{200}{\pi} \quad\text{and}\quad l = 0$$

Sign diagram of $A'(x)$:



$\therefore$ area will be a maximum when the track is a circle.

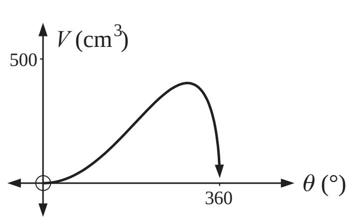
6 **a**
$$\begin{aligned} \text{Arc AC} &= \frac{\theta}{360} \times (2\pi r) \\ &= \frac{\theta}{360}(2 \times \pi \times 10) \\ \text{Arc AC} &= \frac{\pi\theta}{18} \end{aligned}$$

b Now arc AC forms the base of the cone.
$$\begin{aligned} \therefore\quad 2\pi r &= \frac{\theta}{360} \times 2\pi \times 10 \\ \therefore\quad r &= \frac{\theta}{36} \end{aligned}$$

c Height of cone $= \sqrt{10^2 - r^2}$ {Pythagoras}
$$\therefore\quad h = \sqrt{100 - \left(\tfrac{\theta}{36}\right)^2}$$

d
$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ V &= \tfrac{1}{3}\pi\left(\tfrac{\theta}{36}\right)^2 \sqrt{100 - \left(\frac{\theta}{36}\right)^2} \\ &= \frac{\pi\theta^2}{3 \times 36^2}\sqrt{\frac{129\,600 - \theta^2}{36^2}} \\ &= \frac{\pi\theta^2}{139\,968}\sqrt{129\,600 - \theta^2} \end{aligned}$$

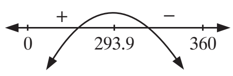
e



f Now
$$\begin{aligned} V'(\theta) &= \frac{2\pi\theta}{139\,968}\left(129\,600-\theta^2\right)^{\frac{1}{2}} + \frac{\pi\theta^2}{139\,968}\left(\tfrac{1}{2}\right)(129\,600-\theta^2)^{-\frac{1}{2}}(-2\theta) \\ &= \frac{\pi\theta}{139\,968}\left(\frac{2\sqrt{129\,600-\theta^2}}{1}) - \frac{\theta^2}{\sqrt{129\,600-\theta^2}}\right) \\ &= \frac{\pi\theta}{139\,968}\left(\frac{2(129\,600-\theta^2)-\theta^2}{\sqrt{129\,600-\theta^2}}\right)\end{aligned}$$

and $V'(\theta) = 0$ when $\theta = 0$ or
$$\begin{aligned} 2(129\,600-\theta^2) &= \theta^2 \\ 259\,200 - 2\theta^2 &= \theta^2 \\ \therefore\quad 3\theta^2 &= 259\,200 \\ \therefore\quad \theta &= \sqrt{86\,400} \quad \{\text{as } \theta > 0\} \\ \therefore\quad \theta &\doteqdot 293.9 \end{aligned}$$

Sign diagram of $V'(\theta)$ is:



$\therefore$ maximum V occurs when $\theta = 293.9^o$

7 **a** Volume of box is
$$\begin{aligned} V &= lbd \\ &= x\times x\times y \\ &= x^2y \end{aligned}$$

But $V = 1\text{ m}^2$
$$\begin{aligned} \therefore\quad 1 &= x^2y \\ y &= \frac{1}{x^2} \end{aligned}$$

b Area of base is x^2 m^2 with cost \$25/m^2 and area of sides is $4xy$ m^2 with cost \$50/m^2.

$\therefore$ total cost is
$$\begin{aligned} C(x) &= 25x^2 + 50(4xy) \\ &= 25x^2 + 200xy \\ &= 25x^2 + 200x\left(\frac{1}{x^2}\right) \\ &= 25x^2 + \frac{200}{x} \\ C(x) &= 25(x^2+8x^{-1}) \text{ dollars} \end{aligned}$$

c $C'(x) = 50x + (-200x^{-2}) = 50x - \dfrac{200}{x^2}$

which is 0 when $50x = \dfrac{200}{x^2}$ i.e., $x^3 = 4$, $\therefore$ $x = \sqrt[3]{4} \doteqdot 1.59$ m

$C''(x) = 50 + 400x^{-3} = 50 + \dfrac{400}{x^3}$ $\therefore$ $C''(x) > 0$ as $x > 0$

$\therefore$ minimum cost occurs when $x \doteqdot 1.59$ and $y \doteqdot \dfrac{1}{(1.59)^2}$

$\therefore$ dimensions are $1.59 \times 1.59 \times 0.397$ m

8 **a** If OB $= x$, then AB $= 2x$

C has coordinates $(x, \dfrac{100}{x^2})$

$\therefore$ BC $= \dfrac{100}{x^2}$

$\therefore$ ABCD is $2x \times \dfrac{100}{x^2}$

b Area ABCD $= 2x \times \dfrac{100}{x^2}$

i.e., $A = \dfrac{200}{x} = 200x^{-1}$

and
$$\begin{aligned} \frac{dA}{dx} &= -200x^{-2} \\ &< 0 \text{ for } x > 0 \end{aligned}$$

$\therefore$ as x increases, area decreases for all $x > 0$.

c Perimeter
$$\begin{aligned} P &= 2(\text{AB}) + 2(\text{BC}) \\ &= 2(2x) + 2\left(\frac{100}{x^2}\right) \\ \therefore\quad P(x) &= 4x + 200x^{-2} \\ P'(x) &= 4 - 400x^{-3} = 4 - \frac{400}{x^3} \end{aligned}$$

which is 0 when
$$\begin{aligned} 4x^3 &= 400 \\ x &= \sqrt[3]{100} \\ x &\doteqdot 4.64 \end{aligned}$$

Now
$$\begin{aligned} P''(x) &= 1200x^{-4} \\ &= \frac{1200}{x^4} \\ &> 0 \quad \text{when} \quad x > 0 \end{aligned}$$

$\therefore$ minimum perimeter occurs when $x = 4.6$

$\therefore$ dimensions of rectangle are 4.64×9.28

9 Let the circle have a radius of x cm. $\therefore$ area of circle $= \pi x^2$
and circumference of circle $= 2\pi x$

$\therefore$ $(24 - 2\pi x)$ cm is used to form the square

$$\therefore \text{ side of the square } = \frac{24 - 2\pi x}{4} = \left(6 - \frac{\pi x}{2}\right) \text{ cm}$$

$$\therefore \text{ area of the square } = \left(6 - \frac{\pi x}{2}\right)^2$$

$$\begin{aligned}\therefore \text{ total area, } A &= \text{area of square} + \text{area of circle}\\ &= \left(6 - \frac{\pi x}{2}\right)^2 + \pi x^2\\ &= 36 - 6\pi x + \frac{\pi^2 x^2}{4} + \pi x^2\end{aligned}$$

$$\therefore \quad \frac{dA}{dx} = -6\pi + \frac{2\pi^2 x}{4} + 2\pi x$$

which is 0 when $\dfrac{2\pi^2 x}{4} + 2\pi x = 6\pi \quad \therefore \quad \dfrac{\pi x}{2} + 2x = 6$

$$\begin{aligned}\therefore \quad \pi x + 4x &= 12\\ \therefore \quad x(\pi + 4) &= 12\\ \therefore \quad x &= \frac{12}{\pi + 4} \text{ cm}\end{aligned}$$

and $\dfrac{d^2A}{dx^2} = \dfrac{2\pi^2}{4} + 2\pi$ which is > 0

$\therefore$ minimum occurs when $x = \dfrac{12}{\pi + 4}$ cm $\doteqdot 1.68$ cm

i.e., when the circumference of the circle $= 2\pi \left[\dfrac{12}{\pi + 4}\right]$ cm $= \dfrac{24\pi}{\pi + 4}$ cm $\doteqdot 10.56$ cm

$\therefore$ make the cut 10.56 cm from RH end or 13.44 cm from LH end.

10 **a** X must lie between A and C (or at A or C).
If $x = 0$, then he rows straight to the shore and runs to C.
If $x = 6$, then he rows straight to C. $\therefore \quad 0 \leqslant x \leqslant 6$

b Now $\text{XC} = 6 - x$

$\therefore$ time to row from B to X

$$= \frac{\text{BX}}{8} = \frac{\sqrt{5^2 + x}}{8}$$

and time to run from X to C

$$= \frac{\text{XC}}{17} = \frac{6 - x}{17}$$

$$\begin{aligned}\therefore \text{ total time } T(x) &= \frac{\sqrt{25 + x^2}}{8} + \frac{6 - x}{17} \text{ hours}\\ &= \tfrac{1}{8}(25 + x^2)^{\frac{1}{2}} + \tfrac{6}{17} - \frac{x}{17}\end{aligned}$$

c Now $\dfrac{dT}{dx} = \tfrac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \tfrac{1}{17}$

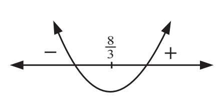
$$\therefore \quad \frac{dT}{dx} = \frac{x}{8\sqrt{25 + x^2}} - \frac{1}{17}$$

$\therefore \quad \dfrac{dT}{dx} = 0$

when $\dfrac{x}{8\sqrt{25 + x^2}} = \tfrac{1}{17}$

$$\begin{aligned}\therefore \quad 17x &= 8\sqrt{25 + x^2}\\ \therefore \quad 289x^2 &= 64(25 + x^2)\\ \therefore \quad 289x^2 &= 1600 + 64x^2\\ \therefore \quad 225x^2 &= 1600\\ \therefore \quad x^2 &= \frac{1600}{225}\\ \therefore \quad x &= \tfrac{40}{15} = \tfrac{8}{3} \text{ km}\end{aligned}$$

Sign diagram of $\dfrac{dT}{dx}$:



Thus, the time taken is a minimum if Peter aims for X such that $x = \frac{8}{3}$ km.

11

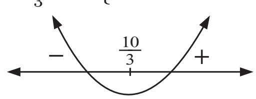
Let MX $= x$ km, then XN $= 5 - x$ km

$\therefore$ AX $= \sqrt{4+x^2}$ km and XB $= \sqrt{1+(5-x)^2}$ km {Pythagoras}

$$\begin{aligned}
\text{Now} \quad P &= \text{AX} + \text{XB} \\
P &= (4+x^2)^{\frac{1}{2}} + (26-10x+x^2)^{\frac{1}{2}} \\
\therefore \quad \frac{dP}{dx} &= \tfrac{1}{2}(4+x^2)^{-\frac{1}{2}}(2x) \ + \ \tfrac{1}{2}(26-10x+x^2)^{-\frac{1}{2}}(2x-10) \\
&= \frac{x}{\sqrt{4+x^2}} + \frac{x-5}{\sqrt{x^2-10x+26}}
\end{aligned}$$

$$\begin{aligned}
\text{Thus,} \quad \frac{dP}{dx} &= 0 \quad \text{when} \\
\frac{x}{\sqrt{4+x^2}} &= \frac{5-x}{\sqrt{x^2-10x+26}} \\
\therefore \quad \frac{x^2}{4+x^2} &= \frac{(5-x)^2}{x^2-10x+26} \qquad \{\text{squaring both sides}\} \\
\therefore \quad x^2(x^2-10x+26) &= (4+x^2)(25-10x+x^2) \\
\therefore \quad x^4 - 10x^3 + 26x^2 &= 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4 \\
\therefore \quad 3x^2 - 40x + 100 &= 0 \\
\therefore \quad (3x-10)(x-10) &= 0 \\
\therefore \quad x &= \tfrac{10}{3} \qquad \{\text{as } x \text{ cannot be } 10\}
\end{aligned}$$

Sign diagram of $\frac{dP}{dx}$ is:



$\therefore$ minimum length pipeline occurs when $x = \frac{10}{3}$ km

12

$$\begin{aligned}
V &= \pi r^2 h \\
\therefore \quad 0.1 &= \pi r^2 h \qquad \{\text{as } 100 \text{ L} = 0.1 \text{ m}^3\} \\
\therefore \quad h &= \frac{0.1}{\pi r^2}
\end{aligned}$$

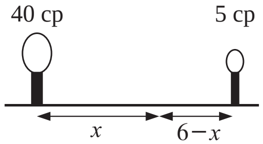
$$\begin{aligned}
\text{Now} \quad A &= \pi r^2 + (2\pi r)h = \pi r^2 + 2\pi r\left(\frac{0.1}{\pi r^2}\right) \\
\text{i.e.,} \quad A(r) &= \pi r^2 + 0.2r^{-1} \\
\therefore \quad A'(r) &= 2\pi r - 0.2r^{-2} = 2\pi r - \frac{0.2}{r^2} \\
\therefore \quad A'(r) &= 0 \quad \text{when} \quad 2\pi r = \frac{0.2}{r^2} \\
r^3 &= \frac{0.2}{2\pi} \\
r &= \sqrt[3]{\tfrac{0.2}{2\pi}} \doteqdot 0.3169 \text{ m} \\
r &\doteqdot 31.7 \text{ cm}
\end{aligned}$$

Now $A''(r) = 2\pi + 0.4r^{-3} = 2\pi + \dfrac{0.4}{r^3}$ which is > 0 as $r > 0$

$\therefore$ minimum area occurs when $r \doteqdot 31.7$ cm and $h \doteqdot \dfrac{1}{10\pi(31.69)} \doteqdot 31.7$ cm

$\therefore \quad r = h \doteqdot 31.7$ cm

13



$I \propto \dfrac{s}{d^2}$ where s is the power of the source and d is the distance from it

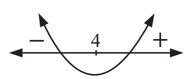
$\therefore\ I = \dfrac{kS}{d^2}$ $\{k$ is a constant$\}$

$$\therefore \text{ intensity due to 40 cp } = \frac{40k}{x^2}$$

$$\text{and intensity due to 5 cp } = \frac{5k}{(6-x)^2}$$

$$\begin{aligned}
\therefore \text{ total intensity, } I &= \frac{40k}{x^2} + \frac{5k}{(6-x)^2} \\
&= k[40x^{-2} + 5(6-x)^{-2}] \\
\text{Now } \frac{dI}{dx} &= k[-80x^{-3} - 10(6-x)^{-3}(-1)] \\
&= k\left[\frac{-80}{x^3} + \frac{10}{(6-x)^3}\right] \\
\therefore \frac{dI}{dx} &= 0 \quad \text{when} \quad \frac{80}{x^3} = \frac{10}{(6-x)^3} \\
\text{i.e., } 8(6-x)^3 &= x^3 \\
\therefore 2(6-x) &= x \qquad \{\text{finding cube roots}\} \\
\therefore 12 - 2x &= x \\
\therefore 12 &= 3x \\
\therefore x &= 4
\end{aligned}$$

Sign diagram of $\dfrac{dI}{dx}$ is:



$\therefore$ the minimum intensity, i.e., the darkest point occurs when $x = 4$ m i.e., at 4 m from the 40 cp lamp.

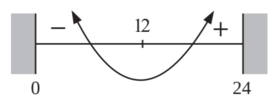
14 **a**

$$\begin{aligned}
AB &= x \text{ m} \\
\therefore BC &= (24 - x) \text{ m} \\
\therefore AC^2 &= AB^2 + BC^2 \quad \{\text{Pythagoras}\} \\
&= x^2 + (24-x)^2 \\
&= x^2 + 576 - 48x + x^2 \\
&= 2x^2 - 48x + 576
\end{aligned}$$

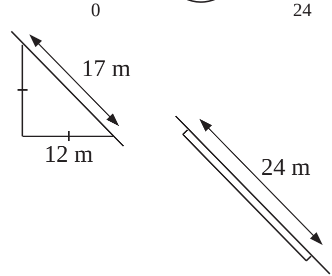
i.e., $[D(x)]^2 = 2x^2 - 48x + 576$ and so $D(x) = \sqrt{2x^2 - 48x + 576}$

b $\therefore \dfrac{d[D(x)]^2}{dx} = 4x - 48$

$\therefore \dfrac{d[D(x)]^2}{dx} = 0$ when $x = 12$ and the sign diagram is:

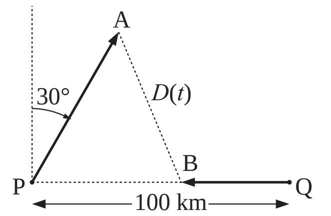


c i.e., when AB = BC = 12 m, $D(x)$ is a minimum, and minimum $D(x) = 12\sqrt{2}$ m $\doteqdot$ 16.97 m.



$D(x)$ is a maximum when either $x = 0$ or $x = 24$, i.e., when the pen ceases to exist and $D(x) = 24$ m.

15 **a** Consider each boat's position t hours after 1.00 pm.



$$AP = 12t \qquad BQ = 8t$$
$$\therefore \quad PB = 100 - 8t$$

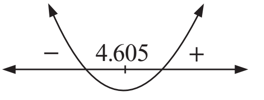
Using the Cosine rule in ΔPAB

$$\begin{aligned} D(t)^2 &= AP^2 + BP^2 - 2AP \times BP \cos 60^o \\ &= (12t)^2 + (100-8t)^2 - 2(12t)(100-8t)\tfrac{1}{2} \\ &= 144t^2 + (100-8t)^2 - 12t(100-8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000 \quad \text{and so} \quad D(t) = \sqrt{304t^2 - 2800t + 10\,000} \end{aligned}$$

b Now $\dfrac{d[D(t)]^2}{dt} = 608t - 2800$

$\therefore \quad \dfrac{d[D(t)]^2}{dt} = 0 \quad$ when $\quad t = \dfrac{2800}{608} \doteqdot 4.605\,26$

and has sign diagram:



$\therefore \quad D(t)$ is a minimum when $t \doteqdot 4.60526$ hours after 1.00 pm

and $\quad [D(t)]^2_{\text{min}} \doteqdot 304\,(4.6053)^2 - 2800\,(4.6053) + 10\,000$

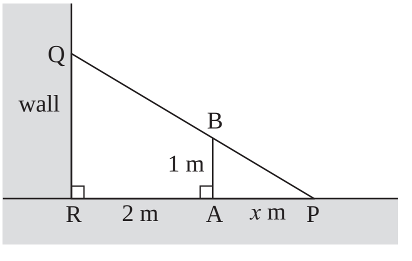
$\therefore \quad [D(t)]^2_{\text{min}} = 3552.63 \text{ km}^2$

c The ships are closest when $\quad t = 4.60526$ hours

i.e., when the time is 4 hours 36 minutes

$\therefore$ time is approximately 5.36 pm

16 **a** Δs PAB and PRQ are similar.



$$\therefore \quad \frac{PA}{PR} = \frac{PB}{PQ} = \frac{AB}{RQ}$$

$$\therefore \quad \frac{x}{x+2} = \frac{1}{QR} \quad \text{and} \quad \therefore \quad QR = \frac{x+2}{x}$$

b Now
$$\begin{aligned} [L(x)]^2 &= RP^2 + QR^2 \quad \{\text{Pythagoras}\} \\ &= (x+2)^2 + \left(\frac{x+2}{x}\right)^2 \\ &= (x+2)^2 \times 1 \; + \; (x+2)^2 \times \frac{1}{x^2} \end{aligned}$$

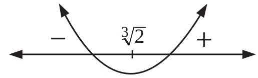
$$\therefore \quad [L(x)]^2 = (x+2)^2 \left[1 + \frac{1}{x^2}\right] \quad \text{as required}$$

$$\therefore \quad [L(x)]^2 = (x+2)^2 \left[1 + x^{-2}\right]$$

c
$$\begin{aligned} \frac{d[L(x)]^2}{dx} &= 2(x+2)\left[1+x^{-2}\right] \; + \; (x+2)^2\left[-2x^{-3}\right] \quad \{\text{product rule}\} \\ &= 2(x+2)\left[1 + x^{-2} - (x+2)x^{-3}\right] \\ &= 2(x+2)[1 + x^{-2} - x^{-2} - 2x^{-3}] \\ &= 2(x+2)(1 - \frac{2}{x^3}) \\ &= 2(x+2)(\frac{x^3-2}{x^3}) \end{aligned}$$

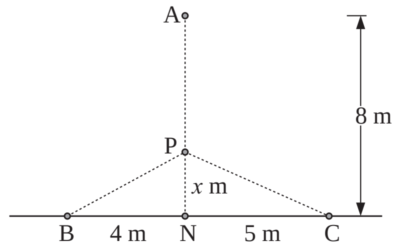
$$\therefore \quad \frac{d[L(x)]^2}{dx} = 0 \quad \text{when} \quad x = \sqrt[3]{2} \doteqdot 1.2599 \qquad \{\text{as } x > 0 \text{ and } L(x) > 0\}$$

d Sign diagram of $L'(x)$ is:



$\therefore$ the ladder is shortest when $x = \sqrt[3]{2}$ m and minimum $L \doteqdot \sqrt{(x+2)^2(1+\frac{1}{x^2})} \doteqdot 4.16$ m

17 Suppose $\text{PN} = x$ m



$\therefore$ length of cable $= \text{PA} + \text{PB} + \text{PC}$

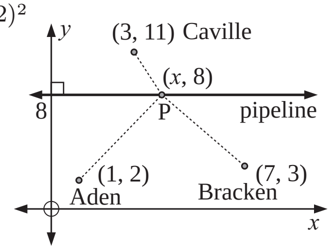
$$\therefore \quad L = 8 - x + \sqrt{x^2+16} + \sqrt{x^2+25}$$

Using technology we graph this and find the minimum,

i.e., the minimum length occurs when $x = 2.577\,98$

i.e., $x \doteqdot 2.578$ m from N

18 Suppose P has coordinates $(x, 8)$



$$\begin{aligned}\therefore \quad \text{PC} &= \sqrt{(x-3)^2+(8-11)^2} \\ &= \sqrt{x^2-6x+9+9} \\ &= \sqrt{x^2-6x+18}\end{aligned} \qquad \begin{aligned}\text{AP} &= \sqrt{(x-1)^2+(8-2)^2} \\ &= \sqrt{x^2-2x+1+36} \\ &= \sqrt{x^2-2x+37}\end{aligned}$$

$$\begin{aligned}\text{PB} &= \sqrt{(x-7)^2+(8-3)^2} \\ &= \sqrt{x^2-14x+49+25} \\ &= \sqrt{x^2-14x+74}\end{aligned}$$

$\therefore$ length of pipeline $L = \sqrt{x^2-6x+18} + \sqrt{x^2-2x+3} + \sqrt{x^2-14x+74}$

Use technology to graph L and find a minimum. This occurs when $x \doteqdot 3.543\,66$,

i.e., P is at $(3.544, 8)$.

19 $y = mx + c$ is the equation of any straight line.

If this line is to pass through $(2a, a)$ then

$$\begin{aligned} a &= m(2a) + c \\ a &= 2am + c \\ \therefore \quad c &= a - 2am \\ \therefore \quad y &= mx + (a - 2am) \quad \text{...... (1)} \end{aligned}$$

This line cuts the x-axis when $y = 0$,

$$\begin{aligned} \therefore \quad 0 &= mx + a - 2am \\ \therefore \quad mx &= 2am - a \\ \therefore \quad x &= \frac{2am - a}{m} \end{aligned}$$

This line cuts the y-axis when $x = 0$,

$$\begin{aligned} \therefore \quad y &= m \times 0 + (a - 2am) \\ \text{i.e.,} \quad y &= a - 2am \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{area of } \Delta\text{AOB}, \quad A(m) &= \tfrac{1}{2}\left(\frac{2am-a}{m}\right)(a-2am) \\ &= \frac{a^2}{2m}(2m-1)(1-2m) \\ &= \frac{a^2}{2}\left(\frac{4m-1-4m^2}{m}\right) \\ \therefore \quad A(m) &= \frac{a^2}{2}\left(4 - m^{-1} - 4m\right) \end{aligned}$$

Now $A'(m) = \dfrac{a^2}{2}(m^{-2} - 4)$ which is 0 when $m^{-2} = 4$ i.e., $m = \pm\frac{1}{2}$

Now since the graphs are in Q_1 the slopes are negative and so $m = -\frac{1}{2}$

$$\text{Now} \quad A''(m) = \frac{a^2}{2} \times -2m^{-3} = \frac{-a^2}{m^3}$$

$$\text{and} \quad A''(-\tfrac{1}{2}) = \frac{-a^2}{-\frac{1}{8}} = 8a^2 \quad \text{which is} > 0$$

$\therefore$ minimum area occurs when $m = -\frac{1}{2}$

$$\text{Using (1)} \quad y = -\tfrac{1}{2}x + \left(a - 2a(-\tfrac{1}{2})\right)$$

$$y = -\tfrac{1}{2}x + 2a$$

Thus the graph meets the x-axis when $y = 0$, $\quad \therefore \; 0 = -\frac{1}{2}x + 2a$

$$\therefore \; \tfrac{1}{2}x = 2a$$

$$\therefore \; x = 4a \qquad \text{i.e., at } (4a, 0)$$

20

$$r^2 + h^2 = s^2$$

$$\therefore \; h^2 = s^2 - r^2$$

$$h = \sqrt{s^2 - r^2}$$

But $V = \frac{1}{3}\pi r^2 h$

$$\therefore \; V = \tfrac{1}{3}\pi r^2\sqrt{s^2 - r^2}$$

$$\therefore \; V^2 = \frac{\pi^2}{9} r^4(s^2 - r^2)$$

$$= \frac{\pi^2}{9}\left(r^4 s^2 - r^6\right)$$

$$\therefore \; \frac{dV^2}{dr} = \frac{\pi^2}{9}\left(4r^3 s^2 - 6r^5\right)$$

$$= \frac{\pi^2}{9}\, 2r^3(2s^2 - 3r^2)$$

$\frac{dV^2}{dr}$ is 0 when

$$2s^2 - 3r^2 = 0$$

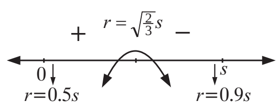
$$\therefore \; 2s^2 = 3r^2 \qquad \{\text{as } r > 0\}$$

$$\therefore \; \frac{s^2}{r^2} = \tfrac{3}{2}$$

$$\frac{s}{r} = \sqrt{\tfrac{3}{2}}$$

i.e., $s : r = \sqrt{\frac{3}{2}} : 1$

Sign diagram of $\dfrac{dV^2}{dr}$ is:



$\therefore$ V is a maximum when $s : r = \sqrt{\frac{3}{2}} : 1 = \sqrt{3} : \sqrt{2}$

21 **a**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \; x^2b^2 + y^2a^2 = a^2b^2 \quad \{\times \text{ by } a^2b^2\}$$

$$\therefore \; a^2y^2 = a^2b^2 - x^2b^2$$

$$\therefore \; y^2 = \frac{a^2b^2 - x^2b^2}{a^2}$$

$$\therefore \; y = \pm\sqrt{b^2 - \frac{b^2}{a^2}x^2}$$

Since A lies in Q_1, $y > 0$

$$\therefore \; y = \sqrt{b^2 - \frac{b^2}{a^2}x^2}$$

$$\therefore \; y = \frac{b}{a}\sqrt{a^2 - x^2}$$

b Seating area is

$$A = 2x \times 2y$$

$$= 4xy$$

$$= 4x\left[\frac{b}{a}\sqrt{a^2 - x^2}\right]$$

$$\therefore \; A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}$$

as required

$$\therefore \; A^2 = \frac{16b^2x^2}{a^2}(a^2 - x^2)$$

$$= \frac{16b^2}{a^2}(a^2x^2 - x^4)$$

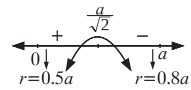
c $\dfrac{dA^2}{dx} = \dfrac{16b^2}{a^2}[2a^2x - 4x^3]$

which is 0 when

$$2a^2x - 4x^3 = 0$$
$$\therefore \quad 2x(a^2 - 2x^2) = 0$$
$$\therefore \quad 2x^2 = a^2 \quad \{\text{as } x > 0\}$$
$$\therefore \quad x = \pm\frac{a}{\sqrt{2}}$$
$$\therefore \quad x = \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in Q}_1\}$$

[Note that A is a maximum when A^2 is a maximum.]

d Sign diagram of $\dfrac{dA^2}{dx}$ is:



$\therefore$ maximum area occurs when $x = \dfrac{a}{\sqrt{2}}$

$$\text{Max. area} = \frac{4b}{a}\frac{a}{\sqrt{2}}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$
$$= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}}$$
$$= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}}$$
$$= 2ab$$

e % occupied $= \dfrac{2ab}{\pi ab} \times 100\% = 63.66\%$

EXERCISE 22I

1 $C(x) = 38\,000 + 250x + x^2$

a
$$\text{Cost} = C(800)$$
$$= 38\,000 + 250(800) + (800)^2$$
$$= \$878\,000$$

$$\text{Average cost} = \frac{\$878\,000}{800}$$
$$= \$1097.50$$

$$\text{Marginal cost} = C'(x)$$
$$= 250 + 2x$$
$$\therefore \quad C'(800) = 250 + 2(800)$$
$$= \$1850$$

b $A(x)$ is minimised when $A(x) = C'(x)$

$$\text{i.e.,} \quad \frac{C(x)}{x} = C'(x)$$
$$C(x) = xC'(x)$$
$$38\,000 + 250x + x^2 = x(250 + 2x)$$
$$38\,000 + 250x + x^2 = 250x + 2x^2$$
$$\therefore \quad 38\,000 = x^2$$
$$\therefore \quad x = \sqrt{38\,000} \quad \{\text{as } x > 0\}$$
$$\therefore \quad x \doteqdot 195$$

i.e., A is minimised when approximately 195 items are produced.

$$\text{Average cost} = \frac{C(x)}{x}$$
$$= \frac{C(195)}{195}$$
$$= \frac{\$124\,775}{195}$$
$$\doteqdot \$639.87$$

2 a
$$C(x) = 295 + 24x - 0.08x^2 + 0.0008x^3$$
$$\text{Average cost,} \quad A(x) = \frac{C(x)}{x} = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$$
$$\text{Marginal cost} = C'(x) = 24 - 0.16x + 0.0024x^2$$

b Average cost is minimised when $C'(x) = A(x)$

$$\therefore \quad 24 - 0.16x + 0.0024x^2 = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$$
$$\therefore \quad 24x - 0.16x^2 + 0.0024x^3 = 295 + 24x - 0.08x^2 + 0.0008x^3$$
$$\therefore \quad 0.0016x^3 - 0.08x^2 - 295 = 0$$

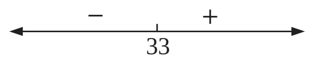
Using technology, $x \doteqdot 79.311$ and so, $x > 79$ items

$$\begin{aligned}\therefore\quad \text{minimum average cost} &= A(79)\\ &= \frac{295}{79} + 24 - 0.08(79) + 0.0008(79)\\ &= \$26.41\end{aligned}$$

c

$$\begin{aligned}C'(x) &= 24 - 0.16x + 0.0024x^2\\ C''(x) &= -0.16 + 0.0048x\\ \text{which is 0 when } 0.0048x &= 0.16\\ \therefore\quad x &= \frac{0.16}{0.0048} \doteqdot 33.33\\ \text{i.e.,}\quad x &= 33 \text{ items}\end{aligned}$$

Sign diagram for $C''(x)$ is:



$\therefore\quad C'(x)$ is a minimum when $x = 33$ items

$\therefore\quad$ minimum marginal cost $= C'(33) = \$21.33$

3 Suppose x fittings are produced daily.

$$\therefore\quad C(x) = 1000 + 2x + \frac{5000}{x} = 1000 + 2x + 5000x^{-1} \text{ dollars}$$

$$\therefore\quad C'(x) = 2 - \frac{5000}{x^2}$$

$$\begin{aligned}C'(x) = 0 \quad\text{when}\quad x^2 &= 2500\\ \text{i.e.,}\quad x &= 50 \qquad \{\text{as } x > 0\}\end{aligned}$$

$$\begin{aligned}C''(x) &= 10\,000x^{-3}\\ &= \frac{10\,000}{x^3}\end{aligned}$$

which is > 0 when $x > 0$

$\therefore$ there is minimum cost when 50 fittings are produced.

4

$$\begin{aligned}C(x) &= 720 + 4x + 0.02x^2 \text{ dollars} \quad\text{and}\quad p(x) = 15 - 0.002x \text{ dollars}\\ \therefore\quad R(x) &= xp(x) = 15x - 0.002x^2 \quad\text{and}\quad R'(x) = 15 - 0.004x\end{aligned}$$

$$\begin{aligned}\text{If}\quad C'(x) &= R'(x) \quad\text{then}\\ 4 + 0.04x &= 15 - 0.004x\\ \therefore\quad 0.044x &= 11\\ \therefore\quad x &= 250\end{aligned}$$

And as $R''(x) = -0.004$ and $C''(x) = 0.04$ i.e., $R''(250) < C''(250)$

then maximum profit is made when 250 items are produced.

5

$$\begin{aligned}C(x) &= \tfrac{1}{4}x^2 + 8x + 200 \quad\text{and}\quad P(x) = 23 - \tfrac{1}{2}x\\ \therefore\quad R(x) &= xP(x) = x(23 - \tfrac{1}{2}x) = 23x - \tfrac{1}{2}x^2 \quad \therefore\quad R'(x) = 23 - x\end{aligned}$$

$$\begin{aligned}\text{If}\quad C'(x) &= R'(x) \quad\text{then}\\ \tfrac{1}{2}x + 8 &= 23 - x\\ \therefore\quad \tfrac{3}{2}x &= 15\\ \therefore\quad x &= 10\end{aligned}$$

And as $R''(x) = -1$ and $C''(x) = \frac{1}{2}$ i.e., $R''(10) < C''(10)$

then maximum profit is made when 10 blankets/day are produced.

6 **a** Let the demand function be $p(x) = A + Bx$

$$\begin{aligned} \text{Now} \quad p(800) &= 150 \\ \therefore \quad A + 800B &= 150 \text{ ...... (1)} \\ \text{and} \quad p(840) &= 145 \\ \therefore \quad A + 840B &= 145 \text{ ...... (2)} \\ (2) - (1) \text{ gives} \quad 40B &= -5 \quad \text{and so} \quad B = -\tfrac{1}{8} \\ \text{and} \quad A &= 150 - 800B = 150 - 800\left(-\tfrac{1}{8}\right) = 250 \end{aligned}$$

i.e., the demand function is $p(x) = 250 - \frac{1}{8}x$

b Revenue $R(x) = xp(x) = x\left(250 - \dfrac{x}{8}\right) = 250x - \dfrac{x^2}{8}$

Now $R'(x) = 250 - \dfrac{2x}{8}$ which is 0 when $\dfrac{x}{4} = 250$ i.e., $x = 1000$

and as $R''(x) = -\frac{1}{4} < 0$, maximum revenue occurs when $x = 1000$

$\therefore$ need to sell at $p(1000) = 250 - \dfrac{1000}{8} = \$125$

and must have a rebate of $\$150 - \$125 = \$25$

c $C(x) = 20\,000 + 30x$ dollars

If $C'(x) = R'(x)$ then $30 = 250 - \dfrac{x}{4}$

$$\therefore \quad \frac{x}{4} = 220$$

$$\therefore \quad x = 880$$

And as $R''(x) = -\frac{1}{4}$ and $C''(x) = 0$ i.e., $R''(880) < C''(880)$

then maximum profit is made when 880 DVD players/week are produced.

$\therefore$ need to sell 880 DVDs and the cost per item $= p(880)$

$$\begin{aligned} &= \$(250 - \tfrac{880}{8}) \\ &= \$250 - \$110 \\ &= \$140.00 \end{aligned}$$

i.e., offer a rebate of $\$150 - \$140 = \$10$.

7 Cost/hour = running costs + other costs $= \dfrac{v^2}{10} + 62.5$

Now $C(v) = \dfrac{\frac{v^2}{10} + 62.5}{v}$ $\quad \{\text{cost/km} = \dfrac{\text{cost/hour}}{\text{km/hour}}\}$

$\therefore \quad C(v) = \dfrac{v}{10} + \dfrac{62.5}{v} = 0.1v + 62.5v^{-1}$

$\therefore \quad C'(v) = 0.1 - 62.5v^{-2}$

which is 0 when $0.1 = \dfrac{62.5}{v^2}$

$$\therefore \quad v^2 = 625$$

$$\therefore \quad v = 25 \quad \{\text{as } v > 0\}$$

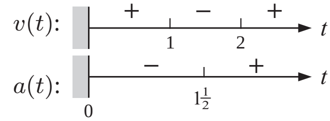
and $C''(x) = 62.5 \times 2v^{-3} = \dfrac{125}{v^3}$ which is > 0 when $v > 0$

$\therefore$ minimum cost/km occurs when $v = 25$ km/h

REVIEW SET 22A

1 **a** $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, $t \geqslant 0$

$$\begin{aligned} v(t) &= 6t^2 - 18t + 12 \\ &= 6(t^2 - 3t + 2) \\ &= 6(t-2)(t-1) \text{ cm s}^{-1} \end{aligned}$$



and $\begin{aligned} a(t) &= 12t - 18 \\ &= 6(2t-3) \text{ cm s}^{-2} \end{aligned}$

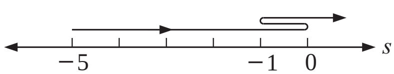
b When $t = 0$, $\quad s(0) = -5$ cm, $v(0) = 12$ cm s^{-1}, $a(0) = -18$ cm s^{-2}

c When $t = 2$, $\quad s(2) = -1$ cm, $v(2) = 0$ cm s^{-1}, $a(2) = 6$ cm s^{-2}

i.e., when $t = 2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d Particle changes direction when $t = 1$ and $t = 2$ $\quad s(1) = 0$ cm $\quad s(2) = -1$ cm

e



f Speed is increasing when $1 \leqslant t \leqslant \frac{3}{2}$ and $t \geqslant 2$
$\{v(t)$ and $a(t)$ have the same sign$\}$

2 $C(v) = 10v + \dfrac{90}{v}$ dollars/hour

a **i** $t = 2$ hours at $v = 15$ kmph

$\therefore$ cost $= \$\left(10 \times 15 + \frac{90}{15}\right) \times 2$
$= \$312.00$

ii $t = 5$ hours at $v = 24$ kmph

$\therefore$ cost $= \$\left(10 \times 24 + \frac{90}{24}\right) \times 5$
$= \$1218.75$

b $C'(v) = 10 - 90v^{-2} = 10 - \dfrac{90}{v^2}$

i If $v = 10$ kmph

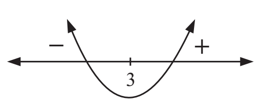
$\therefore$ $C'(10) = 10 - \dfrac{90}{10^2}$
$= 10 - 0.9$
$= \$9.10$ per kmph

ii If $v = 6$ kmph

$\therefore$ $C'(6) = 10 - \dfrac{90}{6^2}$
$= \$7.50$ per kmph

c Now $C'(v) = 10 - \dfrac{90}{v^2} = \dfrac{10v^2 - 90}{v^2}$

$\therefore$ $C'(v) = 0$ when $v^2 = 9$
i.e., $v = 3$ $\{$as $v > 0\}$

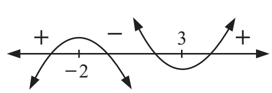


$\therefore$ minimum cost occurs when $v = 3$ kmph

3 **a**

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 36x + 7 \\ \therefore\ f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x-3)(x+2) \end{aligned}$$

and $f'(x)$ has sign diagram:

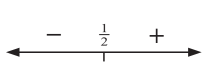


and $f(-2) = 51$, $f(3) = -74$

So there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$

$$\begin{aligned} f''(x) &= 12x - 6 \\ &= 6(2x - 1) \end{aligned}$$

and $f''(x)$ has sign diagram:

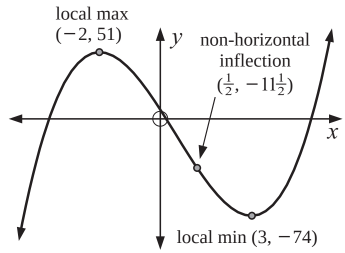


and $f(\frac{1}{2}) = -\frac{23}{2}$ so there is a non-horizontal inflection at $(\frac{1}{2}, -\frac{23}{2})$

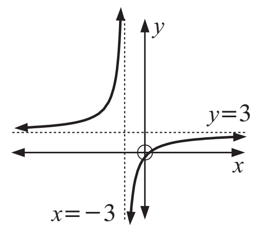
b $f(x)$ is increasing when $x \leqslant -2$ or $x \geqslant 3$, and decreasing when $-2 \leqslant x \leqslant 3$.

c $f(x)$ is concave up when $x \geqslant \frac{1}{2}$, and concave down when $x \leqslant \frac{1}{2}$.

d



4 $f(x) = \dfrac{3x-2}{x+3}$

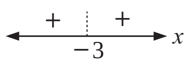


a The vertical asymptote is $x = -3$ $\{$when $x + 3 = 0\}$

b Cuts x-axis at $(\frac{2}{3}, 0)$, y-axis at $(0, -\frac{2}{3})$

c $f'(x) = \dfrac{3(x+3)-(3x-2)(1)}{(x+3)^2} = \dfrac{11}{(x+3)^2}$

which has sign diagram:



d No stationary points exist since $f'(x)$ is never 0.

5 **a** Now if OD $= x$, the coordinates of C are $(x, k - x^2)$

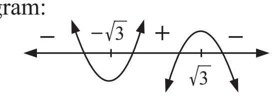
$\therefore$ area of ABCD $= 2x \times (k - x^2)$

i.e., $A = 2kx - 2x^3$

b $\dfrac{dA}{dx} = 2k - 6x^2$ and $\dfrac{dA}{dx} = 0$ when AD $= 2\sqrt{3}$ i.e., $x = \sqrt{3}$

$$\begin{aligned}\therefore\ 2k - 6(\sqrt{3})^2 &= 0\\ \therefore\ 2k - 18 &= 0\\ \therefore\ 2k &= 18\\ \therefore\ k &= 9\end{aligned}$$

Checking $\dfrac{dA}{dx} = 18 - 6x^2$
$= 6(3 - x^2)$ which has sign diagram:
$= 6(\sqrt{3}+x)(\sqrt{3}-x)$



i.e., maximum occurs at $x = \sqrt{3}$ i.e., when AD $= 2\sqrt{3}$

6 $f(x) = x^3 + x^2 + 2x - 4$

a y-intercept is -4

b x-intercepts occur when $x^3 + x^2 + 2x - 4 = 0$

$\therefore$ $(x-1)(x^2+2x+4) = 0$ given $x = 1$ is one root

Now $x^2 + 2x + 4 = 0$ has $\Delta < 0$ $\therefore$ the only x-intercept is 1.

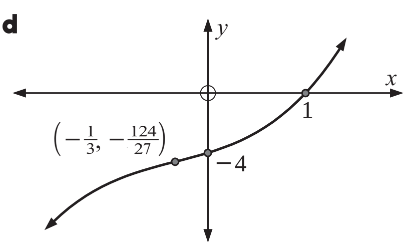
c $f'(x) = 3x^2 + 2x + 2$ has $\Delta = 4 - 24 = -20$ which is < 0, $\therefore$ $f'(x)$ is never 0, so there are no stationary points

$f''(x) = 6x + 2$

$\therefore$ $f''(x) = 0$ when $x = -\frac{1}{3}$

$\therefore$ a non-horizontal inflection occurs at $\left(-\frac{1}{3}, -\frac{124}{27}\right)$

d



7 **a** $A = 500 \text{ m}^2$

$\therefore\ xy = 500$

$\therefore\ y = \dfrac{500}{x}, \quad x > 0$

b $y = 500x^{-1}$

$\therefore\ \dfrac{dy}{dx} = -500x^{-2} = -\dfrac{500}{x^2}$

and since $x^2 \geqslant 0$ for all x,

$\dfrac{dy}{dx} < 0$ for all x

c As the width of the rectangle increases, the length must decrease to maintain a constant area.

8 $H(t) = 6\left(1 - \dfrac{2}{t+3}\right) = 6 - 12(t+3)^{-1}$ m, $\quad t > 0$

a When $t = 0$, $\quad H(0) = 6\left(1 - \frac{2}{3}\right) = 6\left(\frac{1}{3}\right) = 2$ m

b $H(3) = 6(1 - \frac{2}{6}) = 6(\frac{2}{3}) = 4$ m

$H(6) = 6(1 - \frac{2}{9}) = 6(\frac{7}{9}) = \frac{14}{3}$ m

$H(9) = 6(1 - \frac{2}{12}) = 6(\frac{5}{6}) = 5$ m

c $H'(t) = \dfrac{12}{(t+3)^2}$

$\therefore\ H'(0) = \frac{12}{9} = \frac{4}{3}$ m yr^{-1} $\qquad H'(6) = \frac{12}{81} = \frac{4}{27}$ m yr^{-1}

$H'(3) = \frac{12}{36} = \frac{1}{3}$ m yr^{-1} $\qquad H'(9) = \frac{12}{144} = \frac{1}{12}$ m yr^{-1}

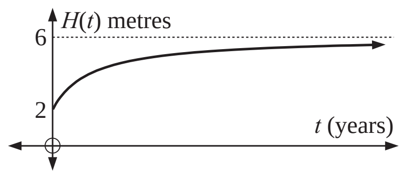
d Since $H'(t) = \dfrac{12}{(t+3)^2}$

$H'(t) > 0$ for all $t \geqslant 0$

i.e., $H(t)$ continually increases.

i.e., the tree is continually growing.

e



9 **a** Volume $= lbd$

$\therefore\ x^2 y = 1$

$\therefore\ y = \dfrac{1}{x^2}, \quad x > 0$

b area $= x^2 + 4xy$

$\therefore$ cost $= (x^2 + 4xy) \times 2 = 2x^2 + 8xy$

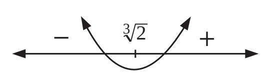
But $y = \dfrac{1}{x^2}$

$\therefore\ C = 2x^2 + \dfrac{8}{x}$ dollars

c Thus $\dfrac{dC}{dx} = 4x - 8x^{-2} = 4x - \dfrac{8}{x^2} = \dfrac{4(x^3 - 2)}{x^2}$

which is 0 when $x = \sqrt[3]{2}$ m

and $\dfrac{dC}{dx}$ has sign diagram:



$\therefore$ minimum cost when $x = \sqrt[3]{2}$ m i.e., $x = 1.26$

$\therefore\ y = \dfrac{1}{x^2} \doteqdot 0.630$

and the box is 1.26 m by 1.26 m by 0.630 m

10 **a** AC $= 2x$ m

Now, ABC is an isosceles triangle.

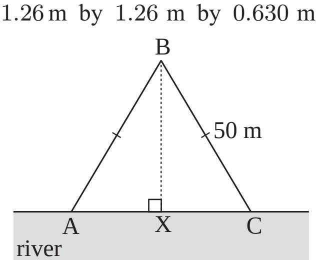
$\therefore$ XC $= x$

But, BC^2 = BX^2 + XC^2 {Pythagoras}

$\therefore\ 2500 = \text{BX}^2 + x^2$

$\therefore\ \text{BX} = \sqrt{2500 - x^2}$

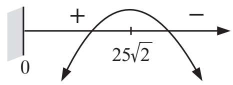
$\therefore\ A(x) = \frac{1}{2}(2x)\sqrt{2500 - x^2} = x\sqrt{2500 - x^2}$



b Now $[A(x)]^2 = x^2(2500 - x^2)$

i.e., $A^2 = 2500x^2 - x^4$

$\therefore\ \dfrac{dA^2}{dx} = 5000x - 4x^3 = 4x(1250 - x^2) = 2x(\sqrt{1250} + x)(\sqrt{1250} - x)$

Sign diagram for $\dfrac{dA^2}{dx}$ is: 

$\therefore$ maximum area occurs when $x = 25\sqrt{2}$ m $\doteqdot 35.4$ m

The corresponding maximum area $\doteqdot 1250$ m^2.

11 $s(t) = 15t - \dfrac{60}{(t-1)^2}$ cm, $t \geqslant 0$

a $\therefore\ s(t) = 15t - 60(t-1)^{-2}$ cm

$\therefore\ v(t) = 15 + 120(t-1)^{-3}$ cm s^{-1}

$\therefore\ a(t) = -360(t-1)^{-4}$ cm s^{-2}

b When $t = 3$, $s(t) = 30$ cm

$v(t) = 30$ cm s^{-1}

$a(t) = -22.5$ cm s^{-2}

The particle is 30 cm right of O, travelling right at 30 cm s^{-1} and is slowing down at 22.5 cm s^{-2}

c $v(t) = 15 + \dfrac{120}{(t-1)^3}$ cm s^{-1}

$\therefore\ v(t) = 0$

when $15 + \dfrac{120}{(t-1)^3} = 0$

$\therefore\ 15(t-1)^3 + 120 = 0$

$\therefore\ (t-1)^3 = -8$

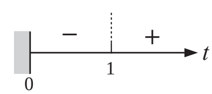
$\therefore\ t = -1$

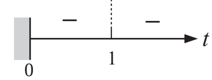
$a(t) = -360(t-1)^{-4} = \dfrac{-360}{(t-1)^4}$ cm s^{-2}

where $(t-1)^4$ is always positive.

$\therefore\ a(t) < 0$ for all $t > 0$

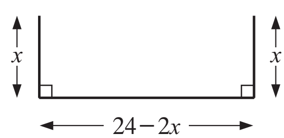
Sign diagrams:

$v(t)$: 

$a(t)$: 

So, speed increases for $0 \leqslant t < 1$ as $v(t)$, $a(t)$ have the same sign.

12 Suppose the sheet is bent x cm from each end.
To maximise the water carried we need to maximise the area of cross-section.

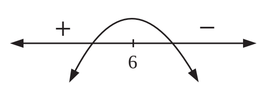


$A = x(24 - 2x)$

$\therefore\ A = 24x - 2x^2$

and $\dfrac{dA}{dx} = 24 - 4x$

which is 0 when $x = 6$

Sign diagram of $\dfrac{dA}{dx}$ is: 

$\therefore$ maximum water when $x = 6$ cm

i.e., bends must be made 6 cm from each end.

REVIEW SET 22B

1 $W(t) = 5000 - \dfrac{4900}{t^2+1}$ g, $t \geqslant 0$

a When $t = 0$, $W(0) = 5000 - \dfrac{4900}{1} = 100$ g

b **i** When $t = 1$,

$W(1) = 5000 - \dfrac{4900}{2}$

$= 2550$ g

ii When $t = 4$,

$W(4) = 5000 - \dfrac{4900}{17}$

$\doteqdot 4711.8$ g

iii When $t = 26$,

$W(26) = 5000 - \dfrac{4900}{677}$

$\doteqdot 4992.8$ g

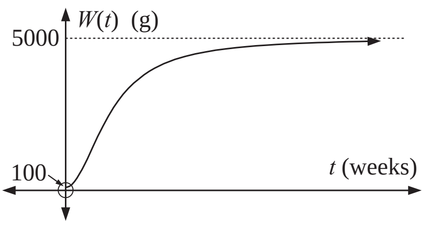
c $W'(t) = 4900(t^2+1)^{-2} \times 2t = \dfrac{9800t}{(t^2+1)^2}$

i $t = 0$ days $= 0$ weeks
$\therefore \quad W'(0) = 0$

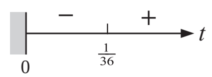
ii $t = 10$ days $= \frac{10}{7}$ weeks
$\therefore \quad W'(\frac{10}{7}) = \dfrac{9800\left(\frac{10}{7}\right)}{\left((\frac{10}{7})^2+1\right)^2}$
$\doteqdot 1514$ g/week

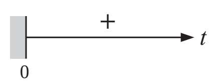
iii $t = 20$ days $= \frac{20}{7}$ weeks
$\therefore \quad W'(\frac{20}{7}) = \dfrac{9800\left(\frac{20}{7}\right)}{\left((\frac{20}{7})^2+1\right)^2}$
$\doteqdot 333.5$ g/week

d



2 $x(t) = 3t - \sqrt{t}$ cm, $\quad t \geqslant 0$





a $x(t) = 3t - t^{\frac{1}{2}}$

$\therefore \quad v(t) = 3 - \frac{1}{2}t^{-\frac{1}{2}} = 3 - \dfrac{1}{2\sqrt{t}} = \dfrac{6\sqrt{t}-1}{2\sqrt{t}}$

which is 0 when $\sqrt{t} = \frac{1}{6}$, i.e., $t = \frac{1}{36}$

and $\quad a(t) = \frac{1}{4}t^{-\frac{3}{2}} = \dfrac{1}{4t\sqrt{t}}$ which is always positive

b When $t = 0$, $\quad x(0) = 0$ cm
$v(0)$ is infinitely large and negative, so is not defined
$a(0)$ is infinitely large and positive, so is not defined

i.e., the particle is at O, moving left and slowing down.

c When $t = 9$, $\quad x(9) = 24$ cm $\quad v(9) = \frac{17}{6}$ cm s^{-1} $\quad a(9) = \frac{1}{108}$ cm s^{-2}

the particle is 24 cm right of O, moving right at $\frac{17}{6}$ cm s^{-1} and increasing its speed.

d The particle reverses direction when $t = \frac{1}{36}$ seconds.

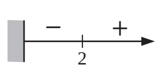
$x\left(\frac{1}{36}\right) = \frac{3}{36} - \frac{1}{6} = -\frac{3}{36} = -\frac{1}{12}$, i.e., it is $\frac{1}{12} \doteqdot 0.083$ cm to the left of O.

e The particle's speed decreases when $v(t)$ and $a(t)$ have different signs,
i.e., for $0 \leqslant t \leqslant \frac{1}{36}$.

3 **a** $G = \frac{1}{3}(t-2)^2 + 5$

$\therefore \quad \dfrac{dG}{dt} = \frac{2}{3}(t-2)^1$

which is 0 when $t = 2$



$\therefore \quad G$ is increasing for $t \geqslant 2$

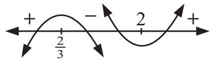
b $\dfrac{dG}{dt} > 10 \quad \therefore \quad \frac{2}{3}(t-2) > 10$
$\therefore \quad t - 2 > 15$
$\therefore \quad t > 17$

4 $f(x) = x^3 - 4x^2 + 4x$
$= x(x^2 - 4x + 4)$
$= x(x-2)^2$

a Cuts the y-axis when $x = 0$ i.e., at (0, 0)

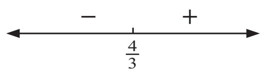
Cuts the x-axis when $y = 0$ i.e., $x(x-2)^2 = 0$,
i.e., when $x = 0$ or 2
i.e., at (0, 0) and (2, 0)

b $f'(x) = 3x^2 - 8x + 4$
$= (3x-2)(x-2)$
which is 0 when $x = \frac{2}{3}$ or 2

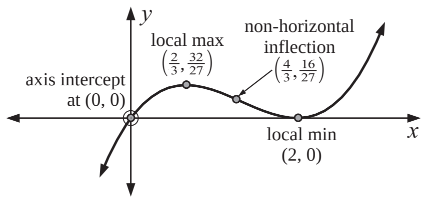
Sign diagram of $f'(x)$ is: 

$\therefore$ there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$,
and a local minimum at (2, 0)

$f''(x) = 6x - 8 = 2(3x-4)$

Sign diagram of $f''(x)$ is: 

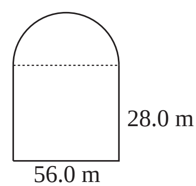
$\therefore$ there is a non-horizontal inflection at $(\frac{4}{3}, \frac{16}{27})$

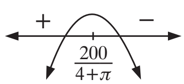
c 

5 **a**
$$P = 200 \text{ m}$$
$$\therefore \quad P = 2x + 2y + \pi x$$
$$\therefore \quad 200 = 2x + 2y + \pi x$$
$$\therefore \quad 2y = 200 - 2x - \pi x$$
$$\therefore \quad y = 100 - x - \tfrac{\pi}{2}x$$

b
$$\begin{aligned}\text{Area of lawn} &= 2x \times y + \tfrac{1}{2}\pi x^2 \\ &= 2x\left[100 - x - \tfrac{\pi}{2}x\right] + \tfrac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \pi x^2 + \tfrac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \tfrac{\pi}{2}x^2\end{aligned}$$
$$\therefore \quad A = 200x - \left(2 + \tfrac{\pi}{2}\right)x^2 \text{ m}^2$$

c
$$\frac{dA}{dx} = 200 - 2\left(2 + \tfrac{\pi}{2}\right)x$$
$$= 200 - (4+\pi)x \quad \text{which is 0 when} \quad (4+\pi)x = 200$$
$$x = \frac{200}{4+\pi}$$



and the sign diagram for $\dfrac{dA}{dx}$ is: 

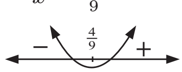
$\therefore$ maximum area occurs when $x = \dfrac{200}{4+\pi} \doteqdot 28.00$ m

6 $f(x) = x\sqrt{x} - x$

a $f(x)$ has meaning provided $x \geqslant 0$

b Cuts y-axis when $x = 0$, i.e., at (0, 0)
Cuts x-axis when $x\sqrt{x} - x = 0$
$$\therefore \quad x(\sqrt{x} - 1) = 0$$
i.e., when $x = 0$ or $\sqrt{x} - 1 = 0$
$$\therefore \quad x = 0 \quad \text{or} \quad \sqrt{x} = 1$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = 1$$
i.e., at (0, 0) and (1, 0)

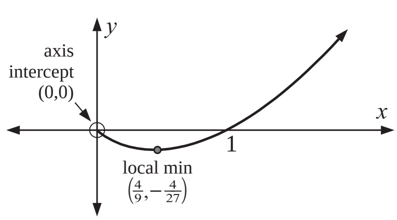
c
$$f(x) = x^{\frac{3}{2}} - x$$
$$\therefore \quad f'(x) = \tfrac{3}{2}x^{\frac{1}{2}} - 1$$
which is 0 when $\frac{3}{2}\sqrt{x} = 1$
$$\therefore \quad \sqrt{x} = \tfrac{2}{3}$$
$$x = \tfrac{4}{9}$$

Sign diagram of $f'(x)$ is: 

$\therefore$ there is a local minimum at $(\frac{4}{9}, -\frac{4}{27})$

and $f''(x) = \frac{2}{3}x^{-\frac{1}{2}} = \dfrac{3}{4\sqrt{x}}$

Since $f''(x) \neq 0$ no inflections.

d 

7 $f(x) = \dfrac{x^2-1}{x^2+1}$

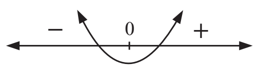
a Cuts x-axis when $y = 0$
i.e., $x^2 - 1 = 0$, $\therefore$ $x = \pm 1$
i.e., at (1, 0) and (-1, 0)
Cuts y-axis when $x = 0$, i.e., (0, -1)

b As $x^2 \geqslant 0$, $x^2 + 1$ can never be 0.
$\therefore$ there are no vertical asymptotes.

c $f'(x) = \dfrac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2}$

$= \dfrac{4x}{(x^2+1)^2}$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at (0, -1)

d $f''(x) = \dfrac{4(x^2+1)^2 - 4x \times 2(x^2+1)(2x)}{(x^2+1)^4}$

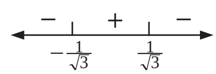
$= \dfrac{4(x^2+1) - 16x^2}{(x^2+1)^3}$

$= \dfrac{4 - 12x^2}{(x^2+1)^3}$

$= \dfrac{4(1+\sqrt{3}x)(1-\sqrt{3}x)}{(x^2+1)^3}$

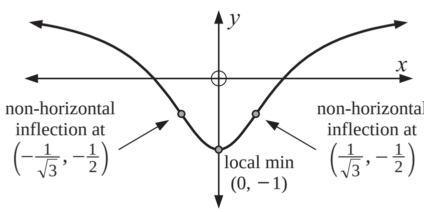
$\therefore$ $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$

and sign diagram is:



$\therefore$ $f(x)$ has nonstationary inflections at $x = \pm\sqrt{\frac{1}{3}}$.

e



8 $f(x) = \dfrac{x-2}{x^2+x-2} = \dfrac{x-2}{(x+2)(x-1)}$

a Vertical asymptotes are: $x = -2$ and $x = 1$ {when the denominator is 0}

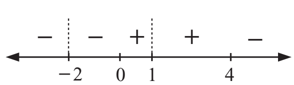
b $f'(x) = \dfrac{1(x^2+x-2) - (x-2)(2x+1)}{(x^2-x-2)^2}$

$= \dfrac{x^2+x-2-2x^2+3x+2}{(x+2)^2(x-1)^2}$

$= \dfrac{4x - x^2}{(x+2)^2(x-1)^2}$

$= \dfrac{-x(x-4)}{(x^2+x-2)^2}$

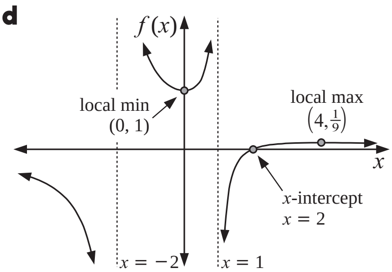
which has sign diagram:



$\therefore$ there is a local minimum at (0, 1),
and a local maximum at (4, $\frac{1}{9}$)

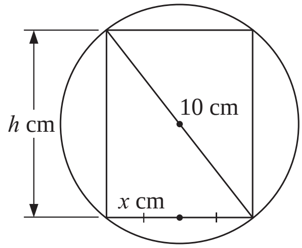
c Cuts y-axis when $x = 0$ i.e., at (0, 1)
Cuts x-axis when $y = 0$ i.e., at (2, 0)

d



e For $\dfrac{x-2}{x^2+x-2} = p$ to have 2 real distinct roots the horizontal line $y = p$ has to cut the curve in exactly two difference places, i.e., $p < 0$ or $p > 1$ or $0 < p < \frac{1}{9}$

9 **a**



Let the height of the cylinder be h cm.

$\therefore \quad (2x)^2 + h^2 = 10^2 \quad$ {Pythagoras}

$\therefore \quad h = \sqrt{100 - 4x^2}$

$\therefore \quad V(x) = \text{area of base} \times \text{height}$

$= \pi x^2 \times \sqrt{100 - 4x^2}$

So, $\quad V(x) = \pi x^2 \sqrt{100 - 4x^2}\ \text{cm}^2$

b Now $\quad V^2 = \pi^2 x^4 (100 - 4x^2)$

$= \pi^2 (100x^4 - 4x^6)$

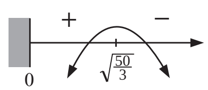
$\therefore \quad \dfrac{dV^2}{dx} = \pi^2 (400x^3 - 24x^5)$

$= 8\pi^2 x^3 (50 - 3x^2)$

$= 8\pi^2 x^3 (\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x)$

which is 0 when $\quad x = \sqrt{\frac{50}{3}} \quad$ {as $x > 0$}

and $\dfrac{dV^2}{dx}$ has sign diagram:



$\therefore \quad$ maximum V occurs when $\quad x = \sqrt{\frac{50}{3}} \doteqdot 4.08$

$\therefore \quad$ radius $= 4.08$ cm, $\quad$ height $= \sqrt{100 - 4\left(\frac{50}{3}\right)} \doteqdot 5.77$ cm

10 **a** Δs LQX and XPM are similar.

$\therefore \quad \dfrac{\text{LQ}}{\text{XP}} = \dfrac{\text{LX}}{\text{XM}} = \dfrac{\text{QX}}{\text{PM}}$

$\therefore \quad \dfrac{\text{LQ}}{1} = \dfrac{8}{\text{PM}}$

$\therefore \quad \text{LQ} = \dfrac{8}{\text{PM}}$

$\therefore \quad \text{LQ} = \dfrac{8}{x}$ km

b $\quad L(x) = \text{LX} + \text{XM}$

$= \sqrt{\left(\dfrac{8}{x}\right)^2 + 8^2} + \sqrt{1^2 + x^2}$

$= 8\sqrt{\dfrac{1}{x^2} + 1} + \sqrt{x^2 + 1}$

$= \dfrac{8}{x}\sqrt{1 + x^2} + 1\sqrt{1 + x^2}$

$= \sqrt{x^2 + 1}\left(\dfrac{8}{x} + 1\right)$

$\therefore \quad L(x) = \sqrt{x^2 + 1}\left(1 + \dfrac{8}{x}\right)$

and $\quad L^2 = \left(x^2 + 1\right)\left(1 + \dfrac{8}{x}\right)^2$

c $\quad \dfrac{d[L(x)]^2}{dx} = 2x\left(1 + \dfrac{8}{x}\right)^2 + (x^2 + 1)2\left(1 + \dfrac{8}{x}\right)\left(-\dfrac{8}{x^2}\right) \quad$ {product rule}

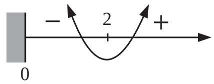
$= 2\left(1 + \dfrac{8}{x}\right)\left[x\left(1 + \dfrac{8}{x}\right) - (x^2 + 1)\left(\dfrac{8}{x^2}\right)\right]$

$= 2\left(1 + \dfrac{8}{x}\right)\left[x + 8 - 8 - \dfrac{8}{x^2}\right]$

$= 2\left(\dfrac{x + 8}{x}\right)\left(\dfrac{x^3 - 8}{x^2}\right)$

$\dfrac{d[L(x)]^2}{dx} = 0 \quad$ when $\quad x = -8 \quad$ or $\quad x^3 = 8$

but $\quad x > 0, \quad \therefore \quad$ when $\quad x = 2$

Sign diagram for $\dfrac{d[L(x)]^2}{dx}$ is: 

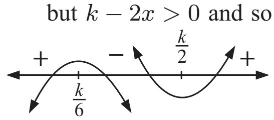
$\therefore$ minimum $L(x)$ occurs when $x = 2$ and the shortest length is $\sqrt{2^2+1}\left(1+\frac{8}{2}\right)$

$$= 5\sqrt{5}$$
$$\doteqdot 11.2 \text{ km}$$

11 When the box is manufactured its base is $(k-2x)$ by $(k-2x)$ and height is x cm.

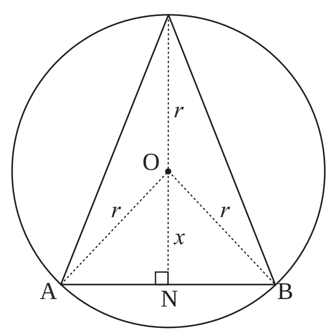
$$\therefore \quad V = x(k-2x)(k-2x)$$
$$\text{i.e.,} \quad V = x(k-2x)^2$$
$$\therefore \quad \frac{dV}{dx} = 1(k-2x)^2 + x\,2(k-2x)^1(-2)$$
$$= (k-2x)^2 - 4x(k-2x)$$
$$= (k-2x)[k-2x-4x]$$
$$= (k-2x)(k-6x)$$

which is 0 when $x = \frac{k}{6}$ or $\frac{k}{2}$, but $k-2x>0$ and so $x<\frac{k}{2}$.

and the sign diagram of $\dfrac{dV}{dx}$ is: 

$\therefore$ maximum capacity occurs when $x = \dfrac{k}{6}$.

12 **a** ON $= x$, OC $= r$

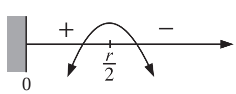


$$\therefore \quad \text{OB} = r \quad \text{(radii)}$$
$$\text{AN} = \sqrt{r^2-x^2}$$
$$\therefore \quad A(x) = \tfrac{1}{2}\text{AB} \times \text{CN} = \text{AN} \times \text{CN}$$
$$\therefore \quad A(x) = \sqrt{r^2-x^2}\,(r+x)$$

b Now $A^2 = (r^2-x^2)(r+x)^2 = (r-x)(r+x)^3$

$$\therefore \quad \frac{dA^2}{dx} = (-1)(r+x)^3 + (r-x)3(r+x)^2(1)$$
$$= (r+x)^2\,[-(r+x)+3(r-x)]$$
$$= (r+x)^2\,[-r-x+3r-3x] = (r+x)^2\,[2r-4x]$$

which is 0 only when $x = \dfrac{r}{2}$ $\{$as $x>0\}$

Sign diagram for $\dfrac{dA^2}{dx}$ is: 

$\therefore$ maximum area occurs when $x = \dfrac{r}{2}$

and in this case CN $= r + \frac{r}{2} = \frac{3}{2}r$

and AN $= \sqrt{r^2 - \frac{r^2}{4}} = \sqrt{\frac{3r^2}{4}} = \frac{\sqrt{3}}{2}r$

$$\therefore \quad \tan\measuredangle\text{CAN} = \frac{\text{CN}}{\text{AN}} = \frac{\frac{3}{2}r}{\frac{\sqrt{3}}{2}r} = \sqrt{3}$$

$$\therefore \quad \measuredangle\text{CAN} = 60^o$$

$\therefore$ Δ is equilateral $\{$as it is isosceles with base angles $60^o\}$

Chapter 23

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXERCISE 23A

1 **a**
$$f(x) = e^{4x}$$
$$\therefore\ f'(x) = 4e^{4x}$$

b
$$f(x) = e^x + 3$$
$$\therefore\ f'(x) = e^x + 0 = e^x$$

c
$$f(x) = \exp(-2x)$$
$$\therefore\ f(x) = e^{-2x}$$
$$\therefore\ f'(x) = -2e^{-2x}$$

d
$$f(x) = e^{\frac{x}{2}}$$
$$\therefore\ f'(x) = \tfrac{1}{2}e^{\frac{x}{2}}$$

e
$$f(x) = 2e^{-\frac{x}{2}}$$
$$\therefore\ f'(x) = 2e^{-\frac{x}{2}}\left(-\tfrac{1}{2}\right) = -e^{-\frac{x}{2}}$$

f
$$f(x) = 1 - 2e^{-x}$$
$$\therefore\ f'(x) = 0 - 2e^{-x}(-1) = 2e^{-x}$$

g
$$f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$$
$$\therefore\ f'(x) = 4e^{\frac{x}{2}}\left(\tfrac{1}{2}\right) - 3e^{-x}(-1) = 2e^{\frac{x}{2}} + 3e^{-x}$$

h
$$f(x) = \frac{e^x + e^{-x}}{2}$$
$$\therefore\ f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$
$$\therefore\ f'(x) = \frac{e^x}{2} + \frac{e^{-x}(-1)}{2} = \frac{e^x - e^{-x}}{2}$$

i
$$f(x) = e^{-x^2}$$
$$\therefore\ f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

j
$$f(x) = e^{\frac{1}{x}}$$
$$\therefore\ f'(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{e^{\frac{1}{x}}}{x^2}$$

k
$$f(x) = 10\left(1 + e^{2x}\right) = 10 + 10e^{2x}$$
$$\therefore\ f'(x) = 0 + 10e^{2x}(2) = 20e^{2x}$$

l
$$f(x) = 20\left(1 - e^{-2x}\right) = 20 - 20e^{-2x}$$
$$\therefore\ f'(x) = 0 - 20e^{-2x}(-2) = 40e^{-2x}$$

m
$$f(x) = e^{2x+1}$$
$$\therefore\ f'(x) = e^{2x+1} \times 2 = 2e^{2x+1}$$

n
$$y = e^{\frac{x}{4}}$$
$$\therefore\ \frac{dy}{dx} = e^{\frac{x}{4}} \times \tfrac{1}{4} = \tfrac{1}{4}e^{\frac{x}{4}}$$

o
$$y = e^{1-2x^2}$$
$$\therefore\ \frac{dy}{dx} = e^{1-2x^2}(-4x) = -4xe^{1-2x^2}$$

p $y = e^{-0.02x}$ $\quad\therefore\ \dfrac{dy}{dx} = e^{-0.02x} \times (-0.02) = -0.02e^{-0.02x}$

2 **a**
$$f(x) = xe^x \qquad u = x,\quad v = e^x,\quad u' = 1,\quad v' = e^x$$
$$\therefore\ f'(x) = 1e^x + e^x x \qquad \{\text{product rule}\}$$
$$= e^x + xe^x$$

b
$$f(x) = x^3e^{-x} \qquad u = x^3,\quad v = e^{-x},\quad u' = 3x^2,\quad v' = -e^{-x}$$
$$\therefore\ f'(x) = 3x^2e^{-x} + x^3(-e^{-x}) \qquad \{\text{product rule}\}$$
$$= 3x^2e^{-x} - x^3e^{-x}$$

c
$$f(x) = \frac{e^x}{x} \qquad u = e^x,\quad v = x,\quad u' = e^x,\quad v' = 1$$
$$\therefore\ f'(x) = \frac{e^x x - e^x(1)}{x^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{xe^x - e^x}{x^2}$$

d
$$f(x) = \frac{x}{e^x} \qquad u = x, \quad v = e^x, \quad u' = 1, \quad v' = e^x$$
$$\therefore \quad f'(x) = \frac{1e^x - xe^x}{(e^x)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{e^x(1-x)}{(e^x)^2}$$
$$= \frac{1-x}{e^x}$$

e
$$f(x) = x^2e^{3x} \qquad u = x^2, \quad v = e^{3x}, \quad u' = 2x, \quad v' = 3e^{3x}$$
$$\therefore \quad f'(x) = 2xe^{3x} + 3x^2e^{3x}$$

f
$$f(x) = \frac{e^x}{\sqrt{x}} \qquad u = e^x, \quad v = x^{\frac{1}{2}}, \quad u' = e^x, \quad v' = \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$\therefore \quad f'(x) = \frac{e^x\sqrt{x} - \dfrac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \quad \{\text{quotient rule}\}$$
$$= \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$$

g
$$f(x) = \sqrt{x}e^{-x} \qquad u = x^{\frac{1}{2}}, \quad v = e^{-x}, \quad u' = \tfrac{1}{2}x^{-\frac{1}{2}}, \quad v' = -e^{-x}$$
$$\therefore \quad f'(x) = \frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x} \quad \{\text{product rule}\}$$

h
$$f(x) = \frac{e^x + 2}{e^{-x} + 1} \qquad u = e^x + 2, \quad v' = e^{-x} + 1, \quad u' = e^x, \quad v' = -e^{-x}$$
$$\therefore \quad f'(x) = \frac{e^x(e^{-x} + 1) - (e^x + 2)\left(-e^{-x}\right)}{(e^{-x} + 1)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2}$$
$$= \frac{2 + e^x + 2e^{-x}}{(e^{-x} + 1)^2}$$

3 **a**
$$f(x) = (e^x + 2)^4 = u^4 \quad \text{where} \quad u = e^x + 2$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3(e^x)$$
$$\therefore \quad f'(x) = 4(e^x + 2)^3(e^x) = 4e^x(e^x + 2)^3$$

b
$$f(x) = \frac{1}{1 - e^{-x}} = u^{-1} \quad \text{where} \quad u = 1 - e^{-x}$$
$$\therefore \quad f'(x) = \frac{dy}{du}\frac{du}{dx} = -u^{-2}(e^{-x})$$
$$\therefore \quad f'(x) = -\frac{e^{-x}}{(1 - e^{-x})^2}$$

c
$$f(x) = \sqrt{e^{2x} + 10} = u^{\frac{1}{2}} \quad \text{where} \quad u = e^{2x} + 10$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{2}u^{-\frac{1}{2}}(2e^{2x})$$
$$\therefore \quad f'(x) = \frac{e^{2x}}{\sqrt{e^{2x} + 10}}$$

d $f(x) = \dfrac{1}{(1-e^{3x})^2} = u^{-2}$ where $u = 1 - e^{3x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -2u^{-3}(-3e^{3x}) = \dfrac{6e^{3x}}{u^3}$

$\therefore \quad f'(x) = \dfrac{6e^{3x}}{(1-e^{3x})^3}$

e $f(x) = \dfrac{1}{\sqrt{1-e^{-x}}} = u^{-\frac{1}{2}}$ where $u = 1 - e^{-x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = -\tfrac{1}{2}u^{-\frac{3}{2}}(e^{-x}) = \dfrac{-e^{-x}}{2u^{\frac{3}{2}}}$

$\therefore \quad f'(x) = \dfrac{-e^{-x}}{2(1-e^{-x})^{\frac{3}{2}}}$

f $f(x) = x\sqrt{1-2e^{-x}} = x\, u^{\frac{1}{2}}$ where $u = 1 - 2e^{-x}$

$\therefore \quad f'(x) = 1\, u^{\frac{1}{2}} + x\, \tfrac{1}{2}u^{-\frac{1}{2}}\, \dfrac{du}{dx}$ {product and chain rules}

$$= 1\,\sqrt{u} + x\, \tfrac{1}{2}u^{-\frac{1}{2}}\, 2e^{-x}$$

$$= \frac{\sqrt{1-2e^{-x}}}{1} + \frac{xe^{-x}}{\sqrt{1-2e^{-x}}}$$

$$= \frac{(1-2e^{-x}) + xe^{-x}}{\sqrt{1-2e^{-x}}}$$

$\therefore \quad f'(x) = \dfrac{1-2e^{-x}+xe^{-x}}{\sqrt{1-2e^{-x}}}$

4 **a** $y = Ae^{kx}$

i $\dfrac{dy}{dx} = Ae^{kx}(k) = k(Ae^{kx}) = ky$

ii $\dfrac{d^2y}{dx^2} = k\dfrac{dy}{dx} = k(ky) = k^2y$

b Prediction: $\dfrac{d^ny}{dx^n} = k^ny$

5

$$y = 2e^{3x} + 5e^{4x}$$

$$\therefore \quad \frac{dy}{dx} = 6e^{3x} + 20e^{4x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$$

Now $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = \left(18e^{3x}+80e^{4x}\right) - 7\left(6e^{3x}+20e^{4x}\right) + 12\left(2e^{3x}+5e^{4x}\right)$

$$= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}$$

$$= e^{3x}\,[18-42+24] + e^{4x}\,[80-140+60]$$

$$= e^{3x}(0) + e^{4x}(0)$$

$$= 0$$

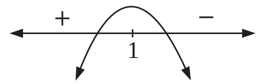
i.e., $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$

6 **a** $y = xe^{-x}$ $\quad u = x, \quad v = e^{-x}, \quad u' = 1, \quad v' = -e^{-x}$

$\therefore \dfrac{dy}{dx} = 1e^{-x} - xe^{-x}$ $\quad$ {product rule}

$= e^{-x}(1-x)$

$= \dfrac{1-x}{e^x}$ $\quad$ which has sign diagram:



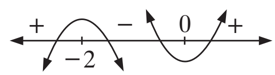
$\therefore$ at $x = 1$, $y = 1e^{-1} = \dfrac{1}{e}$ we have a maximum turning point.

$\therefore$ has a local maximum at $(1, \dfrac{1}{e})$

b $y = x^2e^x$ $\quad u = x^2, \quad v = e^x, \quad u' = 2x, \quad v' = e^x$

$\therefore \dfrac{dy}{dx} = 2xe^x + x^2e^x$ $\quad$ {product rule}

$= xe^x(2+x)$ $\quad$ which has sign diagram:



$\therefore$ at $x = -2$, $y = 4e^{-2}$, we have a maximum turning point

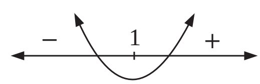
and at $x = 0$, $y = 0$, we have a minimum turning point.

$\therefore$ we have a local maximum at $(-2, \dfrac{4}{e^2})$, and a local minimum at (0, 0).

c $y = \dfrac{e^x}{x}$ $\quad u = e^x, \quad v = x, \quad u' = e^x, \quad v' = 1$

$\therefore \dfrac{dy}{dx} = \dfrac{e^x x - e^x(1)}{x^2}$ $\quad$ {quotient rule}

$= \dfrac{e^x(x-1)}{x^2}$ $\quad$ which has sign diagram:



$\therefore$ at $x = 1$, $y = \dfrac{e}{1} = e$ we have a minimum turning point.

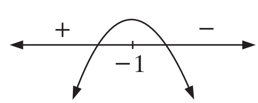
$\therefore$ we have a local minimum at $(1, e)$

d $y = e^{-x}(x+2)$ $\quad u = e^{-x}, \quad v = x+2, \quad u' = -e^{-x}, \quad v' = 1$

$\therefore \dfrac{dy}{dx} = -e^{-x}(x+2) + e^{-x}$ $\quad$ {product rule}

$= e^{-x}(-x-2+1)$

$= e^{-x}(-x-1)$ $\quad$ which has sign diagram:



$\therefore$ at $x = -1$, $y = e(-1+2) = e$ we have a maximum turning point.

$\therefore$ we have a local maximum at $(-1, e)$

EXERCISE 23B

1 **a** $N = 50e^{2t}$

$\therefore \ln N = \ln(50e^{2t})$ $\quad$ $\{\ln ab = \ln a + \ln b\}$

$= \ln 50 + \ln e^{2t}$ $\quad$ {as $\ln e^n = n$}

$\therefore \ln N = \ln 50 + 2t$

b $P = 8.69e^{-0.0541t}$

$\therefore \ln P = \ln(8.69e^{-0.0541t})$

$= \ln 8.69 + \ln e^{-0.0541t}$

$\therefore \ln P = \ln 8.69 - 0.0541t$

c $S = a^2e^{-kt}$

$\therefore \ln S = \ln\left(a^2e^{-kt}\right)$

$= \ln a^2 + \ln e^{-kt}$

$\therefore \ln S = 2\ln a - kt$ $\quad$ $\{\ln a^n = n\ln a\}$

2 **a** $\ln D \doteq 2.1 + 0.69t$
$\therefore \; D \doteq e^{2.1+0.69t}$
$\doteq e^{2.1} \times e^{0.69t}$
$\therefore \; D \doteq 8.166e^{0.69t}$

b $\ln G \doteq -31.64 + 0.0173t$
$\therefore \; G \doteq e^{-31.64+0.0173t}$
$G \doteq e^{-31.64} \times e^{0.0173t}$
$\therefore \; G \doteq 1.815 \times 10^{-14} \times e^{0.0173t}$

c $\ln P = \ln g - 2t$
$= \ln g - \ln(e^{2t})$
$= \ln\left(\dfrac{g}{e^{2t}}\right)$
$\therefore \; P = \dfrac{g}{e^{2t}}$
$\therefore \; P = ge^{-2t}$

d $\ln F = 2\ln x - 0.03t$
$= \ln x^2 - \ln e^{0.03t}$
$= \ln(x^2 e^{0.03t})$
$\therefore \; F = x^2 e^{-0.03t}$

3 **a** $\ln e^2 = 2\ln e$
$= 2(1)$
$= 2$

b $\ln\sqrt{e} = \ln e^{\frac{1}{2}}$
$= \frac{1}{2}\ln e$
$= \frac{1}{2}$

c $\ln\left(\dfrac{1}{e}\right) = \ln e^{-1}$
$= -1\ln e$
$= -1$

d $\ln\left(\dfrac{1}{\sqrt{e}}\right) = \ln e^{-\frac{1}{2}}$
$= -\frac{1}{2}\ln e$
$= -\frac{1}{2}(1)$
$= -\frac{1}{2}$

e $e^{\ln 3} = 3$

f $e^{2\ln 3} = e^{\ln 3^2}$
$= e^{\ln 9}$
$= 9$

g $e^{-\ln 5} = e^{\ln 5^{-1}}$
$= e^{\ln \frac{1}{5}}$
$= \frac{1}{5}$

h $e^{-2\ln 2} = e^{\ln 2^{-2}}$
$= e^{\ln \frac{1}{4}}$
$= \frac{1}{4}$

4 **a** $\ln 5 + \ln 6$
$= \ln(5 \times 6)$
$= \ln 30$

b $3\ln 4 - 2\ln 2$
$= \ln 4^3 - \ln 2^2$
$= \ln \dfrac{4^3}{2^2}$
$= \ln 16$

c $2\ln 5$
$= \ln 5^2$
$= \ln 25$

d $2 + 3\ln 2$
$= \ln e^2 + \ln 2^3$
$= \ln(8e^2)$

5 **a** Let $2 = e^x$
$\therefore \; \ln 2 = x$
$\therefore \; 2 = e^{\ln 2}$

b Let $10 = e^x$
$\therefore \; \ln 10 = x$
$\therefore \; 10 = e^{\ln 10}$

c Let $a = e^x$
$\therefore \; \ln a = x$
$\therefore \; a = e^{\ln a}$

d Let $a^x = e^x$
$\therefore \; \ln a^x = x$
$\therefore \; x\ln a = x$
$\therefore \; a^x = e^{x\ln a}$

6 **a** $e^x = 2$
$\therefore \; \ln e^x = \ln 2$
$x = \ln 2$

b $e^x = -2$ has no solutions
as $e^x > 0$ for all x

c $e^x = 0$ has no solutions
as $e^x > 0$ for all x

d $e^{2x} = 2e^x$
$\therefore \; e^x(e^x - 2) = 0$
$\therefore \; e^x = 2 \quad \{\text{as } e^x > 0\}$
$\therefore \; x = \ln 2$

e $e^x = e^{-x}$
$\therefore \; x = -x$
$\therefore \; 2x = 0$
$\therefore \; x = 0$

f $e^{2x} - 5e^x + 6 = 0$
$\therefore \; (e^x - 3)(e^x - 2) = 0$
$\therefore \; e^x = 3 \text{ or } 2$
$\therefore \; x = \ln 3 \text{ or } \ln 2$

g
$$\begin{aligned} e^x + 2 &= 3e^{-x} \\ \therefore\ e^{2x} + 2e^x &= 3 \quad \{\times e^x\} \\ \therefore\ e^{2x} + 2e^x - 3 &= 0 \\ \therefore\ (e^x+3)(e^x-1) &= 0 \\ \therefore\ e^x &= -3 \text{ or } 1 \\ \therefore\ e^x &= 1 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 1 \\ \therefore\ x &= 0 \end{aligned}$$

h
$$\begin{aligned} 1 + 12e^{-x} &= e^x \\ \therefore\ e^x + 12 &= e^{2x} \quad \{\times e^x\} \\ \therefore\ e^{2x} - e^x - 12 &= 0 \\ \therefore\ (e^x-4)(e^x+3) &= 0 \\ \therefore\ e^x &= 4 \text{ or } -3 \\ \therefore\ e^x &= 4 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 4 \end{aligned}$$

i
$$\begin{aligned} e^x + e^{-x} &= 3 \\ \therefore\ e^{2x} + 1 &= 3e^x \qquad \{\times e^x\} \\ \therefore\ e^{2x} - 3e^x + 1 &= 0 \\ \therefore\ e^x &= \frac{3 \pm \sqrt{9-4}}{2} \\ \therefore\ e^x &= \frac{3 \pm \sqrt{5}}{2} \\ \therefore\ x &= \ln\left(\frac{3+\sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3-\sqrt{5}}{2}\right) \\ &\doteqdot 0.962 \quad \text{or} \quad -0.962 \end{aligned}$$

7 **a**
$$\begin{aligned} \ln x + \ln(x+2) &= \ln 8 \\ \therefore\ \ln[x(x+2)] &= \ln 8 \\ \therefore\ x(x+2) &= 8 \\ \therefore\ x^2 + 2x &= 8 \\ \therefore\ x^2 + 2x - 8 &= 0 \\ \therefore\ (x+4)(x-2) &= 0 \\ \therefore\ x &= 2 \text{ or } -4 \\ \therefore\ x &= 2 \end{aligned}$$
$\{$since $\ln(-4)$ is undefined$\}$

b
$$\begin{aligned} \ln(x-2) - \ln(x+3) &= \ln 2 \\ \therefore\ \ln\left(\frac{x-2}{x+3}\right) &= \ln 2 \\ \therefore\ \frac{x-2}{x+3} &= 2 \\ \therefore\ x - 2 &= 2(x+3) \\ \therefore\ x - 2 &= 2x + 6 \\ \therefore\ -8 &= x \end{aligned}$$
As $\ln(-8)$ is undefined, no solutions exist.

8 **a** $y_1 = e^x$, $y_2 = e^{2x} - 6$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ e^x &= e^{2x} - 6 \\ \therefore\ e^{2x} - e^x - 6 &= 0 \\ \therefore\ (e^x-3)(e^x+2) &= 0 \\ \therefore\ e^x &= 3 \text{ or } -2 \\ \therefore\ e^x &= 3 \qquad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 3 \qquad \text{and } y = e^x = 3 \end{aligned}$$
$\therefore$ meet at $(\ln 3, 3)$

b $y_1 = 2e^x + 1$, $y_1 = 7 - e^x$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ 2e^x + 1 &= 7 - e^x \\ \therefore\ 3e^x &= 6 \\ \therefore\ e^x &= 2 \\ \therefore\ x &= \ln 2 \qquad \text{and} \quad y = 7 - e^x = 7 - 2 = 5 \end{aligned}$$
$\therefore$ meet at $(\ln 2, 5)$

c $y_1 = 3 - e^x$, $y_2 = 5e^{-x} - 3$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ 3 - e^x &= 5e^{-x} - 3 \\ \therefore\ 3e^x - e^{2x} &= 5 - 3e^x \qquad \{\times \text{ by } e^x\} \\ \therefore\ e^{2x} - 6e^x + 5 &= 0 \\ \therefore\ (e^x-5)(e^x-1) &= 0 \\ \therefore\ e^x &= 1 \text{ or } 5 \\ \therefore\ x &= 0 \text{ or } \ln 5 \\ \text{and}\quad y &= 3 - e^0 \qquad \text{or} \qquad 3 - e^{\ln 5} \\ &= 3 - 1 \qquad\qquad\ \ = 3 - 5 \\ &= 2 \qquad\qquad\qquad = -2 \end{aligned}$$
i.e., meet at $(0, 2)$ and $(\ln 5, -2)$

9 **a** $f(x) = 3 - e^x$

cuts x-axis when $y = 0$ i.e., $3 - e^x = 0$

$\therefore\ e^x = 3$

$\therefore\ x = \ln 3$ i.e., A is $(\ln 3,\ 0)$

cuts y-axis when $x = 0$ $\therefore\ y = 3 - e^0 = 3 - 1 = 2$ i.e., B is (0, 2)

b $f'(x) = -e^x$ where

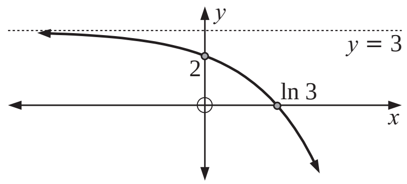
Now $e^x > 0$ for all x, $\therefore\ f'(x) < 0$ for all x

$\therefore\ f(x)$ is decreasing for all x

c $f''(x) = -e^x$ where $e^x > 0$ for all x, $\therefore\ f'(x) < 0$ for all x

$\therefore\ f(x)$ is concave down for all x

d



e as $x \to +\infty$, $f(x) \to -\infty$,

as $x \to -\infty$, $e^x \to 0$ $\therefore\ 3 - e^x \to 3$,

i.e., $f(x) \to 3$ (below)

$\therefore\ y = 3$ is a horizontal asymptote.

10 **a** $y = e^x - 3e^{-x}$

cuts the x axis at P when $y = 0$ $\therefore\ e^x - 3e^{-x} = 0$

$\therefore\ e^{2x} - 3 = 0$ $\{\times$ each term by $e^x\}$

$\therefore\ e^{2x} = 3$

$\therefore\ 2x = \ln 3$

$\therefore\ x = \frac{1}{2}\ln 3$ $\therefore$ P is $(\frac{1}{2}\ln 3,\ 0)$

cuts the y-axis at Q when $x = 0$ $\therefore\ y = e^0 - 3e^0 = 1 - 3 = -2$ $\therefore$ Q is $(0,\ -2)$

b $$\frac{dy}{dx} = e^x + 3e^{-x} = e^x + \frac{3}{e^x}$$

Since $e^x > 0$ for all x

$\frac{dy}{dx} > 0$ for all x

$\therefore$ the function is increasing for all x

c $$\frac{dy}{dx} = e^x + 3x^{-x}$$

$$\therefore\ \frac{d^2y}{dx^2} = e^x - 3e^{-x} = y$$

Above x-axis $y > 0$ $\therefore\ \frac{d^2y}{dx^2} > 0$

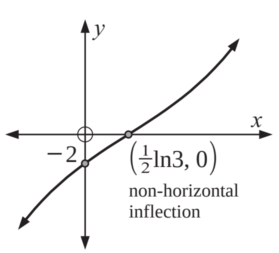
$\therefore$ the function is concave up

Below x-axis $y < 0$ $\therefore\ \frac{d^2y}{dx^2} < 0$

$\therefore$ the function is concave down

i.e., a non-horizontal inflection occurs when $y = 0$

d



11 **a** For $f(x) = e^x - 3$ and $g(x) = 3 - \dfrac{5}{e^x}$

the x-intercept occurs when $y = 0$ $\quad \therefore\ e^x - 3 = 0$
$\therefore\ e^x = 3$
$\therefore\ x = \ln 3$
i.e., at $(\ln 3, 0)$

and $3 - \dfrac{5}{e^x} = 0$
$\therefore\ \dfrac{3e^x - 5}{e^x} = 0$
$\therefore\ 3e^x - 5 = 0$
$\therefore\ e^x = \frac{5}{3}$
$\therefore\ x = \ln \frac{5}{3}$
i.e., at $(\ln \frac{5}{3}, 0)$

the y-intercept occurs when $x = 0$ $\quad \therefore\ y = e^0 - 3$
$y = -2$
i.e., at $(0, -2)$

and $y = 3 - \dfrac{5}{e^0}$
$= 3 - 5$
$= -2$
i.e., $(0, -2)$

b as $x \to +\infty$ $\quad f(x) \to +\infty$ $\qquad$ as $x \to +\infty$ $\quad g(x) \to 3$ (below)
$x \to -\infty$ $\quad f(x) \to -3$ (above) $\qquad$ $x \to -\infty$ $\quad g(x) \to -\infty$

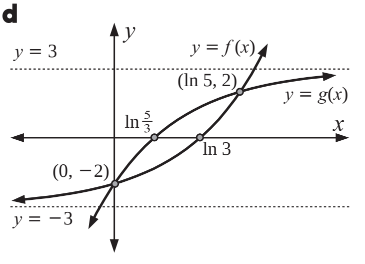
c $f(x)$ and $g(x)$ meet when $e^x - 3 = 3 - 5e^{-x}$

$$\therefore\ e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$$
$$\therefore\ e^{2x} - 6e^x + 5 = 0$$
$$\therefore\ (e^x - 5)(e^x - 1) = 0$$
i.e., when $e^x = 5$ or 1
$$\therefore\ x = \ln 5 \text{ or } 0$$

when $x = \ln 5$, $\quad f(x) = e^{\ln 5} - 3 = 5 - 3 = 2$
when $x = 0$, $\quad f(x) = e^0 - 3 = 1 - 3 = -2$

$\therefore\ f(x)$ and $g(x)$ meet at $(0, -2)$ and $(\ln 5, 2)$

d



EXERCISE 23C

1 **a**
$$y = \ln(7x)$$
$$\therefore\ y = \ln 7 + \ln x$$
$$\therefore\ \frac{dy}{dx} = 0 + \frac{1}{x}$$
$$= \frac{1}{x}$$

or

$$y = \ln(7x)$$
$$\therefore\ \frac{dy}{dx} = \frac{7}{7x} \begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$
$$= \frac{1}{x}$$

b
$$y = \ln(2x + 1)$$
$$\therefore\ \frac{dy}{dx} = \frac{2}{2x + 1} \begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$

c
$$y = \ln(x - x^2)$$
$$\therefore\ \frac{dy}{dx} = \frac{1 - 2x}{x - x^2} \begin{array}{l}\leftarrow f'(x)\\ \leftarrow f(x)\end{array}$$

d
$$y = 3 - 2\ln x$$
$$\therefore\ \frac{dy}{dx} = 0 - 2\left(\frac{1}{x}\right)$$
$$= -\frac{2}{x}$$

e
$$y = x^2 \ln x$$
$$\therefore\ \frac{dy}{dx} = 2x \ln x + x^2\left(\frac{1}{x}\right)$$
$$= 2x \ln x + x$$

f
$$y = \frac{\ln x}{2x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)2x - \ln x(2)}{(2x)^2} = \frac{2-2\ln x}{4x^2} = \frac{1-\ln x}{2x^2}$$

g
$$y = e^x \ln x$$
$$\therefore \quad \frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$$

h
$$y = (\ln x)^2$$
$$\therefore \quad \frac{dy}{dx} = 2(\ln x)^1\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$$

i
$$y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$$
$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right) = \frac{1}{2x\sqrt{\ln x}}$$

j
$$y = e^{-x}\ln x$$
$$\therefore \quad \frac{dy}{dx} = -e^{-x}\ln x + e^{-x}\left(\frac{1}{x}\right) = \frac{e^{-x}}{x} - e^{-x}\ln x$$

k
$$y = \sqrt{x}\ln 2x$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln 2x + \sqrt{x}\left(\frac{1}{x}\right) = \frac{\ln 2x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

l
$$y = \frac{2\sqrt{x}}{\ln x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}}\ln x - 2\sqrt{x}\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\frac{1}{\sqrt{x}}\ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} = \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$$

m
$$y = 3 - 4\ln(1-x)$$
$$\therefore \quad \frac{dy}{dx} = 0 - 4 \times \frac{-1}{1-x} = \frac{4}{1-x}$$

n
$$y = \sqrt{x}\ln 4x$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln 4x + \sqrt{x}\left(\frac{1}{x}\right) = \frac{\ln 4x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

o
$$y = x\ln(x^2+1)$$
$$\therefore \quad \frac{dy}{dx} = \ln(x^2+1) + x\left(\frac{2x}{x^2+1}\right) = \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

2 **a**
$$y = x\ln 5$$
$$\therefore \quad \frac{dy}{dx} = \ln 5$$

b
$$y = \ln(x^3) = 3\ln x$$
$$\therefore \quad \frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

c
$$y = \ln(x^4 + x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{4x^3+1}{x^4+x}$$

d
$$y = \ln(10-5x)$$
$$\therefore \quad \frac{dy}{dx} = \frac{-5}{10-5x} = \frac{1}{x-2}$$

e
$$\begin{aligned} y &= [\ln(2x+1)]^3 \\ \therefore \frac{dy}{dx} &= 3\,[\ln(2x+1)]^2 \times \frac{2}{2x+1} \\ &= \frac{6\,[\ln(2x+1)]^2}{2x+1} \end{aligned}$$

f
$$\begin{aligned} y &= \frac{\ln(4x)}{x} \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{4}{4x}\right)x - \ln(4x)\times 1}{x^2} \\ &= \frac{1-\ln(4x)}{x^2} \end{aligned}$$

g
$$\begin{aligned} y &= \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\ \therefore y &= -\ln x \\ \therefore \frac{dy}{dx} &= -\frac{1}{x} \end{aligned}$$

h
$$\begin{aligned} y &= \ln(\ln x) \\ \therefore \frac{dy}{dx} &= \frac{\frac{1}{x}}{\ln x} \\ &= \frac{1}{x\ln x} \end{aligned}$$

i
$$y = \frac{1}{\ln x} = [\ln x]^{-1} \qquad \therefore \frac{dy}{dx} = -1\,[\ln x]^{-2} \times \frac{1}{x} = \frac{-1}{x\,[\ln x]^2}$$

3 **a**
$$\begin{aligned} y &= \ln\sqrt{1-2x} \\ &= \ln(1-2x)^{\frac{1}{2}} \\ &= \tfrac{1}{2}\ln(1-2x) \\ \therefore \frac{dy}{dx} &= \tfrac{1}{2} \times \frac{-2}{1-2x} \\ &= \frac{1}{2x-1} \end{aligned}$$

b
$$\begin{aligned} y &= \ln\frac{1}{(2x+3)} \\ &= -\ln(2x+3) \\ \therefore \frac{dy}{dx} &= -\frac{2}{2x+3} \end{aligned}$$

c
$$\begin{aligned} y &= \ln(e^x\sqrt{x}) \\ &= \ln e^x + \ln x^{\frac{1}{2}} \\ &= \ln e^x + \tfrac{1}{2}\ln x \\ &= x + \tfrac{1}{2}\ln x \\ \therefore \frac{dy}{dx} &= 1 + \tfrac{1}{2}\frac{1}{x} = 1 + \frac{1}{2x} \end{aligned}$$

d
$$\begin{aligned} y &= \ln(x\sqrt{2-x}) \\ &= \ln x + \ln(2-x)^{\frac{1}{2}} \\ &= \ln x + \tfrac{1}{2}\ln(2-x) \\ \therefore \frac{dy}{dx} &= \frac{1}{x} + \tfrac{1}{2}\left(\frac{-1}{2-x}\right) \\ &= \frac{1}{x} - \frac{1}{2(2-x)} \end{aligned}$$

e
$$\begin{aligned} y &= \ln\left(\frac{x+3}{x-1}\right) \\ &= \ln(x+3) - \ln(x-1) \\ \therefore \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1} \end{aligned}$$

f
$$\begin{aligned} y &= \ln\left(\frac{x^2}{3-x}\right) \\ &= \ln x^2 - \ln(3-x) \\ &= 2\ln x - \ln(3-x) \\ \therefore \frac{dy}{dx} &= \frac{2}{x} - \frac{-1}{3-x} \\ &= \frac{2}{x} + \frac{1}{3-x} \end{aligned}$$

g
$$\begin{aligned} f(x) &= \ln\big((3x-4)^3\big) \\ &= 3\ln(3x-4) \\ \therefore f'(x) &= 3 \times \frac{3}{3x-4} \\ &= \frac{9}{3x-4} \end{aligned}$$

h
$$\begin{aligned} f(x) &= \ln\big(x(x^2+1)\big) \\ &= \ln x + \ln(x^2+1) \\ \therefore f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1} \end{aligned}$$

i
$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\ &= \ln(x^2+2x) - \ln(x-5) \\ f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5} \end{aligned}$$

j
$$\begin{aligned} f(x) &= \ln\left(\frac{x^3}{(2-3x)^2}\right) \\ &= 3\ln x - 2\ln(2-3x) \\ \therefore\ f'(x) &= \frac{3}{x} - 2\left(\frac{-3}{2-3x}\right) \\ &= \frac{3}{x} + \frac{6}{2-3x} \end{aligned}$$

k
$$\begin{aligned} f(x) &= \ln\left(\frac{\sqrt{x+1}}{x(x+2)}\right) \\ &= \tfrac{1}{2}\ln(x+1) - \ln x - \ln(x+2) \\ \therefore\ f'(x) &= \tfrac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x} - \frac{1}{x+2} \end{aligned}$$

l
$$\begin{aligned} f(x) &= \ln\left(\frac{(x+1)(2x-1)}{3x^2}\right) \\ &= \ln(x+1) + \ln(2x-1) - \ln(3x^2) \\ \therefore\ f'(x) &= \frac{1}{x+1} + \frac{2}{2x-1} - \frac{6x}{3x^2} \\ &= \frac{1}{x+1} + \frac{2}{2x-1} - \frac{2}{x} \end{aligned}$$

4 **a**
$$\begin{aligned} y &= 2^x \\ &= (e^{\ln 2})^x \\ &= e^{x\ln 2} \end{aligned} \qquad \therefore\ \begin{aligned} \frac{dy}{dx} &= e^{x\ln 2} \times \ln 2 \\ &= 2^x \ln 2 \end{aligned}$$

b
$$\begin{aligned} y &= a^x \\ &= (e^{\ln a})^x \\ &= e^{x\ln a} \end{aligned} \qquad \therefore\ \begin{aligned} \frac{dy}{dx} &= e^{x\ln a} \times \ln a \\ &= a^x \ln a \end{aligned}$$

5 $f(x) = \ln(2x-1) - 3$

a when $y = 0$,
$$\begin{aligned} \ln(2x-1) &= 3 \\ \therefore\ 2x-1 &= e^3 \\ \therefore\ 2x &= e^3+1 \\ \therefore\ x &= \frac{e^3+1}{2} \doteqdot 10.54 \end{aligned}$$
i.e., the x intercept is $\dfrac{e^3+1}{2}$.

b $f(0) = \ln(-1)$ cannot be found as $\ln(-1)$ is not defined. $\therefore$ there is no y-intercept.

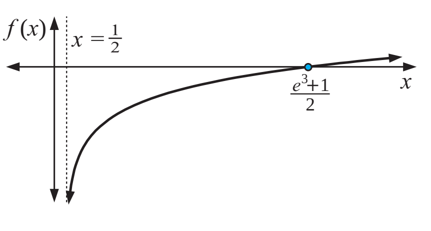
c
$$f'(x) = \frac{2}{2x-1}, \quad \therefore\ f'(1) = \frac{2}{2-1} = 2$$
$\therefore$ slope of tangent $= 2$

d $\ln(2x-1)$ has meaning provided
$$\begin{aligned} 2x-1 &> 0 \\ \text{i.e., } 2x &> 1 \\ x &> \tfrac{1}{2} \end{aligned}$$
$\therefore$ $f(x)$ has meaning provided $x > \frac{1}{2}$

e
$$\begin{aligned} f'(x) &= 2(2x-1)^{-1} \\ \therefore\ f''(x) &= -2(2x-1)^{-2}2 = \frac{-4}{(2x-1)^2} \end{aligned}$$
provided $f(x)$ has meaning i.e., provided $x > \frac{1}{2}$, $f''(x) < 0$

$\therefore$ $f(x)$ is always concave down for $x > \frac{1}{2}$

f



6 **a** $f(x) = x \ln x$

$f(x)$ is defined provided $\ln x$ is defined i.e., $x > 0$

b $f(x) = x \ln x$ Let $u = x$, $v = \ln x$, $u' = 1$, $v = \dfrac{1}{x}$

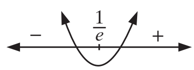
$$\therefore \quad f'(x) = (1)\ln x + x\left(\frac{1}{x}\right) \qquad \{\text{product rule}\}$$
$$= \ln x + 1$$

which is 0 when $\ln x = -1$

i.e., $x = e^{-1}$

i.e., $x = \dfrac{1}{e}$, and $f\left(\dfrac{1}{e}\right) = \dfrac{1}{e}\ln e^{-1} = -\dfrac{1}{e}$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$. i.e., the smallest value of $\dfrac{\ln x}{x}$ is $\dfrac{1}{e}$.

7 Consider $f(x) = \dfrac{\ln x}{x}$ Let $u = \ln x$, $v = x$, $u' = \dfrac{1}{x}$, $v' = 1$

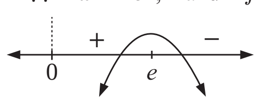
$$\therefore \quad f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{1 - \ln x}{x^2}$$

Now $f'(x) = 0$ when $1 - \ln x = 0$

$\therefore$ $\ln x = 1$

$\therefore$ $x = e^1$, and $f(e) = \dfrac{\ln e}{e} = \dfrac{1}{e}$

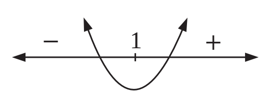
Sign diagram of $f'(x)$ is:



$\therefore$ there is a max. turning point at $\left(e, \dfrac{1}{e}\right)$, i.e., $f(x) \leqslant \dfrac{1}{e}$ for all x, i.e., $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all x

8 $f(x) = x - \ln x$

$\therefore$ $f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$ and the sign diagram of $f'(x)$ is:



$\therefore$ $f(x)$ has a minimum turning point at $(1, 1 - \ln 1)$, i.e., $(1, 1)$

$\therefore$ $f(x) \geqslant 1$ for all $x > 0$

$x - \ln x \geqslant 1$

i.e., $\ln x \leqslant x - 1$ for all $x > 0$

EXERCISE 23D

1 $f(x) = e^{-x}$ $\therefore$ $f(1) = e^{-1}$ i.e., the point of contact is $(1, \frac{1}{e})$.

Now $f'(x) = -e^{-x}$

$f'(1) = -e^{-1} = \dfrac{-1}{e}$ and is the slope of the tangent at $(1, \frac{1}{e})$.

$\therefore$ tangent has equation $\dfrac{y - \frac{1}{e}}{x - 1} = -\dfrac{1}{e}$ $\therefore$ $e(y - \frac{1}{e}) = -(x - 1)$

$\therefore$ $ey - 1 = -x + 1$

i.e., $x + ey = 2$ or $y = -\dfrac{1}{e}x + \dfrac{2}{e}$

2 $y = \ln(2-x)$, so when $x = -1$, $y = \ln 3$ i.e., the point of contact is $(-1, \ln 3)$.

$$\text{Now } \frac{dy}{dx} = \frac{-1}{2-x}$$

$\therefore$ when $x=-1$, $\dfrac{dy}{dx} = -\dfrac{1}{2+1} = -\frac{1}{3}$ and is the slope of the tangent at $(-1, \ln 3)$.

$\therefore$ tangent has equation $\dfrac{y - \ln 3}{x+1} = -\frac{1}{3}$ i.e., $3(y - \ln 3) = -(x+1)$

$$\therefore\ 3y - 3\ln 3 = -x - 1$$
$$\text{i.e., } x + 3y = 3\ln 3 - 1$$

3 $y = x^2 e^x$, so when $x = 1$, $y = e$, i.e., the point of contact is $(1, e)$.

$$\text{Now } \frac{dy}{dx} = x^2 e^x + 2xe^x$$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = e + 2e = 3e$ and is the slope of the tangent at $(1, e)$.

$\therefore$ the tangent has equation $\dfrac{y-e}{x-1} = 3e$ i.e., $y - e = 3e(x-1)$

$$\therefore\ y - e = 3ex - 3e$$
$$\therefore\ y - 3ex = -2e$$
$$\text{i.e., } 3ex - y = 2e$$

Cuts the x-axis when $y = 0$ $\therefore$ $3ex = 2e$ i.e., $x = \frac{2}{3}$ i.e., at A$(\frac{2}{3}, 0)$

Cuts the y-axis when $x = 0$ i.e., $-y = 2e$ i.e., at B$(0, -2e)$

4
$$y = \ln\sqrt{x} = \ln x^{\frac{1}{2}} = \tfrac{1}{2}\ln x$$

$\therefore$ when $y = -1$, $-1 = \frac{1}{2}\ln x$

$$\therefore\ \ln x = -2$$
$$\therefore\ x = e^{-2}$$

$\therefore$ $x = \dfrac{1}{e^2}$ i.e., the point of contact is $\left(\dfrac{1}{e^2}, -1\right)$

Now $\dfrac{dy}{dx} = \frac{1}{2}\dfrac{1}{x} = \dfrac{1}{2x}$ $\therefore$ at $\left(\dfrac{1}{e^2}, -1\right)$, $\dfrac{dy}{dx} = \dfrac{1}{2e^{-2}} = \dfrac{e^2}{2}$

$\therefore$ the tangent has slope $= \dfrac{e^2}{2}$ and the normal has slope $= -\dfrac{2}{e^2}$

$\therefore$ the normal has equation $\dfrac{y+1}{x - \frac{1}{e^2}} = -\dfrac{2}{e^2}$ i.e., $e^2(y+1) = -2\left(x - \dfrac{1}{e^2}\right)$

$$\therefore\ e^2 y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore\ 2x + e^2 y = -e^2 + \frac{2}{e^2} \quad \text{or} \quad y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$$

5
$$y = e^x$$

When $x = a$, $y = e^a$, i.e., the point of contact is (a, e^a).

Now $\dfrac{dy}{dx} = e^x$ and so at (a, e^a), $\dfrac{dy}{dx} = e^a$ and is the slope of the tangent at (a, e^a).

$\therefore$ the tangent has equation $\dfrac{y - e^a}{x - a} = e^a$ i.e., $y - e^a = e^a(x-a)$ ......$*$

if the tangent is through the origin, (0, 0) must satisfy $*$ $\therefore$ $0 - e^a = e^a(0 - a)$

$$\therefore\ -e^a = -ae^a$$
$$\therefore\ e^a(a-1)0 = 0$$
$$\therefore\ a = 1 \quad \{\text{as } e^a > 0\}$$

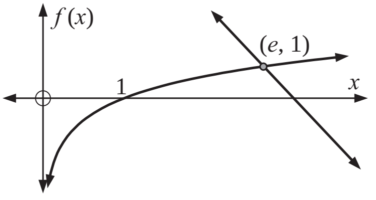
So the equation of the tangent is $y - e = ex - e$ i.e., $y = ex$

6 **a** $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \dfrac{1}{x}$ which is > 0 for all $x > 0$, $\therefore$ $f(x)$ is increasing on $x > 0$.

$f''(x) = -x^{-2} = \dfrac{-1}{x^2}$ which is < 0 for all $x > 0$, $\therefore$ $f(x)$ is concave down on $x > 0$.

c



At $y = 1$, $1 = \ln x$

$\therefore$ $x = e^1 = e$,

i.e., the point of contact is $(e, 1)$

and $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ at $(e, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

$\therefore$ the slope of the tangent is $\dfrac{1}{e}$, and the slope of the normal is $-e$

$\therefore$ the equation of the normal is $\dfrac{y-1}{x-e} = -e$ $\therefore$ $y - 1 = -e(x - e)$

$\therefore$ $y - 1 = -ex + e^2$

$\therefore$ $ex + y = 1 + e^2$

7 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when $3e^{-x} = 2 + e^x$

$\therefore$ $3 = 2e^x + e^{2x}$ $\{\times \text{ by } e^x\}$

$\therefore$ $e^{2x} + 2e^x - 3 = 0$

$\therefore$ $(e^x + 3)(e^x - 1) = 0$

$\therefore$ $e^x = -3$ or 1 $\{\text{as } e^x > 0\}$

$\therefore$ $x = 0$ $\{\text{as } e^x > 0\}$

and when $x = 0$, $y = 3e^0 = 3$

When $y = 2 + e^x$, $\dfrac{dy}{dx} = e^x$ $\therefore$ at $(0, 3)$, $\dfrac{dy}{dx} = e^0 = 1$

i.e., the slope of the tangent is 1

and $1 = \tan\theta$ where θ is the angle between the tangent and the x-axis, $\therefore$ $\theta = 45^o$

When $y = 3e^{-x}$, $\dfrac{dy}{dx} = -3e^{-x}$ $\therefore$ at $(0, 3)$, $\dfrac{dy}{dx} = -3$

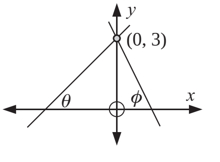
and $\tan\phi = 3$ where ϕ is the angle between the tangent and the x-axis

$\therefore$ $\phi = \tan^{-1}(3)$

$\doteqdot 71.57^o$

$\therefore$ angle between the tangents $\doteqdot (180 - (71.57 + 45))^o$

$\doteqdot 63.43^o$



8 **a** $W = 200e^{\frac{t}{2}}$

i When $t = 0$,
$W = 200$ g

ii When $t = \frac{1}{2}$,
$W = 200e^{\frac{t}{2}}$
$= 200e^{\frac{1}{4}}$
$= 200e^{0.25}$
$= 256.8$ g

iii When $t = 1\frac{1}{2}$,
$W = 200e^{\frac{t}{2}}$
$= 200e^{0.75}$
$\doteqdot 423.4$ g

b When $W = 1000$,

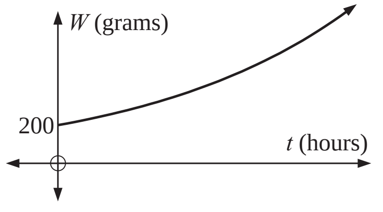
$$\begin{aligned} 200e^{\frac{t}{2}} &= 1000 \\ \therefore \quad e^{\frac{t}{2}} &= 5 \\ \therefore \quad \tfrac{t}{2} &= \ln 5 \\ \therefore \quad t &= 2\ln 5 \\ \therefore \quad t &\doteqdot 3.219 \\ \text{i.e.,} \quad t &\doteqdot 3 \text{ hours } 13 \text{ min} \end{aligned}$$

c
$$\begin{aligned} W &= 200e^{\frac{t}{2}} \\ \therefore \quad \frac{dW}{dt} &= \tfrac{1}{2} \times 200e^{\frac{t}{2}} \\ &= 100e^{\frac{t}{2}} \end{aligned}$$

i when $t = 0$,
$$\begin{aligned} \frac{dW}{dt} &= 100 \times e^0 \\ &= 100 \text{ g/h} \end{aligned}$$

ii when $t = 2$,
$$\begin{aligned} \frac{dW}{dt} &= 100 \times e^{\frac{2}{2}} \\ &= 100e \\ &= 271.8 \text{ g/h} \end{aligned}$$

d



9 **a** $W = 20e^{-kt}$ and when $t = 50$ hours, $W = 10$ g

$$\begin{aligned} \therefore \quad 20e^{-50k} &= 10 \\ \therefore \quad e^{-50k} &= \tfrac{1}{2} \\ \therefore \quad -50k &= \ln \tfrac{1}{2} = -\ln 2 \\ \therefore \quad k &= \tfrac{1}{50}\ln 2 \doteqdot 0.0139 \end{aligned}$$

b **i** When $t = 0$,
$$\begin{aligned} W &= 20e^{-kt} \\ &= 20e^0 \\ \therefore \quad W &= 20 \text{ g} \end{aligned}$$

ii When $t = 24$,
$$\begin{aligned} W &= 20e^{-24k} \\ &= 20e^{-24\frac{\ln 2}{50}} \\ &\doteqdot 14.34 \text{ g} \end{aligned}$$

iii When
$$\begin{aligned} t &= 1 \text{ week} \\ &= 7 \times 24 \text{ hours} \\ &= 168 \text{ hours} \end{aligned}$$
$$\begin{aligned} W &= 20e^{-168\frac{\ln 2}{50}} \\ &\doteqdot 1.948 \text{ g} \end{aligned}$$

c When $W = 1$ g,

$$\begin{aligned} 20e^{-\frac{\ln 2}{50} \times t} &= 1 \\ \therefore \quad e^{-\frac{\ln 2}{50} \times t} &= 0.05 \\ \therefore \quad -\frac{\ln 2}{50} \times t &= \ln 0.05 \\ \therefore \quad t &= \frac{-50 \ln 0.05}{\ln 2} \doteqdot 216.1 \text{ hours} \end{aligned}$$

d
$$\begin{aligned} &\frac{dW}{dt} \\ &= 20e^{-kt}(-k) \\ &= \left(-20\frac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50}t} \end{aligned}$$

i When $t = 100$ hours,
$$\begin{aligned} \frac{dW}{dt} &= \left(\frac{-20\ln 2}{50}\right) e^{-2\ln 2} \\ &\doteqdot -0.0693 \text{ g/h} \end{aligned}$$

ii When $t = 1000$ hours,
$$\begin{aligned} \frac{dW}{dt} &= \left(\frac{-20\ln 2}{50}\right) e^{-2\ln 2} \\ &\doteqdot -2.64 \times 10^{-7} \text{ g/h} \end{aligned}$$

e
$$\begin{aligned} \frac{dW}{dt} &= -k(20e^{-kt}) \\ &= -kW \end{aligned} \qquad \therefore \quad \frac{dW}{dt} \; \alpha \; W$$

10 $T = 5 + 95e^{-kt}$ ^{o}C

a $T = 20^o$C when $t = 15$

$$\begin{aligned} \therefore \quad 20 &= 5 + 95e^{-15k} \\ \therefore \quad 15 &= 95e^{-15k} \\ \therefore \quad \ln\left(\tfrac{15}{95}\right) &= -15k \\ \therefore \quad k &= \frac{\ln\left(\frac{15}{95}\right)}{-15} \end{aligned}$$

i.e., $k \doteqdot 0.1231$

b When $t = 0$,

$$\begin{aligned} T &= 5 + 95e^0 \\ \therefore \quad T &= 5 + 95 \\ &= 100^o \text{ C} \end{aligned}$$

c

$$\begin{aligned} \frac{dT}{dt} &= 0 + 95e^{-kt}(-k) \\ &= -(95e^{kt})k \\ &= -k(T-5) \end{aligned}$$

$$\therefore \quad \frac{dT}{dt} \propto T - 5$$

d

$$\begin{aligned} \frac{dT}{dt} &= -95e^{-kt} \times k \\ &\doteqdot -11.6902e^{-0.1231t} \end{aligned}$$

i When $t = 0$,

$$\frac{dT}{dt} \doteqdot -11.69$$

$\therefore$ temperature is decreasing at 11.69^{o}C/min

ii When $t = 10$,

$$\begin{aligned} \frac{dT}{dt} &\doteqdot -11.6902e^{-1.231} \\ &\doteqdot -3.415 \end{aligned}$$

$\therefore$ temperature is decreasing at 3.415^{o}C/min

iii When $t = 20$,

$$\begin{aligned} \frac{dT}{dt} &\doteqdot -11.6902e^{-2.462} \\ &\doteqdot -0.998 \end{aligned}$$

$\therefore$ temperature is decreasing at 0.998^{o}C/min

11 $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

a Planted when $t = 0$

$$\begin{aligned} \therefore \quad H(0) &= 20\ln(2) + 30 \\ &\doteqdot 43.86 \text{ cm} \end{aligned}$$

b When $H = 1$ m $= 100$ cm

$$\begin{aligned} \therefore \quad 20\ln(3t+2) + 30 &= 100 \\ \therefore \quad 20\ln(3t+2) &= 70 \\ \therefore \quad \ln(3t+2) &= 3.5 \\ \therefore \quad 3t+2 &= e^{3.5} \\ \therefore \quad 3t &= e^{3.5} - 2 \\ \therefore \quad t &= \frac{e^{3.5}-2}{3} \text{ years} \\ \therefore \quad t &\doteqdot 10.37 \text{ years} \end{aligned}$$

c

$$\begin{aligned} \frac{dH}{dt} &= \frac{20}{(3t+2)} \times 3 \\ &= \frac{60}{3t+2} \quad \text{cm/year} \end{aligned}$$

i When $t = 3$,

$$\frac{dH}{dt} = \frac{60}{11} \doteqdot 5.454$$

$\therefore$ it is growing at 5.45 cm/year

ii When $t = 10$,

$$\frac{dH}{dt} = \frac{60}{32} = 1.875$$

$\therefore$ it is growing at 1.88 cm/year

12 $A = s(1 - e^{-kt})$, $t \geqslant 0$

a When $t = 0$,

$$\begin{aligned} A &= s(1 - e^0) \\ &= s(1-1) \\ &= 0 \end{aligned}$$

b When $t = 3$, $A = 5$ and $s = 10$

$$\begin{aligned} \therefore \quad 5 &= 10(1 - e^{-3k}) \\ \therefore \quad 0.5 &= 1 - e^{-3k} \\ \therefore \quad e^{-3k} &= 0.5 \\ \therefore \quad e^{3k} &= 2 \\ \therefore \quad 3k &= \ln 2 \\ \therefore \quad k &= \tfrac{1}{3}\ln 2 \\ \therefore \quad k &\doteqdot 0.231 \end{aligned}$$

c When $t = 5$ and $s = 10$

$$\begin{aligned}\frac{dA}{dt} &= ske^{-kt}\\ &= 10\left(\tfrac{1}{3}\ln 2\right)\left(e^{-\frac{5}{3}\ln 2}\right)\\ &\doteqdot 0.7278 \quad \text{units/hour}\end{aligned}$$

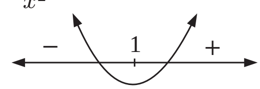
d Now
$$\begin{aligned}\frac{dA}{dt} &= sk\left(e^{-kt}\right)\\ &= k\left(se^{-kt}\right)\\ &= -k\left(-se^{-kt}\right)\\ &= -k(A-s)\\ \therefore \quad \frac{dA}{dt} &\propto A - s\end{aligned}$$

13 $f(x) = \dfrac{e^x}{x}$

a No x-intercept since $e^x \neq 0$, i.e., $f(x) \neq 0$, and no y-intercept since $\dfrac{e^0}{0}$ is undefined.

b as $x \to +\infty$ $f(x) \to \infty$, and as $x \to -\infty$ $f(x) \to 0$ (below)

c By the quotient rule, $f'(x) = \dfrac{e^x x - e^x(1)}{x^2} = \dfrac{e^x(x-1)}{x^2}$ with sign diagram:



and when $x = 1$, $f(x) = \dfrac{e^1}{1} = e$

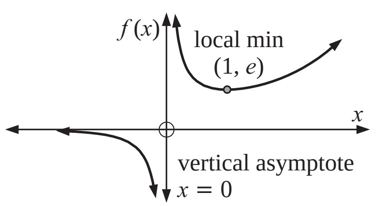
$\therefore$ local minimum when $x = 1$

i.e., local minimum at $(1, e)$.

Note: Vertical asymptote exists at $x = 0$,

i.e., as $x \to 0$ (from above), $y \to +\infty$, as $x \to 0$ (from below), $y \to -\infty$

d



e Now $f'(x) = \dfrac{e^x(x-1)}{x^2}$

$\therefore \quad f'(-1) = \dfrac{e^{-1}(-1-1)}{(-1)^2} = -\dfrac{2}{e}$

$\therefore$ slope of the tangent is $= -\dfrac{2}{e}$

and when $x = -1$, $y = \dfrac{e^{-1}}{-1} = -\dfrac{1}{e}$

$\therefore$ equation of tangent is $\dfrac{y - \left(-\frac{1}{e}\right)}{x - (-1)} = -\dfrac{2}{e}$ i.e., $\dfrac{y + \frac{1}{e}}{x+1} = -\dfrac{2}{e}$

$$\begin{aligned}\therefore \quad e\left(y + \frac{1}{e}\right) &= -2(x+1)\\ ey + 1 &= -2x - 2\\ \therefore \quad 2x + ey &= -3\end{aligned}$$

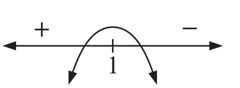
14 $f(x) = \dfrac{x}{e^x}$

a Cuts x and y-axis at (0, 0)

b as $x \to +\infty$, $f(x) \to 0$ (above), as $x \to -\infty$, $f(x) \to -\infty$

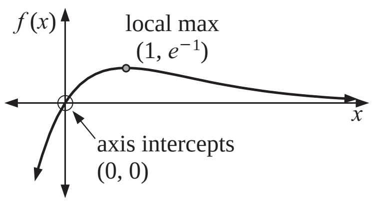
c
$$\begin{aligned}f'(x) &= \frac{(1)e^x - xe^x}{(e^x)^2} \quad \{\text{quotient rule}\}\\ &= \frac{e^x(1-x)}{(e^x)^2}\\ &= \frac{1-x}{e^x}\end{aligned}$$

which has sign diagram:



and when $x = 1$, $f(x) = \dfrac{1}{e^1} = \dfrac{1}{e}$, $\therefore$ there is a local maximum at $\left(1, \dfrac{1}{e}\right)$

d



e When $x = 2$, $f(2) = \dfrac{2}{e^2}$

and the slope of the tangent $= \dfrac{1-2}{e^2} = -\dfrac{1}{e^2}$, $\therefore$ slope of the normal $= e^2$

$\therefore$ equation of the normal is $\dfrac{y - \frac{2}{e^2}}{x-2} = e^2$ i.e., $y - \dfrac{2}{e^2} = e^2(x-2)$

i.e., $e^2x - y = 2e^2 - \dfrac{2}{e^2}$

cuts the x-axis when $y = 0$

$\therefore$ $e^2x = 2e^2 - \dfrac{2}{e^2}$ i.e., at $x = 2 - \dfrac{2}{e^4}$

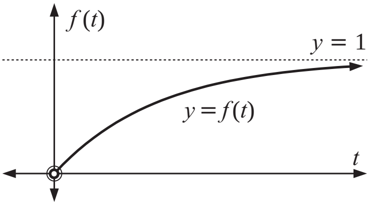
15 **a** $f(t) = 1 - e^{-at}$

$\therefore$ $f'(t) = ae^{-at} = \dfrac{a}{e^{at}}$

$\therefore$ for all positive a, $f'(t) > 0$

$\therefore$ there are no stationary points.

b when $a = 1$, $f(t) = 1 - e^{-t}$



c when $a = 1$ $f(t) = 1 - e^{-t}$

$\therefore$ $f'(t) = e^{-t}$

$\therefore$ $f''(t) = -e^{-t}$

$\therefore$ for all values of t, $f'(t) > 0$ and $f''(t) < 0$ $\{e^{-t} > 0\}$

$\therefore$ the curve is always concave down and $f(t)$ is always increasing.

16 **a** $x(t) = 100\left(2 - e^{-\frac{t}{10}}\right) = 200 - 200e^{-\frac{t}{10}}$ cm, $t > 0$

$\therefore$ $v(t) = x'(t) = -100e^{-\frac{t}{10}}(-\frac{1}{10}) = 10e^{-\frac{t}{10}}$ cm/sec

$\therefore$ $a(t) = v'(t) = -\frac{10}{10}e^{-\frac{t}{10}}$ $\therefore$ $a(t) = -e^{-\frac{t}{10}}$ cm/sec^2

b when $t = 0$,

$x(0) = 100(2-1) = 100$ cm

$v(0) = 10e^0 = 10$ cm/sec

$a(0) = -e^0 = -1$ cm/sec^2

c when $t = 5$,

$x(5) = 100(2 - e^{-\frac{1}{2}}) \doteqdot 139.3$ cm

$v(5) = 10e^{-\frac{1}{2}} \doteqdot 6.065$ cm/sec

$a(5) = -e^{-\frac{1}{2}} \doteqdot -0.6065$ cm/sec^2

d when $x(t) = 150$ cm

$100(2 - e^{-\frac{t}{10}}) = 150$

$\therefore$ $2 - e^{-\frac{t}{10}} = 1.5$

$\therefore$ $e^{-\frac{t}{10}} = 0.5$

$\therefore$ $-\frac{t}{10} = \ln 0.5$

$\therefore$ $t = -10\ln 0.5$

i.e., $t \doteqdot 6.93$ seconds

e Now for all $t > 0$,

$v(t) = 10e^{-\frac{t}{10}} > 0$ {as $e^{-\frac{t}{10}} > 0$}

and $a(t) = -e^{-\frac{t}{10}} < 0$

$\therefore$ speed is decreasing for all t, and velocity is decreasing for all t as $a(t) < 0$.

17 **a** $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geqslant 0$

$v(t) = 100 - 40e^{-\frac{t}{5}}$ cm/sec

$a(t) = 8e^{-\frac{t}{5}}$ cm/sec^2

b When $t = 0$,

$s(0) = 200$ cm

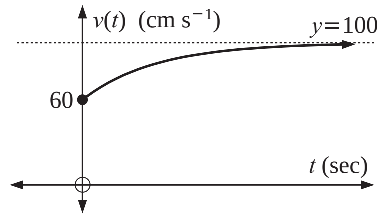
$v(0) = 60$ cm/sec

$a(0) = 8$ cm/sec^2

c as $t \to +\infty$, $e^{-\frac{t}{5}} \to 0$,

$\therefore$ $v(t) \to 100$ cm/sec (below)

d



e When $v(t) = 80$ cm/sec,

$$100 - 40e^{-\frac{t}{5}} = 80$$
$$\therefore\quad -40e^{-\frac{t}{5}} = -20$$
$$\therefore\quad e^{-\frac{t}{5}} = 0.5$$
$$\therefore\quad -\frac{t}{5} = \ln 0.5$$
$$\therefore\quad t = -5\ln 0.5$$
$$\therefore\quad t \doteqdot 3.47 \text{ seconds}$$

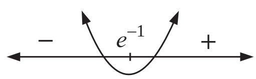
18 **a** $A(t) = t\ln t + 1, \quad 0 < t \leqslant 5$

$$\therefore\quad A'(t) = \ln t + t \times \frac{1}{t} + 0 \quad \{\text{product rule}\}$$
$$= \ln t + 1$$

which is 0 when $\ln t = -1$

i.e., $t = e^{-1}$

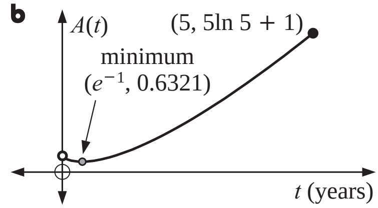
and the sign diagram of $A'(t)$ is:



$\therefore$ $A(t)$ is a minimum when $t = \dfrac{1}{e} \doteqdot 0.3679$ years

i.e., at 4.41 months old.

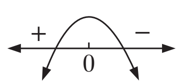
b



19 **a** $f(x) = \dfrac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$, $\quad\therefore\quad f'(x) = \dfrac{1}{\pi\sqrt{2}}\,e^{-\frac{1}{2}x^2}(-x) = \dfrac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$

which is 0 only when $x = 0$, and when $x = 0$, $f(0) = \dfrac{1}{\pi\sqrt{2}}$

and $f'(x)$ has sign diagram:



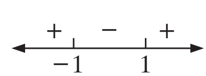
$\therefore$ there is a local maximum at $\left(0, \dfrac{1}{\pi\sqrt{2}}\right)$

and the function is increasing for $x \leqslant 0$, and decreasing for $x \geqslant 0$

b $f'(x) = \dfrac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2} = \dfrac{1}{\pi\sqrt{2}}\left(-xe^{-\frac{1}{2}x^2}\right)$

$$\therefore\quad f''(x) = \frac{1}{\pi\sqrt{2}}\left((-1)\,e^{-\frac{1}{2}x^2} + (-x)\,e^{-\frac{1}{2}x^2}\,(-x)\right) \quad \{\text{product rule}\}$$
$$= \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}\left(x^2 - 1\right)$$
$$= \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}(x+1)(x-1)$$

which has sign diagram:



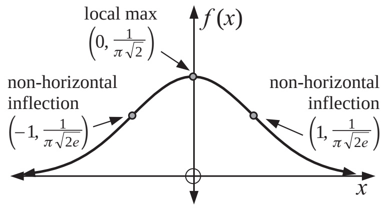
When $x = 1$, $f(x) = \dfrac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}} = \dfrac{1}{\pi\sqrt{2e}}$ and also when $x = -1$, $f(x) = \dfrac{1}{\pi\sqrt{2e}}$

$\therefore$ there are points of inflection at $\left(1, \dfrac{1}{\pi\sqrt{2e}}\right)$ and $\left(-1, \dfrac{1}{\pi\sqrt{2e}}\right)$

c as $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0$ (above),
$\therefore$ $f(x) \to 0$ (above),

as $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0$ (above),
$\therefore$ $f(x) \to 0$ (above)

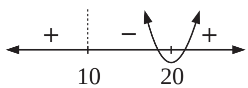
d



20 $C(x) = 4\ln x + \left(\dfrac{30-x}{10}\right)^2$, $x \geqslant 10$ $\therefore$

$$C'(x) = \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\tfrac{1}{10}\right) = \frac{4}{x} - \frac{30-x}{50} = \frac{200 - x(30-x)}{50x} = \frac{200 - 30x + x^2}{50x} = \frac{(x-10)(x-20)}{50x}$$

which has sign diagram:



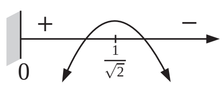
$\therefore$ the minimum cost occurs when $x = 20$, i.e., 20 kettles/day are produced

21 Let coordinates of D be $(x, 0)$, where $x > 0$. $\therefore$ the coordinates of C are (x, e^{-x^2})

$\therefore$ area ABCD $= 2xe^{-x^2}$ $\therefore$

$$\frac{dA}{dx} = 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule}\}$$
$$= 2e^{-x^2}\left(1 - 2x^2\right)$$
$$= 2e^{-x^2}(1+\sqrt{2}x)(1-\sqrt{2}x)$$

and $\dfrac{dA}{dx}$ has sign diagram:



$\therefore$ area is a maximum when $x = \frac{1}{\sqrt{2}}$, and so C is $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

22

$$P(x) = R(x) - C(x)$$
$$\therefore \quad P(x) = \left[1000\ln\left(1 + \frac{x}{400}\right) + 600\right] - [x(1.5) + 300]$$
$$= 1000\ln(1 + 0.0025x) - 1.5x + 300$$
$$\therefore \quad P'(x) = 1000\left(\frac{0.0025}{1+0.0025x}\right) - 1.5 = \frac{2.5}{1+0.0025x} - 1.5$$

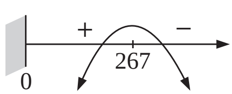
which is 0 when $\dfrac{2.5}{1+0.0025x} = \dfrac{3}{2}$

$\therefore$ $3 + 0.0075x = 5$

$\therefore$ $0.0075x = 2$

$\therefore$ $x = \dfrac{2}{0.0075} \doteqdot 266.7$ and $P(266) \doteqdot 410.83$
$P(267) \doteqdot 410.83$

Sign diagram of $P'(x)$:



$\therefore$ to maximise profit, 266 or 267 torches/day should be produced.

23 **a** $y = ax^2 \quad (a > 0)$ touches $y = \ln x$ when $ax^2 = \ln x$

If they touch at $x = b$

then $ab^2 = \ln b$ ...... (1)

Now if $y = ax^2$ then $\dfrac{dy}{dx} = 2ax$ $\therefore$ at $x = b$, $\dfrac{dy}{dx} = 2ab$

and if $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$ $\therefore$ at $x = b$, $\dfrac{dy}{dx} = \dfrac{1}{b}$

and the slope of the tangent at $x = b$ to the curve $y = ax^2$ is also the slope of the tangent to $y = \ln x$ at $x = b$, $\therefore$ $\dfrac{1}{b} = 2ab$ ...... (2)

b Now $ab^2 = \frac{1}{2}$ {from (2)}

$ab^2 = \ln b$ {from (1)}

$\therefore$ $\ln b = \frac{1}{2}$

$\therefore$ $b = e^{\frac{1}{2}}$ i.e., $b = \sqrt{e}$ and when $x = b = \sqrt{e}$, $y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$

$\therefore$ point of contact is $(\sqrt{e}, \frac{1}{2})$

c $a = \dfrac{1}{2b^2}$ {from (2)} $\therefore$ $a = \dfrac{1}{2(\sqrt{e})^2} = \dfrac{1}{2e}$

d The tangent has slope $2ab = 2\left(\dfrac{1}{2e}\right)\sqrt{e} = \dfrac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$\therefore$ tangent is $\dfrac{y - \frac{1}{2}}{x - \sqrt{e}} = \dfrac{1}{\sqrt{e}}$

$$\therefore \quad y - \tfrac{1}{2} = \frac{1}{\sqrt{e}}\left(x - \sqrt{e}\right)$$
$$\therefore \quad y - \tfrac{1}{2} = \frac{1}{\sqrt{e}}x - 1$$
$$\therefore \quad y = xe^{-\frac{1}{2}} - \tfrac{1}{2}$$

24 $P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$, $0 \leqslant t \leqslant 25$

$= 50\,000(1 + 1000e^{-0.5t})^{-1}$

$\therefore$ $P'(t) = -50\,000(1 + 1000e^{-0.5t})^{-2}\left(-500e^{-0.5t}\right)$

$= 2.5 \times 10^7 e^{-0.5t}(1 + 1000e^{-0.5t})^{-2}$

Now the wasp population is growing the fastest when $\dfrac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.
Maximum occurs when $x = 13.8155$, i.e., $x \doteqdot 13.81$ weeks

25 $f(t) = ate^{bt^2} = (at)\,e^{bt^2}$

$\therefore$ $f'(t) = ae^{bt^2} + ate^{bt^2}(2bt)$ {product rule}

$= ae^{bt^2}(1 + 2bt^2)$

$\therefore$ $f'(t)$ is 0 when $1 + 2bt^2 = 0$ but $t = 2$ at this point,

$\therefore$ $1 + 8b = 0$

$b = -\frac{1}{8}$

$\therefore \quad f(t) = ate^{-\frac{t^2}{8}}$ and $f(2) = 1, \quad \therefore \quad 2ae^{-\frac{4}{8}} = 1$

$\therefore \quad ae^{-\frac{1}{2}} = \frac{1}{2}$ and so $a = \dfrac{\sqrt{e}}{2}$

REVIEW SET 23A

1 **a**

$$\begin{aligned} y &= e^{x^3+2} \\ &= e^u \quad \text{where } u = x^3+2 \\ \therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= e^u \times 3x^2 \\ &= 3x^2e^u \\ &= 3x^2e^{x^3+2} \end{aligned}$$

b

$$y = \frac{e^x}{x^2} = \frac{u}{v}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\ \therefore \quad \frac{dy}{dx} &= \frac{e^x\, x^2 - e^x\, 2x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \frac{e^x(x-2)}{x^3} \end{aligned}$$

2 $y = e^{-x^2}$ and when $x = 1, \quad y = e^{-x^2} = e^{-1} = \dfrac{1}{e}, \quad \therefore$ point of contact is $\left(1, \dfrac{1}{e}\right)$

$$\text{Now} \quad \frac{dy}{dx} = -2xe^{-x^2}$$

$$\therefore \quad \text{at} \quad x = 1, \quad \frac{dy}{dx} = -2e^{-1}$$

$\therefore$ the slope of the tangent $= -\dfrac{2}{e}$ and the slope of the normal $= \dfrac{e}{2}$

$\therefore$ equation of the normal is $\dfrac{y - \frac{1}{e}}{x-1} = \dfrac{e}{2}$ i.e., $2\left(y - \dfrac{1}{e}\right) = e(x-1)$

$$\therefore \quad 2y - \frac{2}{e} = ex - e$$

$$\therefore \quad 2ey - 2 = e^2x - e^2$$

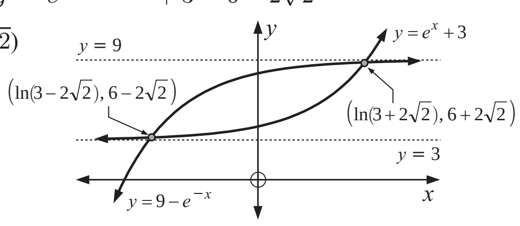
$$\therefore \quad e^2x - 2ey = e^2 - 2 \quad \text{or} \quad y = \frac{e}{2}x - \frac{e}{2} + \frac{1}{e}$$

3 Graphs meet when $e^x + 3 = 9 - e^{-x}$

$$\begin{aligned} \therefore \quad e^{2x} + 3e^x &= 9e^x - 1 \quad \{\times e^x\} \\ \therefore \quad e^{2x} - 6e^x + 1 &= 0 \\ \therefore \quad e^x &= \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2} \\ \therefore \quad x &= \ln(3+2\sqrt{2}) \ \text{ or } \ \ln(3-2\sqrt{2}) \end{aligned}$$

and $y = e^{\ln(3+2\sqrt{2})} + 3 = 6 + 2\sqrt{2}$ or $y = e^{\ln(3-2\sqrt{2})} + 3 = 6 - 2\sqrt{2}$

i.e., graphs meet at $(\ln(3+2\sqrt{2}), 6+2\sqrt{2})$
and $(\ln(3-2\sqrt{2}), 6-2\sqrt{2})$
i.e., $(1.76, 8.83)$ and $(-1.76, 3.17)$



4 **a** $f(x) = \dfrac{e^x}{x-1} \quad \therefore$ when $x = 0, \quad f(x) = \dfrac{e^0}{-1} = -1$

and so the y-intercept is at $(0, -1)$ no x-intercept exist as $\dfrac{e^x}{x-1} \neq 0$

b $f(x)$ is defined for all $x \neq 1$

c Now $f'(x) = \dfrac{e^x(x-1) - e^x(1)}{(x-1)^2}$ {quotient rule}

$= \dfrac{e^x(x-2)}{(x-1)^2}$ and has sign diagram:

$\therefore$ $f(x)$ is decreasing for $x < 1$ and $1 < x \leqslant 2$, and increasing for $x \geqslant 2$.

$$f''(x) = \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1]}{(x-1)^4} \quad \text{\{product and quotient rules\}}$$

$$= \frac{[e^x(x-2+1)](x-1)^2] \;-\; 2e^x(x-2)(x-1)}{(x-1)^4}$$

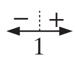
$$= \frac{e^x(x-1)(x-1)^2 \;-\; 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)\left[(x-1)^2 \;-\; 2(x-2)\right]}{(x-1)^4}$$

$$= \frac{e^x(x-1)\left[x^2 - 2x + 1 - 2x + 4\right]}{(x-1)^4}$$

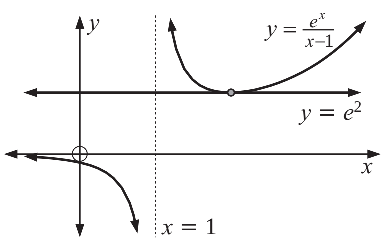
$$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \quad \text{where the quadratic has } \Delta < 0$$

The sign diagram of $f''(x)$ is:



$\therefore$ concave down for all $x \leqslant 1$
and concave up for all $x \geqslant 1$.

d



e Using **c** we have a local minimum at

$(2,\, e^2)$ {as $f(2) = \dfrac{e^2}{2-1} = e^2$}

$\therefore$ the tangent at $x = 2$ is horizontal and is $y = e^2$

5 $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$

a when planted $t = 0$ $\therefore$ $H(0) = 60 + 40\ln(1) = 60 + 40(0) = 60$ cm

b **i** When $H(t) = 150$ cm,

$\therefore$ $60 + 40\ln(2t+1) = 150$

$\therefore$ $40\ln(2t+1) = 90$

$\therefore$ $\ln(2t+1) = \frac{90}{40} = 2.25$

$\therefore$ $2t + 1 = e^{2.25}$

$\therefore$ $2t = e^{2.25} - 1$

$\therefore$ $t = \frac{1}{2}(e^{2.25} - 1)$

$\therefore$ $t \doteqdot 4.244$ years

ii When $H(t) = 300$ cm,

$\therefore$ $60 + 40\ln(2t+1) = 300$

$\therefore$ $40\ln(2t+1) = 240$

$\therefore$ $\ln(2t+1) = 6$

$\therefore$ $2t + 1 = e^6$

$\therefore$ $2t = e^6 - 1$

$\therefore$ $t = \frac{1}{2}(e^6 - 1)$

$\therefore$ $t \doteqdot 201.2$ years

c $H'(t) = 40\left(\dfrac{2}{2t+1}\right) = \dfrac{80}{2t+1}$ cm/year

i When $t = 2$, $H'(2) = \frac{80}{5} = 16$ cm/year

ii When $t = 20$, $H'(20) = \frac{80}{41} \doteqdot 1.95$ cm/year

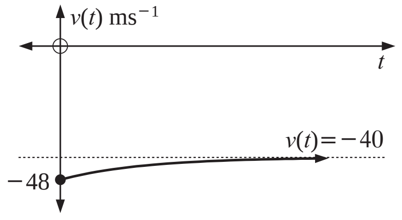
6 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geqslant 0$

a $\therefore\ v(t) = -8e^{-\frac{t}{10}} - 40$ m/sec and $a(t) = 0.8e^{-\frac{t}{10}}$ m/sec^2

b When $t = 0$, $s(0) = 80$ m $v(0) = -48$ m/sec $a(0) = 0.8$ m/sec^2

c as $t \to \infty$, $e^{-\frac{t}{10}} \to 0$, $\therefore\ v(t) \to -40$ m/sec (below)

d



e when $v(t) = -44$ m/sec

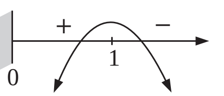
$$-8e^{-\frac{t}{10}} - 40 = -44$$
$$\therefore\ -8e^{-\frac{t}{10}} = -4$$
$$\therefore\ e^{-\frac{t}{10}} = 0.5$$
$$\therefore\ -\tfrac{t}{10} = \ln 0.5$$
$$\therefore\ t = -10\ln 0.5$$
$$\therefore\ t \doteqdot 6.931 \text{ seconds}$$

7 Let the coordinates of B be $(x, 0)$

$\therefore$ the coordinates of A are (x, e^{-x})

$$\therefore\ \text{area OABC} = xe^{-x}$$
$$\therefore\ \frac{dA}{dx} = (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\}$$
$$= e^{-x}(1-x)$$
$$= \frac{1-x}{e^x}$$

and has sign diagram:



$\therefore$ has a local maximum when $x = 1$

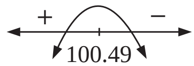
and when $x = 1$, $y = 1e^{-1} = \dfrac{1}{e}$

$\therefore$ the coordinates of A are $\left(1, \dfrac{1}{e}\right)$

8
$$P(x) = R(x) - C(x)$$
$$= \left[200\ln\left(1 + \tfrac{x}{100}\right) + 1000\right] - \left[(x-100)^2 + 200\right]$$
$$= 200\ln(1 + 0.01x) - (x-100)^2 + 800$$
$$\frac{dP}{dx} = 200\left(\frac{0.01}{1+0.01x}\right) - 2(x-100)^1$$
$$= \frac{2}{1+0.01x} - \frac{2(x-100)}{1}$$
$$= \frac{2 - 2(x-100)(1+0.01x)}{1+0.01x}$$
$$= \frac{2 - 2(x + 0.01x^2 - 100 - x)}{1+0.01x}$$
$$= \frac{2 - 0.02x^2 + 200}{1+0.01x}$$
$$= \frac{202 - 0.02x^2}{1+0.01x}$$

which is 0 when $0.02x^2 = 202$

$$\therefore\ x^2 = 10\,100$$
$$\therefore\ x = \sqrt{10\,100} \quad \{\text{as } x > 0\}$$
$$\therefore\ x \doteqdot 100.49$$

and sign diagram of $\dfrac{dP}{dx}$ is: 

$\therefore$ maximum profit when $x \doteqdot 100.49$

Now $P(100) = \$938.63$ and $P(101) = \$938.63$

$\therefore$ maximum profit of $\$938.63$ when 100 or 101 shirts are made.

REVIEW SET 23B

1 **a**
$$\begin{aligned} y &= \ln(x^3 - 3x) \\ \therefore\ y &= \ln u \quad \text{where} \quad u = x^3 - 3x \\ \text{Now}\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u}(3x^2 - 3) \\ \therefore\ \frac{dy}{dx} &= \frac{3x^2 - 3}{x^3 - 3x} \end{aligned}$$

b
$$\begin{aligned} y &= \ln\left(\frac{x+3}{x^2}\right) \\ &= \ln(x+3) - \ln x^2 \\ &= \ln(x+3) - 2\ln x \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x+3} - \frac{2}{x} \end{aligned}$$

2
$$\begin{aligned} y &= \ln(x^2 + 3) \\ \therefore\ y &= \ln u \quad \text{where} \quad u = x^2 + 3 \\ \text{Now}\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{1}{u} \times 2x \\ \therefore\ \frac{dy}{dx} &= \frac{2x}{x^2+3} \end{aligned}$$
and when $x = 0$, $\dfrac{dy}{dx} = 0$, i.e., the slope of the tangent is 0.

But at $x = 0$, $y = \ln(0+3) = \ln 3$

$\therefore$ tangent is $y = \ln 3$ which does not cut the x-axis.

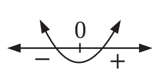
3 **a**
$$\begin{aligned} e^{2x} &= 3e^x \\ \therefore\ e^{2x} - 3e^x &= 0 \\ \therefore\ e^x(e^x - 3) &= 0 \\ \therefore\ e^x &= 0 \text{ or } 3 \\ \therefore\ e^x &= 3 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 3 \end{aligned}$$

b
$$\begin{aligned} e^{2x} - 7e^x + 12 &= 0 \\ \therefore\ (e^x - 3)(e^x - 4) &= 0 \\ \therefore\ e^x &= 3 \text{ or } 4 \\ \therefore\ x &= \ln 3 \text{ or } \ln 4 \end{aligned}$$

4 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

which is 0 when $e^x = 1$
i.e., $x = 0$

Sign diagram of $f'(x)$ is: 

and at $x = 0$, $f(0) = e^0 - 0 = 1$

$\therefore$ there is a local minimum at (0, 1)

b as $x \to +\infty$, $e^x \to \infty$ faster than x $\therefore$ $f(x) \to +\infty$,

as $x \to -\infty$, $e^x \to 0$

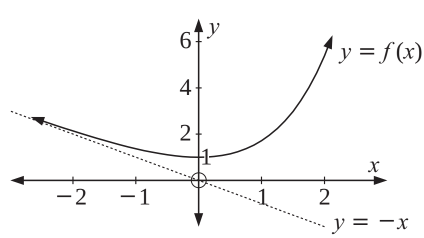
$\therefore$ $f(x) \to -x$ (from above)

c $f''(x) = e^x$

$\therefore$ $f''(x) > 0$ for all x

$\therefore$ $f(x)$ is concave up for all x

d



e Since a local minimum exists at (0, 1), $f(x) \geqslant 1$ for all x.

$$\therefore \quad e^x - x \geqslant 1$$

$$\therefore \quad e^x \geqslant x + 1 \quad \text{for all } x$$

5 **a** $f(x) = \ln(e^x + 3)$

$\therefore \quad y = \ln u \quad$ where $\quad u = e^x + 3$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u} \times e^x \\ &= \frac{e^x}{u} \\ \therefore \quad f'(x) &= \frac{e^x}{e^x + 3} \end{aligned}$$

b

$$\begin{aligned} f(x) &= \ln\left(\frac{(x+2)^3}{x}\right) \\ &= \ln(x+2)^3 - \ln x \\ &= 3\ln(x+2) - \ln x \\ \therefore \quad f'(x) &= \frac{3}{x+2} - \frac{1}{x} \\ &= \frac{3x - (x+2)}{x(x+2)} \\ \therefore \quad f'(x) &= \frac{2x-2}{x(x+2)} \end{aligned}$$

6 **a**

$$\begin{aligned} 3e^x - 5 &= -2e^{-x} \\ \therefore \quad 3e^{2x} - 5e^x &= -2 \qquad \{\times \text{ by } e^x\} \\ \therefore \quad 3e^{2x} - 5e^x + 2 &= 0 \\ \therefore \quad (3e^x - 2)(e^x - 1) &= 0 \\ \therefore \quad e^x &= \tfrac{2}{3} \text{ or } 1 \\ \therefore \quad x &= \ln\tfrac{2}{3} \text{ or } 0 \end{aligned}$$

b

$$\begin{aligned} 2\ln x - 3\ln\left(\frac{1}{x}\right) &= 10 \\ \therefore \quad 2\ln x - 3\ln(x^{-1}) &= 10 \\ \therefore \quad 2\ln x + 3\ln x &= 10 \\ \therefore \quad 5\ln x &= 10 \\ \therefore \quad \ln x &= 2 \\ \therefore \quad x &= e^2 \end{aligned}$$

7 $s(t) = 25t - 10\ln t$ cm $\quad t \geqslant 1$

a

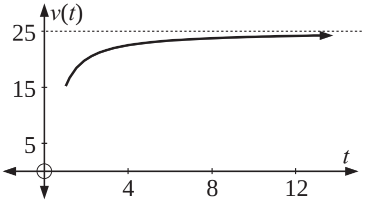
$$\begin{aligned} v(t) &= 25 - \tfrac{10}{t} \text{ cm/min} \\ \therefore \quad a(t) &= 10t^{-2} \\ &= \frac{10}{t^2} \text{ cm/min}^2 \end{aligned}$$

b when $\quad t = e,$

$$\begin{aligned} s(e) &= 25e - 10\ln e = 25e - 10 \text{ cm} \doteqdot 57.96 \text{ cm} \\ v(e) &= 25 - \frac{10}{e} \text{ cm/min} \doteqdot 21.32 \text{ cm/min} \\ a(e) &= \frac{10}{e^2} \text{ cm/min}^2 \doteqdot 1.35 \text{ cm/min}^2 \end{aligned}$$

c as $\quad t \to +\infty, \quad \dfrac{10}{t} \to 0, \quad \therefore \quad v(t) \to 25$ cm/min (below)

d



e when $\quad v(t) = 12$ cm/min,

$$\begin{aligned} 25 - \frac{10}{t} &= 12 \\ \therefore \quad \frac{10}{t} &= 13 \\ \therefore \quad t &= \tfrac{10}{13} \text{ minutes} \end{aligned}$$

8 $$C(x) = 10\ln x + \left(20 - \frac{x}{10}\right)^2 = 10\ln x + 400 - 4x + \frac{x^2}{100}$$

$$\therefore \quad C'(x) = \frac{10}{x} - 4 + \frac{x}{50}$$

$$= \frac{500 - 200x + x^2}{50x}$$

which is 0 when $x^2 - 200x + 500 = 0$

i.e., when $x = \dfrac{200 \pm \sqrt{38\,000}}{2} \doteqdot 2.53$ or 197.47

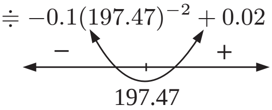
But $x \geqslant 50$, $x \doteqdot 197.47$ and $C(197) \doteqdot 52.92$

$C(198) \doteqdot 52.92$

Now $C''(x) = -10x^{-2} + \frac{1}{50}$, $\therefore$ $C''(197.47) \doteqdot -0.1(197.47)^{-2} + 0.02 \doteqdot 0.02$ which is > 0

$\therefore$ minimum cost when $x \doteqdot 197.46$

i.e., need to produce 197 or 198 clocks/day



Chapter 24
DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

EXERCISE 24A

1 **a**
$$\begin{aligned} y &= \sin(2x) \\ \therefore \frac{dy}{dx} &= \cos(2x)\frac{d}{dx}(2x) \\ &= 2\cos(2x) \end{aligned}$$

b
$$\begin{aligned} y &= \sin x + \cos x \\ \therefore \frac{dy}{dx} &= \cos x - \sin x \end{aligned}$$

c
$$\begin{aligned} y &= \cos(3x) - \sin x \\ \therefore \frac{dy}{dx} &= -\sin(3x) \times 3 - \cos x \\ &= -3\sin(3x) - \cos x \end{aligned}$$

d
$$\begin{aligned} y &= \sin(x+1) \\ \therefore \frac{dy}{dx} &= \cos(x+1)\frac{d}{dx}(x+1) \\ &= 1\cos(x+1) \\ &= \cos(x+1) \end{aligned}$$

e
$$\begin{aligned} y &= \cos(3-2x) \\ \therefore \frac{dy}{dx} &= -\sin(3-2x) \times -2 \\ &= 2\sin(3-2x) \end{aligned}$$

f
$$\begin{aligned} y &= \tan(5x) \\ \therefore \frac{dy}{dx} &= \frac{1}{\cos^2(5x)} \times 5 \\ &= \frac{5}{\cos^2(5x)} \end{aligned}$$

g
$$\begin{aligned} y &= \sin\left(\tfrac{x}{2}\right) - 3\cos x \\ \therefore \frac{dy}{dx} &= \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right) + 3\sin x \end{aligned}$$

h
$$\begin{aligned} y &= 3\tan(\pi x) \\ \therefore \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(\pi x)} \times \pi \\ &= \frac{3\pi}{\cos^2(\pi x)} \end{aligned}$$

i
$$\begin{aligned} y &= 4\sin x - \cos(2x) \\ \therefore \frac{dy}{dx} &= 4\cos x + \sin(2x) \times 2 \\ &= 4\cos x + 2\sin(2x) \end{aligned}$$

2 **a**
$$\begin{aligned} y &= x^2 + \cos x \\ \therefore \frac{dy}{dx} &= 2x - \sin x \end{aligned}$$

b
$$\begin{aligned} y &= \tan x - 3\sin x \\ \therefore \frac{dy}{dx} &= \frac{1}{\cos^2 x} - 3\cos x \end{aligned}$$

c
$$\begin{aligned} y &= e^x \cos x \\ \therefore \frac{dy}{dx} &= e^x(-\sin x) + e^x \cos x \\ &= e^x \cos x - e^x \sin x \end{aligned}$$

d
$$\begin{aligned} y &= e^{-x} \sin x \\ \therefore \frac{dy}{dx} &= -e^{-x}\sin x + e^{-x}\cos x \end{aligned}$$

e
$$\begin{aligned} y &= \ln(\sin x) \\ \therefore \frac{dy}{dx} &= \frac{\cos x}{\sin x} \end{aligned}$$

f
$$\begin{aligned} y &= e^{2x}\tan x \\ \therefore \frac{dy}{dx} &= 2e^{2x}\tan x + e^{2x} \times \frac{1}{\cos^2 x} \\ \therefore \frac{dy}{dx} &= 2e^{2x}\tan x + \frac{e^{2x}}{\cos^2 x} \end{aligned}$$

g
$$\begin{aligned} y &= \sin(3x) \\ \therefore \frac{dy}{dx} &= 3\cos(3x) \end{aligned}$$

h
$$\begin{aligned} y &= \cos(\tfrac{x}{2}) \\ \therefore \frac{dy}{dx} &= -\tfrac{1}{2}\sin(\tfrac{x}{2}) \end{aligned}$$

i
$$\begin{aligned} y &= 3\tan(2x) \\ \therefore\ \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(2x)} \times 2 \\ &= \frac{6}{\cos^2(2x)} \end{aligned}$$

j
$$\begin{aligned} y &= x\cos x \\ \therefore\ \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\ &= \cos x - x\sin x \end{aligned}$$

k
$$\begin{aligned} y &= \frac{\sin x}{x} \\ \therefore\ \frac{dy}{dx} &= \frac{x(\cos x) - \sin x \times 1}{x^2} \\ &= \frac{x\cos x - \sin x}{x^2} \end{aligned}$$

l
$$\begin{aligned} y &= x\tan x \\ \therefore\ \frac{dy}{dx} &= 1 \times \tan x + x \times \frac{1}{\cos^2 x} \\ \therefore\ \frac{dy}{dx} &= \tan x + \frac{x}{\cos^2 x} \end{aligned}$$

3 **a**
$$\begin{aligned} y &= \sin(x^2) \\ \therefore\ \frac{dy}{dx} &= 2x\cos(x^2) \end{aligned}$$

b
$$\begin{aligned} y &= \cos\left(\sqrt{x}\right) \\ &= \cos(x^{\frac{1}{2}}) \\ \therefore\ \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{x}}\sin(\sqrt{x}) \end{aligned}$$

c
$$\begin{aligned} y &= \sqrt{\cos x} \\ &= (\cos x)^{\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}} \end{aligned}$$

d
$$\begin{aligned} y &= \sin^2 x \\ &= (\sin x)^2 \\ \therefore\ \frac{dy}{dx} &= 2\sin x\cos x \end{aligned}$$

e
$$\begin{aligned} y &= \cos^3 x \\ &= (\cos x)^3 \\ \therefore\ \frac{dy}{dx} &= 3\cos^2 x \times (-\sin x) \\ &= -3\sin x\cos^2 x \end{aligned}$$

f
$$\begin{aligned} y &= \cos x\sin(2x) \\ \therefore\ \frac{dy}{dx} &= (-\sin x)\sin(2x) + \cos x(2\cos(2x)) \\ &= -\sin x\sin(2x) + 2\cos x\cos(2x) \end{aligned}$$

g
$$\begin{aligned} y &= \cos(\cos x) \\ \therefore\ \frac{dy}{dx} &= -\sin(\cos x) \times (-\sin x) \\ &= \sin x \times \sin(\cos x) \end{aligned}$$

h
$$\begin{aligned} y &= \cos^3(4x) \\ &= (\cos(4x))^3 \\ \therefore\ \frac{dy}{dx} &= 3(\cos(4x))^2 \times (-4\sin(4x)) \\ &= -12\sin(4x)\cos^2(4x) \end{aligned}$$

i
$$\begin{aligned} y &= \frac{1}{\sin x} \\ &= (\sin x)^{-1} \\ \therefore\ \frac{dy}{dx} &= -1(\sin x)^{-2} \times \cos x \\ &= -\frac{\cos x}{\sin^2 x} \end{aligned}$$

j
$$\begin{aligned} y &= \frac{1}{\cos(2x)} \\ &= (\cos(2x))^{-1} \\ \therefore\ \frac{dy}{dx} &= -1(\cos(2x))^{-2} \times (-2\sin(2x)) \\ &= \frac{2\sin(2x)}{\cos^2(2x)} \end{aligned}$$

k
$$\begin{aligned} y &= \frac{2}{\sin^2(2x)} \\ &= 2(\sin(2x))^{-2} \\ \therefore \quad \frac{dy}{dx} &= -4(\sin(2x))^{-3} \times 2\cos(2x) \\ &= -\frac{8\cos(2x)}{\sin^3(2x)} \end{aligned}$$

l
$$\begin{aligned} y &= \frac{8}{\tan^3\left(\frac{x}{2}\right)} = 8\left[\tan\left(\frac{x}{2}\right)\right]^{-3} \\ \therefore \quad \frac{dy}{dx} &= -24\left[\tan\left(\frac{x}{2}\right)\right]^{-4} \times \frac{1}{2} \times \frac{1}{\cos^2\left(\frac{x}{2}\right)} \\ &= \frac{-12}{\cos^2\left(\frac{x}{2}\right)\tan^4\left(\frac{x}{2}\right)} \end{aligned}$$

4 **a** If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$, $\dfrac{d^2y}{dx^2} = -\sin x$, $\dfrac{d^3y}{dx^3} = -\cos x$ and $\dfrac{d^4y}{dx^4} = \sin x$

b Successive derivatives will cycle through the pattern,

so $\dfrac{d^ny}{dx^n}$ may only take the values found in **a**.

$$\begin{tikzcd} & \sin x \arrow[dr] & \\ -\cos x \arrow[ur] & & \cos x \arrow[dl] \\ & -\sin x \arrow[ul] & \end{tikzcd}$$

5 **a** If $y = \sin(2x+3)$, then $\dfrac{dy}{dx} = 2\cos(2x+3)$ and $\dfrac{d^2y}{dx^2} = -4\sin(2x+3)$

$$\therefore \quad \frac{d^2y}{dx^2} + 4y = -4\sin(2x+3) + 4\sin(2x+3) = 0$$

b
$$\begin{aligned} \text{If} \quad y &= 2\sin x + 3\cos x \\ \text{then} \quad y' &= 2\cos x - 3\sin x \\ \text{and} \quad y'' &= -2\sin x - 3\cos x \\ \therefore \quad y'' + y &= -2\sin x - 3\cos x + 2\sin x + 3\cos x = 0 \end{aligned}$$

c
$$\begin{aligned} y &= \frac{\cos x}{1+\sin x} \\ \therefore \quad \frac{dy}{dx} &= \frac{(-\sin x)(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\ &= \frac{-1-\sin x}{(1+\sin x)^2} \quad \{\text{as} \quad \sin^2 x + \cos^2 x = 1\} \\ &= -\frac{(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{-1}{1+\sin x} \end{aligned}$$

Since $\dfrac{-1}{1+\sin x}$ never equals 0, there are no horizontal tangents.

6 **a**
$$\begin{aligned} y &= \sin x \\ \therefore \quad \frac{dy}{dx} &= \cos x \end{aligned}$$

$\therefore$ At $x = 0$, the tangent has slope $\cos 0 = 1$

$\therefore$ the equation is $\dfrac{y-0}{x-0} = 1$

i.e., $y = x$

b
$$\begin{aligned} y &= \tan x \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{\cos^2 x} \end{aligned}$$

$\therefore$ at $x = 0$, the tangent has slope $\dfrac{1}{\cos^2 0} = 1$

so the equation is $\dfrac{y-0}{x-0} = 1$

i.e., $y = x$

c $y = \cos x$ and at $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

$\therefore \quad \frac{dy}{dx} = -\sin x$

At $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$ and the tangent has slope $-\sin\frac{\pi}{6} = -\frac{1}{2}$

$\therefore$ the normal has slope 2,

so its equation is $\dfrac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

i.e. $y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$

i.e. $2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

d $y = \dfrac{1}{\sin(2x)} = (\sin(2x))^{-1}$

$$\therefore \quad \frac{dy}{dx} = -1(\sin(2x))^{-2} \times 2\cos(2x) = -\frac{2\cos(2x)}{(\sin(2x))^2}$$

At $x = \frac{\pi}{4}$, $y = 1$ and the tangent has slope $-\dfrac{2\cos\frac{\pi}{2}}{\left(\sin\frac{\pi}{2}\right)^2} = 0$

$\therefore$ the slope of the normal is undefined, so the normal is $x = \frac{\pi}{4}$.

7 $d = 9.3 + 6.8\cos(0.507t)$ m $\quad \therefore \quad \dfrac{dd}{dt} = -6.8\sin(0.507t) \times 0.507 = -3.4476\sin(0.507t)$

a When $t = 8$, $\dfrac{dd}{dt} \doteqdot 2.731 > 0$

$\therefore$ the tide is rising.

b When $t = 8$, the tide is rising at the rate of 2.731 m per hour.

8 **a** $V(t) = 340\sin(100\pi t)$

$$\therefore \quad \frac{dV}{dt} = 340\cos(100\pi t) \times 100\pi = 34\,000\pi\cos(100\pi t)$$

When $t = 0.01$,

$$\frac{dV}{dt} = 34\,000\pi \times \cos\pi = -34\,000\pi \text{ units/second}$$

b $V(t)$ is a maximum when $\sin(100\pi t) = 1$.

This occurs when $100\pi t = \frac{\pi}{2}$, and at this time,

$$\frac{dV}{dt} = 34\,000\pi\cos(100\pi t) = 34\,000\pi\cos\left(\tfrac{\pi}{2}\right) = 0 \text{ units/second}$$

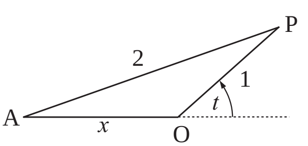
9 **a** The distance from $A(-x, 0)$ to $P(\cos t, \sin t)$ is fixed at 2 m.

$\therefore \quad (\cos t + x)^2 + \sin^2 t = 2^2$

$\therefore \quad (\cos t + x)^2 = 4 - \sin^2 t$

$\therefore \quad x + \cos t = \pm\sqrt{4 - \sin^2 t}$

$\therefore$ since $x > 0$, $x = \sqrt{4 - \sin^2 t} - \cos t$



b Now $\dfrac{dx}{dt} = \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2\sin t\cos t) + \sin t$

$$= \frac{-\sin t\cos t}{\sqrt{4 - \sin^2 t}} + \sin t$$

i When $t = 0$,

$\sin t = 0$ and $\cos t = 1$

$\therefore \quad \dfrac{dx}{dt} = 0 + 0 = 0$

ii When $t = \frac{\pi}{2}$,

$\sin t = 1$ and $\cos t = 0$

$\therefore \quad \dfrac{dx}{dt} = 0 + \sin\frac{\pi}{2} = 1$

iii When $t = \frac{2\pi}{3}$,

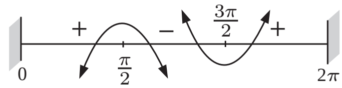
$\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = -\frac{1}{2}$

$$\therefore \quad \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right)}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2} \doteqdot 1.106$$

10 **a** If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$

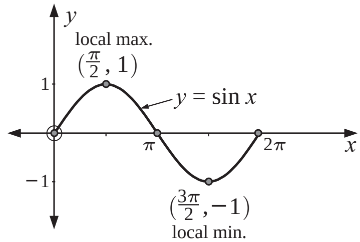
Stationary points occur when $\dfrac{dy}{dx} = 0$, i.e., when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{2}, 1)$.

Local minimum at $(\frac{3\pi}{2}, -1)$.



b If $y = \cos(2x)$ then $\dfrac{dy}{dx} = -2\sin(2x)$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad -2\sin(2x) = 0$$

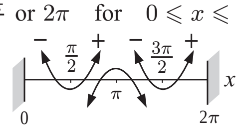
$$\therefore \quad \sin(2x) = 0$$

$$\therefore \quad 2x = k\pi \quad \text{for any integer } k$$

$$\therefore \quad x = \frac{k\pi}{2}$$

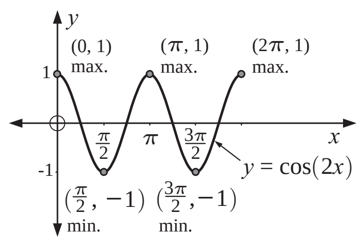
$$\therefore \quad x = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2} \text{ or } 2\pi \quad \text{for} \quad 0 \leqslant x \leqslant 2\pi$$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$.

Local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.



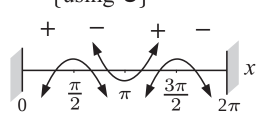
c If $y = \sin^2 x$ then $\dfrac{dy}{dx} = 2\sin x \cos x = \sin(2x)$

$$\therefore \quad \frac{dy}{dx} = 0 \quad \text{when} \quad \sin(2x) = 0$$

i.e., when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or 2π for $0 \leqslant x \leqslant 2\pi$

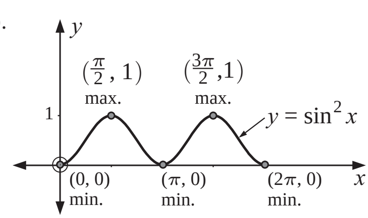
{using **b**}

Sign diagram for $\dfrac{dy}{dx}$ is:



Local mimima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$.

Local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.



11 **a** $f(x) = \dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant 2\pi$

$\therefore$ $f(x)$ is undefined whenever

$\cos x = 0$

i.e., $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b

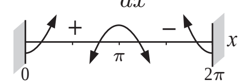
$$f(x) = (\cos x)^{-1}$$

$$\therefore \quad f'(x) = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$$

$\therefore$ $f'(x) = 0$ when $\sin x = 0$

i.e., $x = 0, \pi, 2\pi$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local minima at $(0, 1)$, $(2\pi, 1)$

Local maximum at $(\pi, -1)$

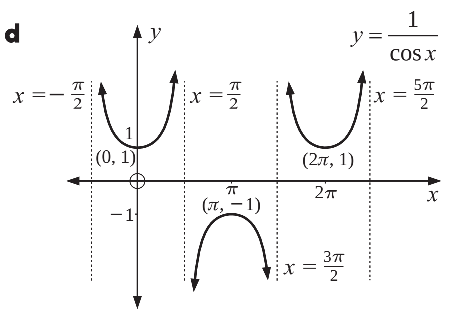
c

$$f(x) = \frac{1}{\cos x}$$

$$\therefore \quad f(x + 2\pi) = \frac{1}{\cos(x + 2\pi)} = \frac{1}{\cos x} = f(x)$$

i.e., $f(x)$ has a period of 2π.

d



12 If $y = \sin(2x) + 2\cos x$

then

$$\frac{dy}{dx} = 2\cos(2x) + (-2\sin x) = 2[1 - 2\sin^2 x] - 2\sin x = 2 - 4\sin^2 x - 2\sin x$$

At the stationary points, $\dfrac{dy}{dx} = 0$.

$$\therefore \quad -4\sin^2 x - 2\sin x + 2 = 0$$

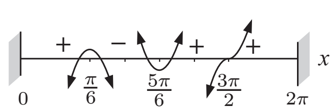
$$\therefore \quad 2\sin^2 x + \sin x - 1 = 0$$

$$\therefore \quad (2\sin x - 1)(\sin x + 1) = 0$$

$$\therefore \quad \sin x = \tfrac{1}{2} \quad \text{or} \quad \sin x = -1$$

$$\therefore \quad x = \tfrac{\pi}{6}, \tfrac{5\pi}{6} \quad \text{or} \quad x = \tfrac{3\pi}{2}$$

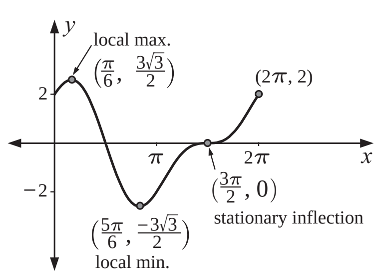
Sign diagram for $\dfrac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$.

Local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$.

Stationary inflection at $(\frac{3\pi}{2}, 0)$.



13 $x(t) = 1 - 2\cos t$ cm

$\therefore \quad v(t) = x'(t) = 2\sin t \quad$ and $\quad a(t) = v'(t) = 2\cos t$

a When $t = 0$

$$\begin{aligned} x(0) &= 1 - 2\cos(0) \\ &= -1 \text{ cm} \\ v(0) &= 2\sin(0) \\ &= 0 \text{ cm/s} \\ a(0) &= 2\cos(0) \\ &= 2 \text{ cm/s}^2 \end{aligned}$$

b When $t = \frac{\pi}{4}$

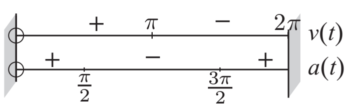
$$\begin{aligned} x(\tfrac{\pi}{4}) &= 1 - \tfrac{2}{\sqrt{2}} = 1 - \sqrt{2} \text{ cm} \\ v(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s} \\ a(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}^2 \end{aligned}$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm/s with increasing speed.

c The particle stops momentarily before it reverses direction. Therefore we need to look for the points where the velocity equals zero, i.e., when

$$\begin{aligned} v(t) = 2\sin t &= 0 \\ \therefore \quad \sin t &= 0 \\ \therefore \quad t &= 0, \pi, 2\pi \quad (0 \leqslant t \leqslant 2\pi) \end{aligned}$$

Sign diagrams for $v(t)$ and $a(t)$ are:



d The particle's speed is increasing when $v(t) = 2\sin t$ and $a(t) = 2\cos t$ have the same sign.

Considering $a(t) = 0$

$$\begin{aligned} \text{i.e.,} \quad 2\cos t &= 0 \\ \cos t &= 0 \\ t &= \tfrac{\pi}{2}, \tfrac{3\pi}{2} \end{aligned}$$

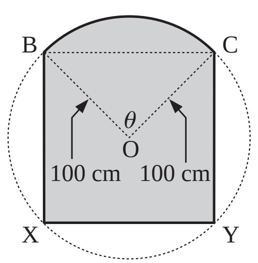
i.e., $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

The particle reverses direction at $t = 0, \ \pi, \ 2\pi \quad (0 \leqslant t \leqslant 2\pi)$

At $t = 0, \quad x(0) = -1$ cm $\qquad$ At $t = \pi, \quad x(\pi) = 3$ cm $\qquad$ At $t = 2\pi, \quad x(2\pi) = -1$ cm

EXERCISE 24B

1



Using the cosine rule in ΔBCO,

$$\begin{aligned} \text{BC}^2 &= 10^2 + 10^2 - 2 \times 10 \times 10\cos\theta \\ \therefore \quad \text{BC} &= \sqrt{200 - 200\cos\theta} \\ \text{and XY} &= \text{BC} \\ \text{Now} \quad \text{BY}^2 &= \text{BX}^2 + \text{XY}^2 \qquad \{\text{Pythagoras}\} \\ \therefore \quad 400 &= \text{BX}^2 + (200 - 200\cos\theta) \\ \therefore \quad \text{BX}^2 &= 200 + 200\cos\theta \\ \therefore \quad \text{BX} &= \sqrt{200 + 200\cos\theta} \end{aligned}$$

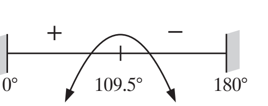
The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of BCYX

$$\begin{aligned} \therefore \quad A &= \tfrac{1}{2}(10)^2\theta + \tfrac{3}{4}[\text{BX} \times \text{BC}] \\ &= 50\theta + \tfrac{3}{4}\sqrt{200 + 200\cos\theta}\sqrt{200 - 200\cos\theta} \\ &= 50\theta + \tfrac{3}{4} \times 200\sqrt{1 + \cos\theta}\sqrt{1 - \cos\theta} \\ &= 50\theta + 150\sqrt{1 - \cos^2\theta} \\ &= 50\theta + 150\sin\theta \\ &= 50(\theta + 3\sin\theta) \quad \text{as required} \end{aligned}$$

$$\therefore \quad \frac{dA}{d\theta} = 50 + 150\cos\theta = 50(1 + 3\cos\theta),$$

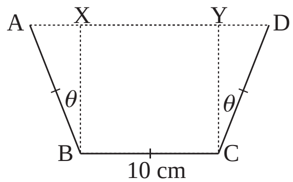
which is zero when $\cos\theta = -\frac{1}{3}$

The sign diagram of $\dfrac{dA}{d\theta}$ is:



$\therefore \quad$ since $0 < \theta < 180$, max A is when $\theta \doteqdot 109.5^o$.

2 **a**



The triangles have height $10\cos\theta$ and width $10\sin\theta$.

$$\begin{aligned}\therefore\ & \text{area } A \\ &= \text{area of } \Delta\text{s} + \text{area of rectangle} \\ &= 2\times\tfrac{1}{2}\times 10\cos\theta\times 10\sin\theta \ + \ 10\times 10\cos\theta \\ &= 100\sin\theta\cos\theta + 100\cos\theta \\ &= 100\cos\theta(1+\sin\theta)\end{aligned}$$

b

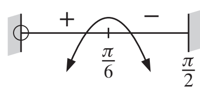
$$\begin{aligned}\frac{dA}{d\theta} &= -100(2\sin\theta-1)(\sin\theta+1) \\ &= 100(-\sin\theta(1+\sin\theta)+\cos\theta\times\cos\theta) \\ &= 100(-\sin\theta-\sin^2\theta+\cos^2\theta) \\ &= 100(-\sin\theta-\sin^2\theta+1-\sin^2\theta) \\ &= -100(2\sin^2\theta+\sin\theta-1) \\ &= -100(2\sin\theta-1)(\sin\theta+1)\end{aligned}$$

$\therefore\ \dfrac{dA}{d\theta} = 0$ when $2\sin\theta - 1 = 0$ *or* $\sin\theta + 1 = 0$

i.e., $\sin\theta = \frac{1}{2}$ *or* $\sin\theta = -1$

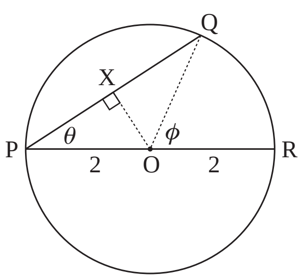
c Using **b**, $\dfrac{dA}{d\theta} = 0$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$.

Sign diagram for $\dfrac{dA}{d\theta}$ is:



so the maximum area is when $\theta = \frac{\pi}{6} = 30^o$

3



$\dfrac{\text{PX}}{2} = \cos\theta, \quad \therefore\ \text{PQ} = 2\text{PX} = 4\cos\theta$

$\therefore$ the time taken to row from P to Q is

$\dfrac{4\cos\theta}{3}$ hours

Now $\phi = 2\theta$ {angle at the centre}

But, arc length $\text{QR}_{\text{arc}} = 2\phi$,

$\therefore\ \text{QR}_{\text{arc}} = 4\theta$,

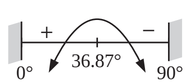
and the time taken to walk from Q to R is $\dfrac{4\theta}{5}$

$\therefore$ total time from P to R, $T = \frac{4}{3}\cos\theta + \dfrac{4\theta}{5}$

$\therefore\ \dfrac{dT}{d\theta} = -\frac{4}{3}\sin\theta + \frac{4}{5}$

$$\begin{aligned}\therefore\ \frac{dT}{d\theta} = 0 \quad \text{when} \quad -\tfrac{4}{3}\sin\theta &= -\tfrac{4}{5} \\ \therefore\ \sin\theta &= \tfrac{3}{5} \\ \therefore\ \theta &\doteqdot 0.6435 \text{ radians} \\ \text{i.e.,}\quad \theta &\doteqdot 36.87^o\end{aligned}$$

sign diagram of $\dfrac{dT}{d\theta}$ is:



i.e., the maximum time is when $\theta \doteqdot 36.87^o$

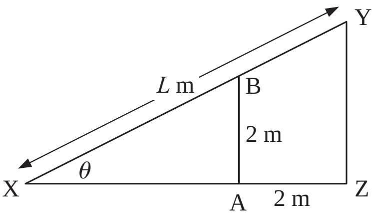
and the maximum time is $\frac{4}{3}\cos 0.6435 + \frac{4}{5}\times 0.6435$

$\doteqdot 1.581$ hours

$\doteqdot 1$ hour 34 min 53 sec

4 **a** $\tan\theta = \dfrac{2}{\text{AX}}$ and $\sin\theta = \dfrac{2}{\text{BX}}$

$\therefore \text{AX} = \dfrac{2}{\tan\theta} = \dfrac{2\cos\theta}{\sin\theta}$ and $\text{BX} = \dfrac{2}{\sin\theta}$



From the similar Δ's, $\dfrac{L}{\text{BX}} = \dfrac{\text{AX}+2}{\text{AX}} = 1 + \dfrac{2}{\text{AX}}$

$$\therefore \quad L = \text{BX} + \frac{2\text{BX}}{\text{AX}}$$

$$\therefore \quad L = \frac{2}{\sin\theta} + \frac{2\left(\frac{2}{\sin\theta}\right)}{\left(\frac{2\cos\theta}{\sin\theta}\right)}$$

$$\therefore \quad L = \frac{2}{\sin\theta} + 2\left(\frac{2}{\sin\theta}\right)\left(\frac{\sin\theta}{2\cos\theta}\right)$$

$$\therefore \quad L = \frac{2}{\sin\theta} + \frac{2}{\cos\theta}, \quad \text{as required.}$$

b Now $L = 2(\cos\theta)^{-1} + 2(\sin\theta)^{-1}$

$$\therefore \quad \frac{dL}{d\theta} = -2(\cos\theta)^{-2}(-\sin\theta) + -2(\sin\theta)^{-2}(\cos\theta)$$

$$= \frac{2\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$= \frac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}$$

c Now $\dfrac{dL}{d\theta} = 0$ when $2\sin^3\theta - 2\cos^3\theta = 0$

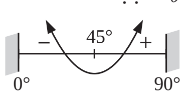
$$\therefore \quad 2\sin^3\theta = 2\cos^3\theta$$

$$\therefore \quad \tan^3\theta = 1$$

$$\therefore \quad \tan\theta = 1$$

$$\therefore \quad \theta = 45^o \quad \text{since} \quad 0 < \theta < 90^o$$

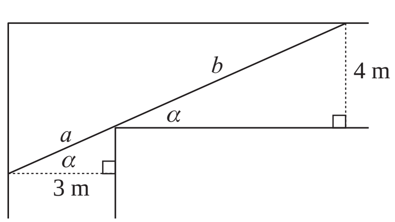
Sign diagram of $\dfrac{dL}{d\theta}$ is:



$\therefore$ the shortest ladder is required when $\theta = 45^o$

and $L_{\min} = \dfrac{2}{\frac{1}{\sqrt{2}}} + \dfrac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ m

5



$\cos\alpha = \dfrac{3}{a}$ and $\sin\alpha = \dfrac{4}{b}$

$\therefore \quad a = \dfrac{3}{\cos\alpha}$ and $b = \dfrac{4}{\sin\alpha}$

Now $L = a + b$

$$\therefore \quad L = \frac{3}{\cos\alpha} + \frac{4}{\sin\alpha}$$

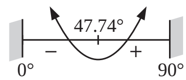
$$\therefore \quad L = 3(\cos\alpha)^{-1} + 4(\sin\alpha)^{-1}$$

$$\therefore \quad \frac{dL}{d\alpha} = -3(\cos\alpha)^{-2}(-\sin\alpha) + 4(-(\sin\alpha)^{-2})\cos\alpha$$

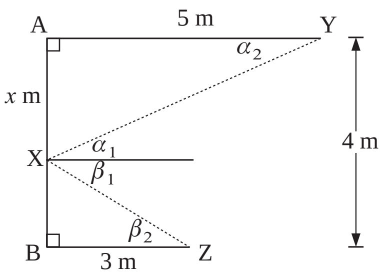
$$= \frac{3\sin\alpha}{\cos^2\alpha} - \frac{4\cos\alpha}{\sin^2\alpha}$$

$$= \frac{3\sin^3\alpha - 4\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}$$

$$\therefore \quad \frac{dL}{d\alpha} = 0 \quad \text{when} \quad 3\sin^3\alpha - 4\cos^3\alpha = 0$$
$$\text{i.e.,} \quad 3\sin^3\alpha = 4\cos^3\alpha$$
$$\tan^3\alpha = \tfrac{4}{3}$$
$$\therefore \quad \tan\alpha = \sqrt[3]{\tfrac{4}{3}} \quad \text{and so} \quad \alpha \doteqdot 47.74^o$$

Sign diagram of $\dfrac{dL}{d\alpha}$ is: 

$\therefore$ AB is minimised when $\alpha = 47.74^o$ and $L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha} \doteqdot 9.866$ m

6 

Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$\therefore \quad \theta = \alpha + \beta$

Let AX $= x$ m

$\therefore$ XB $= (4 - x)$ m

$\therefore \quad \tan\alpha = \dfrac{x}{5} \quad \therefore \quad x = 5\tan\alpha$

and $\tan\beta = \dfrac{4-x}{3}$

$$\text{Now} \quad \tan\theta = \tan(\alpha+\beta)$$
$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\text{i.e.,} \quad \tan\theta = \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5}\left(\frac{4-x}{3}\right)}$$
$$\text{i.e.,} \quad \tan\theta = \frac{3x + 20 - 5x}{15 - x(4-x)} \qquad \{\text{multiplying top and bottom by } 15\}$$
$$\therefore \quad \tan\theta = \frac{20 - 2x}{x^2 - 4x + 15}$$

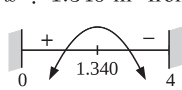
Differentiating both sides with respect to x

$$\frac{1}{\cos^2\theta}\,\frac{d\theta}{dx} = \frac{-2(x^2-4x+15) - (20-2x)(2x-4)}{(x^2-4x+15)^2} \qquad \{\text{Chain and Quotient rules}\}$$
$$\therefore \quad \frac{1}{\cos^2\theta}\,\frac{d\theta}{dx} = \frac{[-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x]}{(x^2-4x+15)^2}$$
$$\therefore \quad \frac{1}{\cos^2\theta}\,\frac{d\theta}{dx} = \frac{[2x^2 - 40x + 50]}{(x^2-4x+15)^2}$$
$$\therefore \quad \frac{d\theta}{dx} = 2\cos^2\theta\,\frac{[x^2 - 20x + 25]}{(x^2-4x+15)^2}$$

Now by inspection, $\theta < 90^o$, so $\dfrac{d\theta}{dx} = 0$ when $x^2 - 20x + 25 = 0$

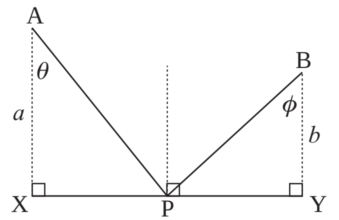
$\therefore \quad x \doteqdot 1.3397$ or $x \doteqdot 18.660$ (where 18.660 is not physically possible)

$\therefore \quad x \doteqdot 1.340$ m from A

Sign diagram for $\dfrac{d\theta}{dx}$ is: 

$\therefore \quad \theta$ is a maximum when $x \doteqdot 1.340$ m from A

7 **a**



$$\frac{a}{\text{AP}} = \cos\theta \quad \text{and} \quad \frac{b}{\text{BP}} = \cos\phi$$

$$\text{AP} = \frac{a}{\cos\theta} \quad \text{and} \quad \text{BP} = \frac{b}{\cos\phi}$$

Now $L = \text{AP} + \text{BP}$

$$\therefore \quad L = \frac{a}{\cos\theta} + \frac{b}{\cos\phi}$$

b

$$L = a(\cos\theta)^{-1} + b(\cos\phi)^{-1}$$

$$\frac{dL}{d\theta} = -a(\cos\theta)^{-2}(-\sin\theta) - b(\cos\phi)^{-2}(-\sin\phi)\frac{d\phi}{d\theta}$$

$$= \frac{a\sin\theta}{\cos^2\theta} + \frac{b\sin\phi}{\cos^2\phi}\frac{d\phi}{d\theta}, \quad \text{as required.}$$

c Now $\dfrac{\text{XP}}{a} = \tan\theta$ and $\dfrac{\text{YP}}{b} = \tan\phi$

$\therefore$ $\text{XP} = a\tan\theta$ and $\text{YP} = b\tan\phi$

$\therefore$ $\text{XY} = a\tan\theta + b\tan\phi$

But XY is a fixed distance, so $a\tan\theta + b\tan\phi$ is a constant,

i.e., $a\tan\theta + b\tan\phi = c$

$$\therefore \quad \frac{a}{\cos^2\theta} + \frac{b}{\cos^2\phi}\frac{d\phi}{d\theta} = 0 \qquad \{\text{differentiating with respect to } \theta\}$$

$$\therefore \quad \frac{b}{\cos^2\phi}\frac{d\phi}{d\theta} = -\frac{a}{\cos^2\theta}$$

$$\therefore \quad \frac{d\phi}{d\theta} = -\frac{a\cos^2\phi}{b\cos^2\theta}, \quad \text{as required.}$$

d

$$\frac{dL}{d\theta} = \frac{a\sin\theta}{\cos^2\theta} + \frac{b\sin\phi}{\cos^2\phi}\left(\frac{-a\cos^2\phi}{b\cos^2\theta}\right) \qquad \{\text{using } \mathbf{b} \text{ and } \mathbf{c}\}$$

$$= \frac{a\sin\theta}{\cos^2\theta} - \frac{a\sin\phi}{\cos^2\theta}$$

$$= \frac{a(\sin\theta - \sin\phi)}{\cos^2\theta}$$

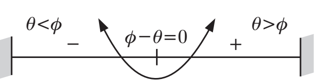
$$\therefore \quad \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin\theta - \sin\phi = 0$$

i.e., when $\sin\phi = \sin\theta$

e Since $\dfrac{dL}{d\theta} = 0$ when $\sin\phi = \sin\theta$,

$L = \text{AP} + \text{PB}$ is either a maximum or minimum when $\phi = \theta$

Sign diagram of $\dfrac{dL}{d\theta}$ is:



$\therefore$ $\text{AP} + \text{PB}$ is a minimum when $\theta - \phi = 0$
i.e., when $\theta = \phi$

$\therefore$ it will be cheapest for the pump house to be located at the point such that $\theta = \phi$.

REVIEW SET 24

1 **a**
$$y = 10x - \sin(10x)$$
$$\therefore \frac{dy}{dx} = 10 - 10\cos(10x)$$

b
$$y = \sin(3x)\cos(2x)$$
$$\therefore \frac{dy}{dx} = 3\cos(3x)\cos(2x) + \sin(3x) \times -2\sin(2x)$$
$$= 3\cos(3x)\cos(2x) - 2\sin(3x)\sin(2x)$$

c
$$\frac{d}{dx}\left(e^{-2x}\sin x\right)$$
$$= -2e^{-2x}\sin x + e^{-2x}\cos x$$
$$= e^{-2x}(\cos x - 2\sin x)$$

d
$$\frac{d}{dx}\left(\ln\left(\frac{1}{\cos x}\right)\right)$$
$$= \frac{d}{dx}(-\ln(\cos x))$$
$$= -\frac{1}{\cos x} \times (-\sin x)$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$

e
$$y = \sin(5x)\ln(2x)$$
$$\therefore \frac{dy}{dx} = 5\cos(5x)\ln(2x) + \sin(5x) \times \frac{2}{2x}$$
$$= 5\cos(5x)\ln(2x) + \frac{\sin(5x)}{x}$$

2
$$y = x\tan x$$
$$\therefore \frac{dy}{dx} = 1 \times \tan x + x \times \frac{1}{\cos^2 x}$$

Now $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} =$ and $\tan\frac{\pi}{4} = 1$

$\therefore$ at $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$ and $\frac{dy}{dx} = 1 + \frac{\pi}{2}$

$\therefore$ the equation of the tangent is $\dfrac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} = 1 + \frac{\pi}{2}$

$$\therefore y - \tfrac{\pi}{4} = (1 + \tfrac{\pi}{2})(x - \tfrac{\pi}{4})$$
$$= x - \tfrac{\pi}{4} + \tfrac{\pi}{2}x - \tfrac{\pi^2}{8}$$
$$\therefore y = (1 + \tfrac{\pi}{2})x - \tfrac{\pi^2}{8}$$
$$\therefore 2y = (2 + \pi)x - \tfrac{\pi^2}{4}$$
$$\therefore (2 + \pi)x - 2y = \tfrac{\pi^2}{4} \quad \text{as required}$$

3 **a**
$$f(x) = 3\sin x - 4\cos(2x)$$
$$\therefore f'(x) = 3\cos x + 8\sin(2x) \quad \text{and} \quad f''(x) = -3\sin x + 16\cos(2x)$$

b
$$f(x) = x^{\frac{1}{2}}\cos(4x)$$
$$\therefore f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) + x^{\frac{1}{2}}(-4\sin(4x))$$
$$= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)$$
$$\text{and} \quad f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) + \tfrac{1}{2}x^{-\frac{1}{2}}(-4\sin 4x)) - \left[2x^{-\frac{1}{2}}\sin(4x) + 4x^{\frac{1}{2}} \times 4\cos(4x)\right]$$
$$= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)$$

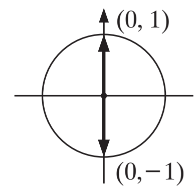
4 $x(t) = 3 + \sin(2t)$ cm, $t \geqslant 0$ sec

a
$$x'(t) = 0 + 2\cos(2t)$$
$$x''(t) = -4\sin(2t)$$

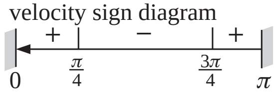
$$\therefore x(0) = 3 \text{ cm}$$
$$x'(0) = 2 \text{ cm/sec}$$
$$x''(0) = 0 \text{ cm/sec}^2$$

$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of 2 cm/sec.

b $x'(t) = 0$ when
$$\begin{aligned} 2\cos(2t) &= 0 \\ \therefore \cos(2t) &= 0 \\ \therefore 2t &= \tfrac{\pi}{2} + k\pi \\ \therefore t &= \tfrac{\pi}{4} \text{ or } \tfrac{3\pi}{4} \quad \{\text{as } 0 \leqslant t \leqslant \pi\} \end{aligned}$$



velocity sign diagram

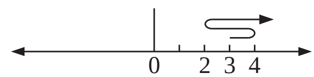


$\therefore$ reverses direction at $t = \frac{\pi}{4}, \frac{3\pi}{4}$

c $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4$

$x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2 \quad x(\pi) = 3 + \sin(2\pi) = 3$

$\therefore$ total distance travelled $= 1 + 2 + 1 = 4$ cm.



5 **a** $f(x) = \sqrt{\cos x}, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore f(x)$ is meaningful when $\cos x \geqslant 0$,

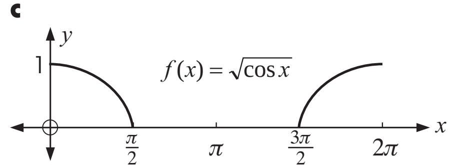
i.e., when $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b
$$\begin{aligned} f(x) &= (\cos x)^{\frac{1}{2}} \\ \therefore f'(x) &= \tfrac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x) \\ &= \frac{-\sin x}{2\sqrt{\cos x}} \end{aligned}$$

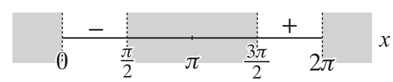
$\therefore f'(x) = 0$ when $-\sin x = 0$

i.e., $x = 0, \pi, 2\pi$ etc.

c



Sign diagram for $f'(x)$ is:



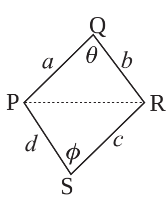
$f(x)$ is increasing for $\frac{3\pi}{2} < x \leqslant 2\pi \qquad f(x)$ is decreasing for $0 \leqslant x < \frac{\pi}{2}$

6 **a**
$$\begin{aligned} y &= x\ln(\sin x) \\ \therefore \frac{dy}{dx} &= 1\ln(\sin x) + x\frac{\cos x}{\sin x} \\ &= \ln(\sin x) + \frac{x\cos x}{\sin x} \end{aligned}$$

b
$$\begin{aligned} y &= \left(e^{\tan x}\right)^{\frac{1}{2}} = e^{\frac{1}{2}\tan x} \\ \therefore \frac{dy}{dx} &= e^{\frac{1}{2}\tan x} \times \tfrac{1}{2}\left(\frac{1}{\cos^2 x}\right) \\ &= \frac{e^{\frac{1}{2}\tan x}}{2\cos^2 x} \end{aligned}$$

c $y = \dfrac{\cos(3x)}{x^{\frac{1}{2}}}$
$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-3\sin(3x)x^{\frac{1}{2}} - \cos(3x)\frac{1}{2}x^{-\frac{1}{2}}}{x^1} \\ &= \frac{-3\sqrt{x}\sin(3x) - \dfrac{\cos(3x)}{2\sqrt{x}}}{x} \end{aligned}$$

7 **a**



Using the cosine rule,

in Δ PQR, $\text{PR}^2 = a^2 + b^2 - 2ab\cos\theta$

in Δ PSR, $\text{PR}^2 = c^2 + d^2 - 2cd\cos\phi$

$\therefore a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 - 2cd\cos\phi$

Now a, b, c and d are constants, so differentiating with respect to ϕ,

$$\therefore 2ab\sin\theta\,\frac{d\theta}{d\phi} = 2cd\sin\phi$$

$$\therefore \frac{d\theta}{d\phi} = \frac{2cd\sin\phi}{2ab\sin\theta} = \frac{cd\sin\phi}{ab\sin\theta} \quad \text{as required}$$

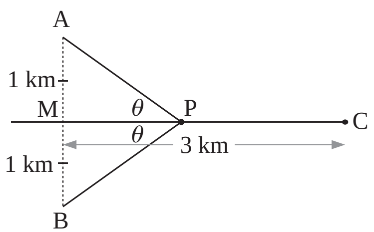
b Area of quadrilateral, A = area of Δ PQR + area of Δ PSR

$$\begin{aligned}
A &= \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\phi \\
\therefore\quad \frac{dA}{d\phi} &= \tfrac{1}{2}ab\cos\theta\,\frac{d\theta}{d\phi} + \tfrac{1}{2}cd\cos\phi \\
&= \tfrac{1}{2}ab\cos\theta\left(\frac{cd\sin\phi}{ab\sin\theta}\right) + \tfrac{1}{2}cd\cos\phi \quad \{\text{using } \textbf{a}\} \\
&= \tfrac{1}{2}cd\left[\frac{\cos\theta\sin\phi}{\sin\theta} + \cos\phi\right] \\
&= \frac{cd}{2\sin\theta}(\sin\phi\cos\theta + \cos\phi\sin\theta) \\
&= \frac{cd}{2\sin\theta}\sin(\phi+\theta)
\end{aligned}$$

$$\therefore\quad \frac{dA}{d\phi} = 0 \quad \text{when} \quad \sin(\phi+\theta) = 0 \quad \text{i.e., when} \quad \phi+\theta = 2\pi$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary,
i.e., when PQRS is a cyclic quadrilateral.

8



a The length of cable required
= (PA + PB + PC) km

i If P is at M, then PA = PB = 1 km
and PC = 3 km
$\therefore$ 5 km of cable is required

ii If P is at C, then PA
= PB = $\sqrt{1^2+3^2} = \sqrt{10}$ km
and PC = 0 km
$\therefore$ $2\sqrt{10}$ km of cable is required

b Now $\sin\theta = \dfrac{1}{\text{AP}} = \dfrac{1}{\text{BP}}$ and $\tan\theta = \dfrac{1}{\text{MP}}$

$$\therefore\quad \text{AP} = \text{BP} = \frac{1}{\sin\theta} \quad\text{and}\quad \text{MP} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

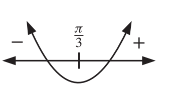
$$\therefore\quad \text{AP} + \text{BP} + \text{CP} = \frac{2}{\sin\theta} + (\text{CM} - \text{MP})$$

$$L = \frac{2}{\sin\theta} + 3 - \frac{\cos\theta}{\sin\theta} \quad \text{as required}$$

c Since $L = 2(\sin\theta)^{-1} + 3 - (\tan\theta)^{-1}$,

$$\begin{aligned}
\frac{dL}{d\theta} &= -2\cos\theta(\sin\theta)^{-2} + (\tan\theta)^{-2}\times\frac{1}{\cos^2\theta} \\
&= \frac{-2\cos\theta}{\sin^2\theta} + \frac{1}{\cos^2\theta}\,\frac{\cos^2\theta}{\sin^2\theta} \\
&= \frac{1-2\cos\theta}{\sin^2\theta} \quad \text{as required}
\end{aligned}$$

$\therefore\quad \dfrac{dL}{d\theta} = 0$ if $\cos\theta = \frac{1}{2}$, i.e., $\theta = \frac{\pi}{3}$ and sign diagram of $\dfrac{dL}{d\theta}$ is:

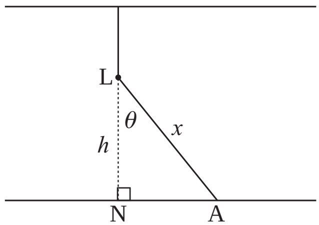


$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

Then $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$,

so $L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3+\sqrt{3})$ km as required

9



a $\sin\theta = \dfrac{\text{NA}}{x} = \dfrac{1}{x}$

$\therefore \quad \dfrac{1}{x^2} = \sin^2\theta$

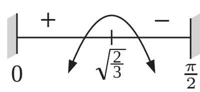
$\therefore$ at A, $I = \dfrac{\sqrt{8}\cos\theta}{x^2}$

$= \sqrt{8}\cos\theta\sin^2\theta$

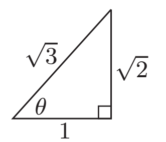
b $$\begin{aligned}\frac{dI}{d\theta} &= \sqrt{8}(-\sin\theta)\sin^2\theta + \sqrt{8}\cos\theta(2\sin\theta\cos\theta)\\ &= \sqrt{8}\sin\theta[2\cos^2\theta - \sin^2\theta]\\ &= \sqrt{8}\sin\theta[2(1-\sin^2\theta) - \sin^2\theta]\\ &= \sqrt{8}\sin\theta[2 - 3\sin^2\theta]\end{aligned}$$

$\dfrac{dI}{d\theta} = 0$ when $\sin\theta = \sqrt{\frac{2}{3}},\ 0 < \theta < \frac{\pi}{2}$

and the sign diagram of $\dfrac{dI}{d\theta}$ is:



$\therefore$ the maximum illumination at A is obtained when $\sin\theta = \sqrt{\frac{2}{3}}$.



$\therefore \quad x = \dfrac{1}{\sin\theta} = \sqrt{\frac{3}{2}}$

$\therefore \quad h = \sqrt{x^2 - \text{NA}^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}}$ i.e., the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.

c $\cos\theta = \dfrac{h}{x}$

$$\therefore \quad I = \frac{\sqrt{8}\cos\theta}{x^2} = \frac{\sqrt{8}\times\frac{h}{x}}{x^2} = \frac{\sqrt{8}h}{x^3}$$

Now $x^2 = h^2 + 1$

$x^3 = \left(h^2+1\right)^{\frac{3}{2}}$

$$\therefore \quad I = \sqrt{8}h\left(h^2+1\right)^{-\frac{3}{2}}$$

$$\begin{aligned}\therefore \quad \frac{dI}{dt} &= \sqrt{8}\,\frac{dh}{dt}\left(h^2+1\right)^{-\frac{3}{2}} - \tfrac{3}{2}\sqrt{8}h\left(h^2+1\right)^{-\frac{5}{2}}\left(2h\,\frac{dh}{dt}\right)\\ &= \sqrt{8}\,\frac{dh}{dt}\left[\left(h^2+1\right)^{-\frac{3}{2}} - 3h^2\left(h^2+1\right)^{-\frac{5}{2}}\right]\end{aligned}$$

Particular case: $\dfrac{dh}{dt} = -0.1 \text{ ms}^{-1}$, and when $h = 1$ m

$$\frac{dI}{dt} = \sqrt{8}(-0.1)\left(2^{-\frac{3}{2}} - 3\times 2^{-\frac{5}{2}}\right) = 0.05$$

$\therefore$ the illumination is increasing at 0.05 units/second.

10 **a** $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$

$\therefore \quad v(t) = -\pi\sin(\pi t)$

So, $v(0) = 0 \text{ cms}^{-1}$

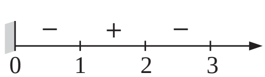
$v(\frac{1}{2}) = -\pi \text{ cms}^{-1}$

$v(1) = 0 \text{ cms}^{-1}$

$v(1\frac{1}{2}) = \pi \text{ cms}^{-1}$

$v(2) = 0 \text{ cms}^{-1}$

sign diagram of $v(t)$:



b The cork is falling when $v(t) \leqslant 0$

i.e., $0 \leqslant t \leqslant 1,\ \ 2 \leqslant t \leqslant 3$, etc.

i.e., $2n \leqslant t \leqslant 2n+1 \quad n = 0, 1, 2, 3, ...$

Chapter 25

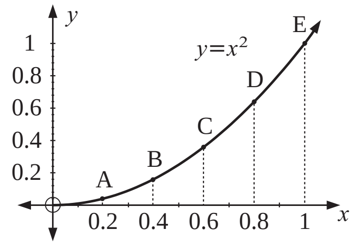
AREAS WITHIN CURVED BOUNDARIES

EXERCISE 25A.1

1 **a** Divide the region into strips of width $\frac{1}{5}$ unit.

O(0, 0) A($\frac{1}{5}$, $\frac{1}{25}$) B($\frac{2}{5}$, $\frac{4}{25}$)

C($\frac{3}{5}$, $\frac{9}{25}$) D($\frac{4}{5}$, $\frac{16}{25}$) E(1, 1)



Now $A < A_1 + A_2 + A_3 + A_4 + A_5$

i.e., $A < \dfrac{\left(0+\frac{1}{25}\right)}{2}\frac{1}{5} + \dfrac{\left(\frac{1}{25}+\frac{4}{25}\right)}{2}\frac{1}{5} + \ldots\ldots + \dfrac{\left(\frac{16}{25}+\frac{25}{25}\right)}{2}\frac{1}{5}$

$\therefore\ A < \frac{1}{10}\left(\frac{85}{25}\right)$

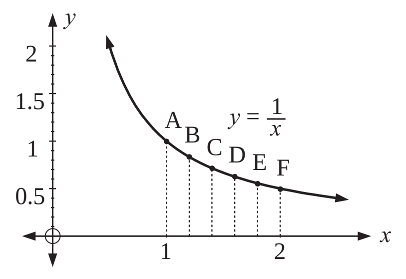
$\therefore\ A < 0.34$

and $A \doteqdot 0.34$ is an estimate of the area

b Divide the region into strips of width $\frac{1}{5}$ unit.

A(1, 1) B($\frac{6}{5}$, $\frac{5}{6}$) C($\frac{7}{5}$, $\frac{5}{7}$)

D($\frac{8}{5}$, $\frac{5}{8}$) E($\frac{9}{5}$, $\frac{5}{9}$) F(2, $\frac{1}{2}$)



Now $A > A_1 + A_2 + A_3 + A_4 + A_5$

i.e., $A > \dfrac{\left(1+\frac{5}{6}\right)}{2}\frac{1}{5} + \dfrac{\left(\frac{5}{6}+\frac{5}{7}\right)}{2}\frac{1}{5} + \ldots\ldots + \dfrac{\left(\frac{5}{9}+\frac{1}{2}\right)}{2}\frac{1}{5}$

$\therefore\ A > \frac{1}{10}\left(\frac{11}{6}+\frac{65}{42}+\frac{75}{56}+\frac{85}{72}+\frac{19}{18}\right)$

$\therefore\ A > 0.6956$

and $A \doteqdot 0.70$ is an estimate of the area

2 **a** Using 10 vertical strips in question **1 a**

$$A < \frac{\left(\frac{0}{100}+\frac{1}{100}\right)\frac{1}{10}}{2} + \frac{\left(\frac{1}{100}+\frac{4}{100}\right)\frac{1}{10}}{2} + \ldots. + \frac{\left(\frac{81}{100}+\frac{100}{100}\right)\frac{1}{10}}{2}$$

$$< \frac{1}{20}\left(\frac{1}{100}+\frac{5}{100}+\frac{13}{100}+\frac{25}{100}+\frac{41}{100}+\frac{61}{100}+\frac{85}{100}+\frac{113}{100}+\frac{145}{100}+\frac{181}{100}\right)$$

$$< \frac{1}{20}\left(\frac{670}{100}\right)$$

$$< 0.335$$

and $A \doteqdot 0.335$ is an estimate of the area

b Using 10 vertical strips in question **1 b**

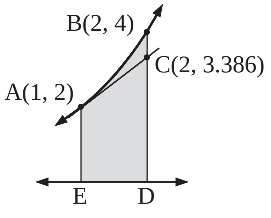
$$A > \frac{\left(\frac{10}{10}+\frac{10}{11}\right)\frac{1}{10}}{2} + \frac{\left(\frac{10}{11}+\frac{10}{12}\right)\frac{1}{10}}{2} + \ldots\ldots + \frac{\left(\frac{10}{19}+\frac{10}{20}\right)\frac{1}{10}}{2}$$

$$\therefore \quad A > \tfrac{10}{20}\left(\left(\tfrac{1}{10}+\tfrac{1}{11}\right)+\left(\tfrac{1}{11}+\tfrac{1}{12}\right)+ \ldots\ldots + \left(\tfrac{1}{19}+\tfrac{1}{20}\right)\right)$$
$$> \tfrac{1}{2}\left(\tfrac{21}{110}+\tfrac{23}{132}+\tfrac{25}{156}+\tfrac{27}{182}+\tfrac{29}{210}+\tfrac{31}{240}+\tfrac{33}{272}+\tfrac{35}{306}+\tfrac{37}{342}+\tfrac{39}{380}\right)$$
$$> 0.693\,771$$

and $A \doteqdot 0.6938$ is an estimate of the area

EXERCISE 25A.2

1 **a**

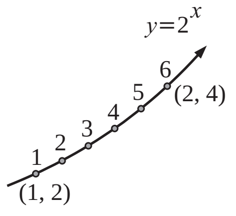


The lower bound is the area ACDE
$= \dfrac{2+3.386}{2}$
$= 2.693$

The upper bound is the area ABDE
$= \dfrac{2+4}{2}$
$= 3$

b Without the tangent and using 5 subdivisions on [1, 2]

$n = 5$



Point	*Coordinates*
1	(1, 2)
2	$(1.2, 2^{1.2})$
3	$(1.4, 2^{1.4})$
4	$(1.6, 2^{1.6})$
5	$(1.8, 2^{1.8})$
6	(2, 4)

$$A_L = \tfrac{1}{5}(2^1) + \tfrac{1}{5}(2^{1.2}) + \tfrac{1}{5}(2^{1.4}) + \tfrac{1}{5}(2^{1.6}) + \tfrac{1}{5}(2^{1.8})$$
$$= \tfrac{1}{5}(2 + 2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8})$$
$$\doteqdot 2.690$$

$$A_U = \tfrac{1}{5}(2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8} + 2^2)$$
$$\doteqdot 3.090$$

c Using the provided software:

n	A_L	A_U
10	2.786 55	2.986 55
50	2.865 44	2.905 44
100	2.875 40	2.895 40
500	2.883 39	2.887 39
5000	2.885 19	2.885 59

The upper and lower sums converge to 2.885 (approx.)

2 **a** $y = 1 + e^x$ [0, 1] choosing 5 subdivisions on this [0, 1].

Point	*Coordinates*
1	(0, 2)
2	$(0.2, 1+e^{0.2})$
3	$(0.4, 1+e^{0.4})$
4	$(0.6, 1+e^{0.6})$
5	$(0.8, 1+e^{0.8})$
6	$(1, 1+e^{1})$

$$A_L = 0.2\left(1+e^0\right) + 0.2(1+e^{0.2}) + \ldots\ldots + 0.2(1+e^{0.8})$$
$$= 0.2\left(5 + (e^0 + e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8})\right)$$
$$\doteqdot 2.552$$

$$\therefore \quad A_U = 0.2(1+e^{0.2}) + 0.2(1+e^{0.4}) + \ldots\ldots + 0.2(1+e^1)$$
$$= 0.2\left(5 + \left(e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8} + e^1\right)\right)$$
$$\doteqdot 2.896$$

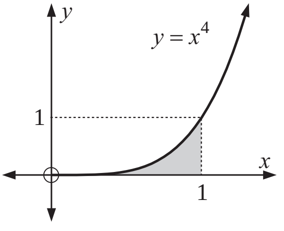
b

n	A_L	A_U
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

c It appears that as $t \to \infty$ both upper and lower sums converge to e, where $e \doteqdot 2.718\,28$

EXERCISE 25B.1

1 **a**

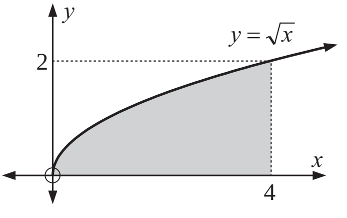


b

n	A_L	A_U
10	0.153 33	0.253 33
100	0.195 03	0.205 03
1000	0.199 50	0.200 50
10 000	0.199 95	0.200 05

c *Guess*: Area $= \int_0^1 x^4 dx = 0.2$

2 **a**

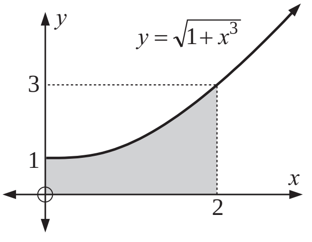


b

n	A_L	A_U
100	5.291 70	5.371 70
10 000	5.332 93	5.333 73

c *Guess*: Area $= \int_0^4 \sqrt{x}\, dx \;= 5.\overline{3}$
$= 5\frac{1}{3}$

3 **a**



b

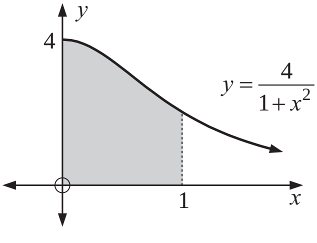
n	A_L	A_U
100	3.221 38	3.261 38
10 000	3.241 11	3.241 51

c A good estimate would be the average

i.e., $\dfrac{3.241\,11 + 3.241\,51}{2}$

$\therefore \int_0^2 \sqrt{1+x^3}dx \doteqdot 3.2413$

4 **a**



b

n	A_L	A_U
100	3.131 58	3.151 58
1000	3.140 59	3.142 59
10 000	3.141 49	3.141 69

c Area $= \int_0^1 \dfrac{4}{1+x^2}\, dx \;= \dfrac{3.141\,69 + 3.141\,49}{2}$
$\doteqdot 3.141\,59$
$\doteqdot 3.1416$
(We suspect this is π)

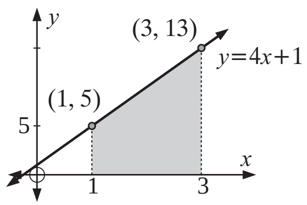
5 For the curve $y = \sqrt[3]{x^2+2}$, $y > 0$ for $0 \leqslant x \leqslant 5$.

Using the software provided

n	A_L	A_U
100	10.159 79	10.246 80
1000	10.198 87	10.207 57
10 000	10.202 78	10.203 65

A good estimate is $\int_0^5 \sqrt[3]{x^2+2}\,dx = \dfrac{10.202\,78 + 10.203\,65}{2} \doteqdot 10.203\,22$

6 **a**



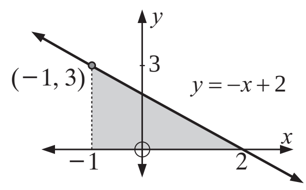
$\int_1^3 (1+4x)\,dx$

$= $ area of the shaded trapezium

$= \left(\dfrac{5+13}{2}\right) \times 2$

$= 18$

b



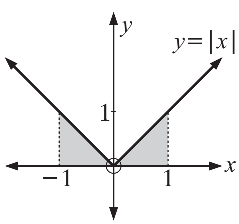
$\int_{-1}^2 (2-x)\,dx$

$= $ area of shaded triangle

$= \frac{1}{2}(3 \times 3)$

$= 4.5$

c



$\int_{-1}^1 |x|\,dx = $ shaded area shown

Due to symmetry we need to only consider the RHS.

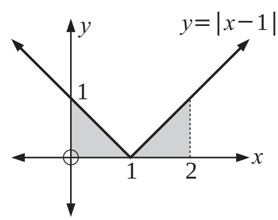
$\therefore \quad \int_0^1 |x|\,dx = $ area of a shaded Δ

$= \frac{1}{2}(1 \times 1)$

$= 0.5$

$\therefore \quad \int_{-1}^1 |x|\,dx = 2 \times 0.5 = 1$

d



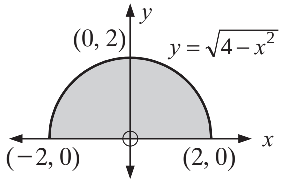
$y = \sqrt{4-x^2}$

$\therefore \quad \int_0^2 |x-1|\,dx = 2\int_1^2 |x-1|\,dx$ {by symmetry}

$= 2 \times \left[\frac{1}{2}(1 \times 1)\right]$

$= 1$

e



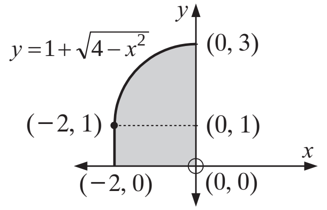
$\int_{-2}^2 \sqrt{4-x^2}\,dx$

$= $ area of semi-circle radius 2

$= \frac{1}{2}\left(\pi \times 2^2\right)$

$= 2\pi$

f



$\int_{-2}^0 \left(1+\sqrt{4-x^2}\right)dx$

$= $ area of rectangle
 $+$ area of quarter of circle with radius 2

$= 2 \times 1 + \frac{1}{4}(\pi \times 2^2)$

$= 2 + \pi$

EXERCISE 25B.2

1 **a** $\int_1^4 \sqrt{x}\,dx = 4.667$

$\int_1^4 (-\sqrt{x})\,dx = -4.667$

b $\int_0^1 x^7\,dx = 0.125 = \frac{1}{8}$

$\int_0^1 (-x^7)\,dx = -0.125 = -\frac{1}{8}$

2 **a** $\int_0^1 x^2 dx = \frac{1}{3}$ **b** $\int_1^2 x^2 dx = \frac{7}{3}$ **c** $\int_0^2 x^2 dx = \frac{8}{3}$ **d** $\int_0^1 3x^2 dx = 1$

3 **a** $\int_0^2 (x^3 - 4x)\,dx = -4$ **b** $\int_2^3 (x^3 - 4x)\,dx = 6\frac{1}{4}$ **c** $\int_0^3 (x^3 - 4x)\,dx = 2\frac{1}{4}$

4 **a** $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ **b** $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

5 **a** $\int_0^1 x^2 dx = \frac{1}{3}$ **b** $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

c $\int_0^1 (x^2 + \sqrt{x})\,dx = 1$

this indicates that $\int_0^1 (x^2 + \sqrt{x})\,dx = \int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx$

and in general $\displaystyle\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x) + g(x))\,dx$

EXERCISE 25B.3

1 **a** $\int_0^3 f(x)\,dx = 2 + 3 + 1.5$
$= 6.5$

b $\int_3^7 f(x)\,dx = -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right)$
$= -9$

c $\int_2^4 f(x)dx = 1.5 - 1.5$
$= 0$

d $\int_0^7 f(x)\,dx = 6.5 - 9$
$= -2.5$

2 **a** $\int_0^4 f(x)\,dx = \frac{1}{2}\pi(2)^2$
$= 2\pi$

b $\int_4^6 f(x)\,dx = -(2 \times 2)$
$= -4$

c $\int_6^8 f(x)\,dx = \frac{1}{2}\pi(1)^2$
$= \frac{\pi}{2}$

d $\int_0^8 f(x)\,dx = 2\pi + (-4) + \frac{\pi}{2}$
$= \frac{5\pi}{2} - 4$

3 **a** $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$
$= \int_2^7 f(x)\,dx$

b $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$
$= \int_1^9 g(x)\,dx$

4 **a** $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx$

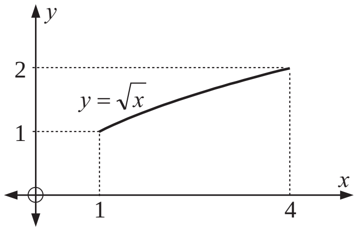
$\therefore \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx - \int_1^3 f(x)\,dx = (-3) - (2) = -5$

b $\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx$

$$\begin{aligned}\therefore \int_2^4 f(x)\,dx &= \int_0^6 f(x)\,dx - \int_4^6 f(x)\,dx - \int_0^2 f(x)\,dx\\ &= (7) - (-2) - (5)\\ &= 7 + 2 - 5\\ &= 4\end{aligned}$$

REVIEW SET 25

1 **a**



x	$\sqrt{x}$
1	1
1.5	1.2247
2	1.4142
2.5	1.5811
3	1.7321
3.5	1.8708
4	2

b $A_L = 0.5\,(1 + 1.2247 + 1.4142 + 1.5811 + 1.7321 + 1.8708)$
$= 0.5(8.8229)$
$\doteqdot 4.411$

2 **a** A is the *upper half* of a circle centre (2, 0) and radius 2.

$$\begin{aligned}\therefore\quad (x-2)^2+(y-0)^2 &= 2^2\\ (x-2)^2+y^2 &= 4\\ y^2 &= 4-(x-2)^2\\ y^2 &= 4-x^2+4x-4\\ y^2 &= 4x-x^2\\ \therefore\quad y &= \pm\sqrt{4x-x^2}\end{aligned}$$

So, $y=\sqrt{4x-x^2}$ 

or $y=-\sqrt{4x-x^2}$ 

$\therefore$ required equation is $y_A = \sqrt{4x-x^2}$

b Now B is the *lower half* of a circle centre (5, 0) and radius 1.

$$\begin{aligned}\therefore\quad (x-5)^2+(y-0)^2 &= 1^2\\ (x-5)^2+y^2 &= 1\\ y^2 &= 1-(x-5)^2\\ y^2 &= 1-x^2+10x-25\\ \therefore\quad y &= \pm\sqrt{10x-x^2-24}\\ \therefore\quad y_B &= -\sqrt{10x-x^2-24}\end{aligned}$$

c $\int_0^4 y_A dx = \frac{1}{2}\pi r^2$ where $r=2$
$= \frac{1}{2}\pi(2)^2$
$= 2\pi$

$\int_4^6 y_B dx = -\frac{1}{2}\pi r^2$ where $r=1$
$= -\frac{1}{2}\pi(1)^2$
$= -\frac{\pi}{2}$

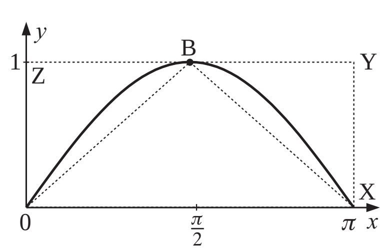
d $\int_0^6 f(x)dx = \int_0^4 y_A dx + \int_4^6 y_B dx$
$= 2\pi + (-\frac{\pi}{2})$
$= \frac{3\pi}{2}$

3 **a** From the graph,

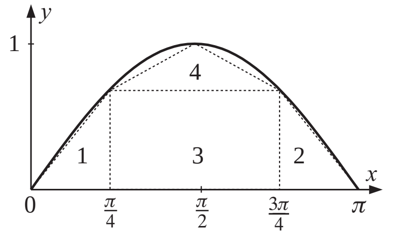
area ΔOBX $<$ area under the curve $<$ area OXYZ

$\therefore\quad \frac{1}{2}\pi(1) < \int_0^\pi \sin x\, dx < \pi(1)$

i.e., $\frac{\pi}{2} < \int_0^\pi \sin x\, dx < \pi$



b If we partition the diagram into triangles as shown:



Area 1 $= \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{16}$

Area 2 $= \frac{\pi\sqrt{2}}{16}$

Area 3 $= \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$

Area 4 $= \frac{1}{2}\left(\frac{\pi}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$

This total area is just less than the area under the arch

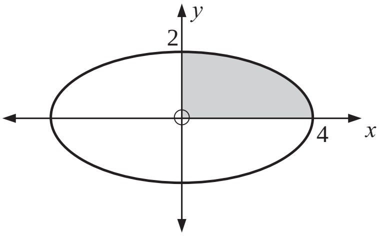
$$\begin{aligned}\text{and total area } &= \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{4} + \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right) \\ &= \frac{\pi\sqrt{2}+\pi\sqrt{2}+4\pi\sqrt{2}+4\pi-2\pi\sqrt{2}}{16} \\ &= \frac{4\pi+4\pi\sqrt{2}}{16} \\ &= \frac{\pi+\pi\sqrt{2}}{4} \\ &= \frac{\pi}{4}(1+\sqrt{2}) \text{ units}^2\end{aligned}$$

4 **a** $$\begin{aligned}\int_0^4 f(x)dx &= \text{area of triangle} + \text{area of } \tfrac{1}{4} \text{ circle} \\ &= \tfrac{1}{2}(2 \times 2) + \tfrac{1}{4}\pi(2)^2 \\ &= 2 + \pi\end{aligned}$$

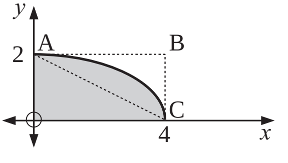
b $$\begin{aligned}\int_4^6 f(x)dx &= -\text{ area of triangle below } x\text{-axis} \\ &= -\tfrac{1}{2}(2 \times 2) \\ &= -2\end{aligned}$$

c $$\begin{aligned}\int_0^6 f(x)dx &= \int_0^4 f(x)dx + \int_4^6 f(x)dx \\ &= (2+\pi) + (-2) \\ &= \pi\end{aligned}$$

5 **a**



b



Now area ΔAOC $<$ shaded area $<$ area ABCD

$$\begin{aligned}\therefore \quad \tfrac{1}{2}(2 \times 4) &< \int_0^4 \tfrac{1}{2}\sqrt{16-x^2}\,dx < 2 \times 4 \\ \therefore \quad 4 &< \int_0^4 \tfrac{1}{2}\sqrt{16-x^2}\,dx < 8 \\ \therefore \quad 8 &< \int_0^4 \sqrt{16-x^2}\,dx < 16 \quad \{(\times 2)\}\end{aligned}$$

6 Using technology

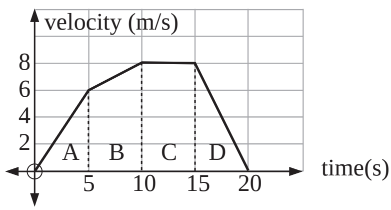
a $\int_0^2 x^3 dx = 4$ **b** $\int_1^3 x^3 dx = 20$

Chapter 26

INTEGRATION

EXERCISE 26A

1



Total distance travelled

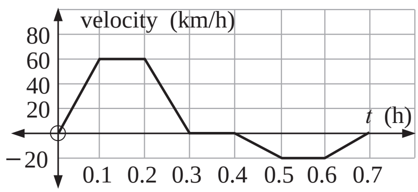
$= \text{area A} + \text{area B} + \text{area C} + \text{area D}$

$= \frac{1}{2}(5 \times 6) + \left(\dfrac{6+8}{2}\right) 5 + 5 \times 8 + \frac{1}{2}(5 \times 8)$

$= 15 + 35 + 40 + 20$

$= 110$ m

2



a **i** The graph above the t-axis indicates that the velocity is positive and the car is travelling forwards.

ii The graph below the t-axis indicates that the velocity is negative and the car is travelling backwards.

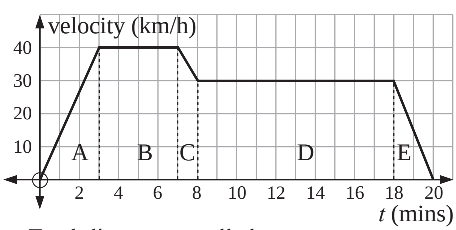
b Final displacement $= \text{area above the } t\text{-axis} - \text{area below the } t\text{-axis}$

$= \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 60 - \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 20$

$= 12 - 4$

$= 8$ km from the starting point in the positive direction

3



Total distance travelled

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$

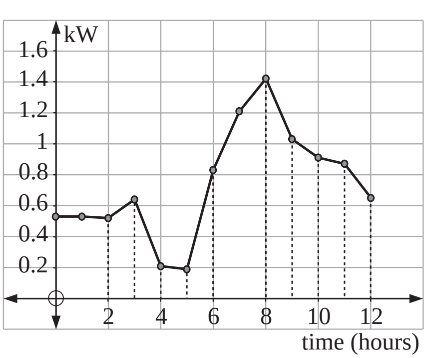
$= \frac{1}{60}\left[\frac{1}{2}(3 \times 40) + (40 \times 4) + \left(\dfrac{40+30}{2}\right) 1 + (10 \times 30) + \frac{1}{2}(2 \times 30)\right]$

$= \frac{1}{60}[60 + 160 + 35 + 300 + 30]$

$= 9.75$ km

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km/h}

4



Total consumption of electricity

$= \text{area under graph}$

$= (2 \times 0.55) + \left(\dfrac{0.55+0.65}{2}\right) 1 + \left(\dfrac{0.65+0.2}{2}\right) 1$

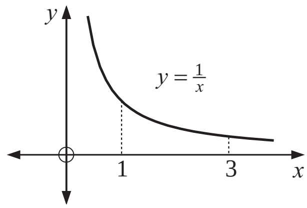
$+1 \times 0.2 + \left(\dfrac{0.2+0.8}{2}\right) 1 + \left(\dfrac{0.8+1.2}{2}\right) 1$

$+ \left(\dfrac{1.2+1.4}{2}\right) 1 + \left(\dfrac{1.4+1}{2}\right) 1 + \left(\dfrac{1+0.9}{2}\right) 1$

$+ \left(\dfrac{0.9+0.85}{2}\right) 1 + \left(\dfrac{0.85+0.65}{2}\right) 1$

$\doteqdot 8.9$ kWh

5



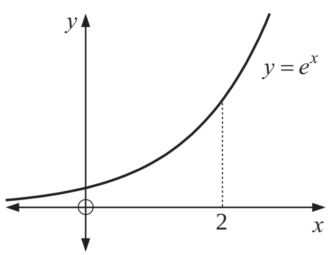
Divide the region into 10 strips of width, $\Delta x = \frac{2}{10} = 0.2$
The endpoints of the sub-intervals are: 1, 1.2, 1.4, ... , 3.0 . Since the function $y = \dfrac{1}{x}$ is decreasing on the interval, the upper rectangle sums use the y-coordinates at the left-hand end of each sub-interval, while the lower rectangle sums use the y-coordinates at the right-hand ends.

a Upper rectangular sums: $\therefore\ A_U = \frac{1}{1}(0.2) + \frac{1}{1.2}(0.2) + \ldots\ldots + \frac{1}{2.8}(0.2)$
$= 0.2\left[1 + \frac{1}{1.2} + \frac{1}{1.4} + \ldots\ldots + \frac{1}{2.8}\right]$
$\doteqdot 1.17$ units^2

b Lower rectangular sums: $\therefore\ A_L = \frac{1}{1.2}(0.2) + \frac{1}{1.4}(0.2) + \frac{1}{1.6}(0.2) + \ldots\ldots + \frac{1}{3.0}(0.2)$
$= 0.2\left[\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \ldots\ldots + \frac{1}{3.0}\right]$
$\doteqdot 1.03$ units^2

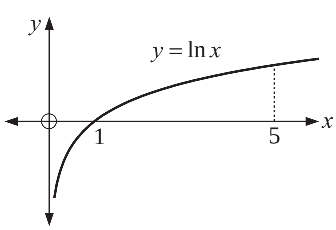
c Midpoints: The midpoints are at $x = 1.1, 1.3, 1.5, 1.7, \ldots\ldots, 2.9$
$\therefore\ A_M = \frac{1}{1.1}(0.2) + \frac{1}{1.3}(0.2) + \ldots\ldots + \frac{1}{2.9}(0.2)$
$= 0.2\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \ldots\ldots + \frac{1}{2.9}\right]$
$\doteqdot 1.10$ units^2

6



10 strips, $\therefore$ width, $\Delta x = \frac{2}{10} = 0.2$
The endpoints of the interval are 0, 0.2, 0.4, 0.6, ... , 2.0
$\therefore$ the midpoints are 0.1, 0.3, 0.5, 0.7, ... , 1.9
$\therefore\ A_M = 0.2e^{0.1} + 0.2e^{0.3} + \ldots\ldots + 0.2e^{1.9}$
$= 0.2[e^{0.1} + e^{0.3} + e^{0.5} + \ldots\ldots + e^{1.9}]$
$\doteqdot 0.2 \times 31.892$
$\doteqdot 6.38$ units^2

7



20 strips, $\therefore$ width, $\Delta x = \frac{4}{20} = 0.2$
Endpoints of the interval are 1.00, 1.2, 1.4, ... , 5.00
$\therefore$ the midpoints are 1.1, 1.3, 1.5, ... , 4.9
$\therefore\ A_M = 0.2[\ln 1.1 + \ln 1.3 + \ldots\ldots + \ln 4.9]$
$\doteqdot 0.2 \times 20.24$
$\doteqdot 4.05$ units^2

8 **a** Using technology
Area $= 6.389\,06$
$\doteqdot 6.389$ units^2

b Using technology
Area $= 4.047\,19$
$\doteqdot 4.047$ units^2

EXERCISE 26C

1 **a** **i** $\dfrac{d}{dx}(x^2) = 2x$
$\therefore\ \dfrac{d}{dx}\left(\frac{1}{2}x^2\right) = x$
$\therefore$ the antiderivative of x is $\frac{1}{2}x^2$

ii $\dfrac{d}{dx}(x^3) = 3x^2$
$\therefore\ \dfrac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$
$\therefore$ the antiderivative of x^2 is $\frac{1}{3}x^3$

iii $\frac{d}{dx}(x^6) = 6x^5$

$\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$

$\therefore$ the antiderivative of x^5 is $\frac{1}{6}x^6$

iv $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$ or $-\frac{1}{x}$

v $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$

$\therefore$ the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3}$

vi $\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$

$\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$

viii $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of of $x^{-\frac{1}{2}}$ is $2x^{\frac{1}{2}} = 2\sqrt{x}$

b the antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$

2 **a** **i** $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

$\therefore \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$

ii $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$

$\therefore$ the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$

iii $\frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$

$\therefore \frac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$

iv $\frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$

$\therefore$ the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$

v $\frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$

$\therefore$ the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$

vi $\frac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$

$\therefore \frac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$

$\therefore$ the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$

b the antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$

3 **a** $\frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\frac{d}{dx}\left(2x^3 + 2x^2\right) = 6x^2 + 4x$

$\therefore$ antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$

b $\frac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$

$\frac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) = e^{3x+1}$

$\therefore$ antiderivative of e^{3x+1} is $\frac{1}{3}e^{3x+1}$

c $\frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

$\therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$

$\therefore$ antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$

d $\frac{d}{dx}((2x+1)^4) = 4(2x+1)^3 \times 2$

$= 8(2x+1)^3$

$\therefore \frac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) = (2x+1)^3$

$\therefore$ antiderivative of $(2x+1)^3$ is $\frac{1}{8}(2x+1)^4$

4 **a**
$$\frac{dy}{dx} = 6$$
$$\frac{d}{dx}(6x+c) = 6$$
$$\therefore \quad y = 6x + c$$

b
$$\frac{dy}{dx} = 4x^2$$
$$\frac{d}{dx}(x^3) = 3x^2$$
$$\frac{d}{dx}(\tfrac{4}{3}x^3 + c) = 4x^2$$
$$\therefore \quad y = \tfrac{4}{3}x^3 + c$$

c $\dfrac{dy}{dx} = 5x - x^2$ Now $\dfrac{d}{dx}(x^2) = 2x$ $\therefore$ $\dfrac{d}{dx}\left(\tfrac{5}{2}x^2\right) = 5x$

and $\dfrac{d}{dx}(x^3) = 3x^2$ $\therefore$ $\dfrac{d}{dx}\left(-\tfrac{1}{3}x^3\right) = -x^2$

Now $\dfrac{d}{dx}\left(\tfrac{5}{2}x^2 - \tfrac{1}{3}x^3 + c\right) = 5x - x^2$, $\therefore$ $y = \tfrac{5}{2}x^2 - \tfrac{1}{3}x^3 + c$

d
$$\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$$
Now $\dfrac{d}{dx}(x^{-1}) = -x^{-2}$
$$\therefore \quad \frac{d}{dx}\left(-\frac{1}{x} + c\right) = x^{-2}$$
$$\therefore \quad y = -\frac{1}{x} + c$$

e
$$\frac{dy}{dx} = e^{-3x}$$
Now $\dfrac{d}{dx}(e^{-3x}) = -3e^{-3x}$
$$\therefore \quad \frac{d}{dx}(-\tfrac{1}{3}e^{-3x} + c) = e^{-3x}$$
$$\therefore \quad y = -\tfrac{1}{3}e^{-3x} + c$$

f $\dfrac{dy}{dx} = 4x^3 + 3x^2$ Now $\dfrac{d}{dx}(x^4 + x^3) = 4x^3 + 3x^2$
$$\therefore \quad \frac{d}{dx}(x^4 + x^3 + c) = 4x^3 + 3x^2$$
$$\therefore \quad y = x^4 + x^3 + c$$

EXERCISE 26D

1 **a** $\int_a^a f(x)\ dx = F(a) - F(a) = 0$
graphically: $\int_a^a f(x)\ dx =$ area of the strip between $x = a$ and $x = a$,
i.e., width $= 0$ $\therefore$ area $= 0$

b
$$\begin{aligned}\int_a^b c\ dx &= F(b) - F(a)\\ &= cb - ca \quad \{\text{antideriv. of } c \text{ is } cx\}\\ &= c(b-a)\end{aligned}$$

c
$$\begin{aligned}\int_b^a f(x)\ dx &= F(a) - F(b)\\ &= -[F(b) - F(a)]\\ &= -\int_a^b f(x)\ dx\end{aligned}$$

d If $\dfrac{d}{dx}F(x) = f(x)$ then
$$\frac{d}{dx}cF(x) = cf(x)$$
$$\begin{aligned}\therefore \quad \int_a^b cf(x)\ dx &= c[F(b) - F(a)]\\ &= c\int_a^b f(x)\ dx\end{aligned}$$

e
$$\begin{aligned}\int_a^b (f(x) + g(x))\ dx &= [F(b) + G(b)] - [F(a) + G(a)]\\ &= [F(b) - F(a)] + [G(b) - G(a)]\\ &= \int_a^b f(x)\ dx + \int_a^b g(x)\, dx\end{aligned}$$

2 **a** $f(x) = x^3$ has antiderivative
$$F(x) = \frac{x^4}{4}$$
$$\begin{aligned}\text{So, area} &= \int_0^1 x^3\ dx\\ &= F(1) - F(0)\\ &= \tfrac{1}{4} - 0\\ &= \tfrac{1}{4} \text{ units}^2\end{aligned}$$

b $f(x) = x^3$ has antiderivative
$$F(x) = \frac{x^4}{4}$$
$$\begin{aligned}\text{So, area} &= \int_1^2 x^3\ dx\\ &= F(2) - F(1)\\ &= \tfrac{16}{4} - \tfrac{1}{4}\\ &= 3\tfrac{3}{4} \text{ units}^2\end{aligned}$$

c $f(x) = x^2 + 3x + 2$ has antiderivative

$F(x) = \dfrac{x^3}{3} + \dfrac{3x^2}{2} + 2x$

So, area $= \int_1^3 (x^2 + 3x + 2)\ dx$
$= F(3) - F(1)$
$= (\frac{27}{3} + \frac{27}{2} + 6) - (\frac{1}{3} + \frac{3}{2} + 2)$
$= 24\frac{2}{3}$ units^2

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

So the area $= \int_0^2 \sqrt{x}\ dx$
$= F(2) - F(0)$
$= \frac{2}{3} \times 2\sqrt{2} - 0$
$= \frac{4\sqrt{2}}{3}$ units^2

e $f(x) = e^x$ has antiderivative

$F(x) = e^x$

So, area $= \int_0^{1.5} e^x\ dx$
$= F(1.5) - F(0)$
$= e^{1.5} - e^0$
$= e^{1.5} - 1$
$\doteqdot 3.482$ units^2

f $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

So, area $= \displaystyle\int_1^4 \frac{1}{\sqrt{x}}\ dx = F(4) - F(1)$
$= 2\sqrt{4} - 2\sqrt{1}$
$= 2$ units^2

g $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

$F(x) = \dfrac{x^4}{4} + \dfrac{2x^3}{3} + \dfrac{7x^2}{2} + 4x$

So the area $= \displaystyle\int_1^{1.25} x^3 + 2x^2 + 7x + 4\ dx$
$= F(1.25) - F(1)$
$= [12.381\,18 - 8.416\,67]$
$\doteqdot 3.965$ units^2

EXERCISE 26E.1

1 $y = x^7$
$\therefore \dfrac{dy}{dx} = 7x^6$

$\therefore \int 7x^6\ dx = x^7 + c_1$
$\therefore 7\int x^6\ dx = x^7 + c_1$
$\therefore \int x^6\ dx = \frac{1}{7}x^7 + c$

2 $y = x^3 + x^2$
$\therefore \dfrac{dy}{dx} = 3x^2 + 2x$

$\therefore \int 3x^2 + 2x\ dx = x^3 + x^2 + c$

3 $y = e^{2x+1}$
$\therefore \dfrac{dy}{dx} = 2e^{2x+1}$

$\therefore \int 2e^{2x+1}\ dx = e^{2x+1} + c_1$
$\therefore 2\int e^{2x+1}\ dx = e^{2x+1} + c_1$
$\therefore \int e^{2x+1}\ dx = \frac{1}{2}e^{2x+1} + c$

4 $y = (2x+1)^4$
$\therefore \dfrac{dy}{dx} = 4(2x+1)^3 \times 2$
$= 8(2x+1)^3$

$\therefore \int 8(2x+1)^3\ dx = (2x+1)^4 + c_1$
$\therefore 8\int (2x+1)^3\ dx = (2x+1)^4 + c_1$
$\therefore \int (2x+1)^3\ dx = \frac{1}{8}(2x+1)^4 + c$

5 $y = x\sqrt{x} = x^{\frac{3}{2}}$
$\therefore \dfrac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \int \frac{3}{2}\sqrt{x}\ dx = x\sqrt{x} + c_1$
$\therefore \frac{3}{2}\int \sqrt{x}\ dx = x\sqrt{x} + c_1$
$\therefore \int \sqrt{x}\ dx = \frac{2}{3}x\sqrt{x} + c$

6 $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right)$

$\therefore \quad \int -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right)\,dx = \frac{1}{\sqrt{x}} + c_1$

$\therefore \quad -\frac{1}{2}\int \frac{1}{x\sqrt{x}}\,dx = \frac{1}{\sqrt{x}} + c_1$

$\therefore \quad \int \frac{1}{x\sqrt{x}}\,dx = -\frac{2}{\sqrt{x}} + c$

7 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$\therefore \quad \frac{d}{dx}(F(x)+G(x)) = f(x)+g(x)$

$$\begin{aligned} \therefore \quad \int (f(x)+g(x))\,dx &= F(x)+G(x)+c \\ &= (F(x)+c_1)+(G(x)+c_2) \\ &= \int f(x)\,dx + \int g(x)\,dx \end{aligned}$$

8 **a** $y = (2x-1)^6$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 6(2x-1)^5 \times 2 \\ &= 12(2x-1)^5 \end{aligned}$$

$\therefore \quad \int 12(2x-1)^5\,dx = (2x-1)^6 + c_1$

$\therefore \quad 12\int (2x-1)^5\,dx = (2x-1)^6 + c_1$

$\therefore \quad \int (2x-1)^5\,dx = \frac{1}{12}(2x-1)^6 + c$

b
$$\begin{aligned} y &= \sqrt{1-4x} \\ &= (1-4x)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \tfrac{1}{2}(1-4x)^{-\frac{1}{2}}(-4) \\ &= -2(1-4x)^{-\frac{1}{2}} \end{aligned}$$

$\therefore \quad \int \frac{-2}{\sqrt{1-4x}}\,dx = \sqrt{1-4x} + c_1$

$\therefore \quad -2\int \frac{1}{\sqrt{1-4x}}\,dx = \sqrt{1-4x} + c_1$

$\therefore \quad \int \frac{1}{\sqrt{1-4x}}\,dx = -\frac{1}{2}\sqrt{1-4x} + c$

c $y = \frac{1}{\sqrt{3x+1}} = (3x+1)^{-\frac{1}{2}}$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= -\tfrac{1}{2}(3x+1)^{-\frac{3}{2}} \times 3 \\ &= -\tfrac{3}{2}(3x+1)^{-\frac{3}{2}} \\ &= -\frac{3}{2(3x+1)^{\frac{3}{2}}} \end{aligned}$$

$\therefore \quad \int -\frac{3}{2(3x+1)^{\frac{3}{2}}}\,dx = \frac{1}{\sqrt{3x+1}} + c_1$

$\therefore \quad -\frac{3}{2}\int \frac{1}{(3x+1)^{\frac{3}{2}}}\,dx = \frac{1}{\sqrt{3x+1}} + c_1$

$\therefore \quad \int \frac{1}{(3x+1)^{\frac{3}{2}}}\,dx = \frac{-2}{3\sqrt{3x+1}} + c$

9 **a** $y = e^{1-3x}$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= e^{1-3x}(-3) \\ &= -3e^{1-3x} \end{aligned}$$

$\therefore \quad \int -3e^{1-3x}\,dx = e^{1-3x} + c_1$

$\therefore \quad -3\int e^{1-3x}\,dx = e^{1-3x} + c_1$

$\therefore \quad \int e^{1-3x}\,dx = -\frac{1}{3}e^{1-3x} + c$

b $y = \ln(4x+1)$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \frac{1}{4x+1} \times 4 \\ &= \frac{4}{4x+1} \end{aligned}$$

$\therefore \quad \int \frac{4}{4x+1}\,dx = \ln(4x+1) + c_1$

$\therefore \quad 4\int \frac{1}{4x+1}\,dx = \ln(4x+1) + c_1$

$\therefore \quad \int \frac{1}{4x+1}\,dx = \frac{1}{4}\ln(4x+1) + c \quad (4x+1>0)$

10 **a** $\dfrac{d}{dx}(e^x - x^2) = e^{x-x^2}(1-2x)$, $\quad\therefore\quad \int e^{x-x^2}(1-2x)\,dx = e^{x-x^2} + c$

b $\dfrac{d}{dx}(\ln(5-3x+x^2)) = \dfrac{2x-3}{5-3x+x^2}$

$\therefore\quad \displaystyle\int \frac{2x-3}{5-3x+x^2}\,dx = \ln|\,5-3x+x^2\,| + c_1$

$\therefore\quad \displaystyle\int \frac{4x-6}{5-3x+x^2}\,dx = 2\ln(5-3x+x^2) + c$ $\quad\{5-3x+x^2>0$ for all x as $a>0$ and $\Delta=-11<0\}$

c $\dfrac{d}{dx}(x^2-5x+1)^{-2}$

$= -2(x^2-5x+1)^{-3}(2x-5)$

$= \dfrac{-4x+10}{(x^2-5x+1)^3}$

$\therefore\quad \displaystyle\int \frac{-4x+10}{(x^2-5x+1)^3}\,dx = (x^2-5x+1)^{-2} + c_1$

$\therefore\quad \displaystyle\int \frac{-2(2x-5)}{(x^2-5x+1)^3}\,dx = (x^2-5x+1)^{-2} + c_1$

$\therefore\quad -2\displaystyle\int \frac{2x-5}{(x^2-5x+1)^3}\,dx = (x^2-5x+1)^{-2} + c_1$

$\therefore\quad \displaystyle\int \frac{2x-5}{(x^2-5x+1)^3}\,dx = -\tfrac{1}{2}(x^2-5x+1)^{-2} + c$

d $\dfrac{d}{dx}(xe^x) = 1\times e^x + xe^x$

$\therefore\quad \int (e^x + xe^x)\,dx = xe^x + c_1$

$\therefore\quad \int e^x\,dx + \int xe^x\,dx = xe^x + c_1$

$\therefore\quad e^x + \int xe^x\,dx = xe^x + c_1$

$\therefore\quad \int xe^x\,dx = xe^x - e^x + c$

e $\dfrac{d}{dx}(2^x) = \dfrac{d}{dx}(e^{\ln 2})^x \quad \{\text{since } 2^x = (e^{\ln 2})^x\}$

$= e^{(\ln 2)x}\times \ln 2$

$= 2^x \ln 2$

$\therefore\quad \int 2^x \ln 2\,dx = 2^x + c_1$

$\therefore\quad \ln 2 \int 2^x\,dx = 2^x + c_1$

$\therefore\quad \int 2^x\,dx = \dfrac{2^x}{\ln 2} + c$

f $\dfrac{d}{dx}(x\ln x) = 1\times \ln x + x\times\dfrac{1}{x}$

$= \ln x + 1$

$\therefore\quad \int(\ln x + 1)\,dx = x\ln x + c_1$

$\therefore\quad \int \ln x\,dx + x = x\ln x + c_1$

$\therefore\quad \int \ln x\,dx = x\ln x - x + c$

EXERCISE 26E.2

1 **a** $\int(x^4 - x^2 - x + 2)\,dx$

$= \tfrac{1}{5}x^5 - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x + c$

b $\displaystyle\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

c $\displaystyle\int\left(2e^x - \frac{1}{x^2}\right)dx$

$= \int\left(2e^x - x^{-2}\right)dx$

$= 2\int e^x\,dx - \dfrac{x^{-1}}{-1} + c$

$= 2e^x + \dfrac{1}{x} + c$

d $\displaystyle\int\left(x\sqrt{x} - \frac{1}{x}\right)dx$

$= \int\left(x^{\frac{3}{2}} - x^{-1}\right)dx$

$= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} - \ln|x| + c$

$= \tfrac{2}{5}x^{\frac{5}{2}} - \ln|x| + c$

e
$$\begin{aligned}&\int(2x+1)^2\ dx\\&=\int(4x^2+4x+1)\ dx\\&=\frac{4x^3}{3}+\frac{4x^2}{2}+x+c\\&=\tfrac{4}{3}x^3+2x^2+x+c\end{aligned}$$

f
$$\begin{aligned}&\int\frac{x^2+x-3}{x}\ dx\\&=\int\left(x+1-\frac{3}{x}\right)\ dx\\&=\frac{x^2}{2}+x-3\ln|x|+c\end{aligned}$$

g
$$\begin{aligned}&\int\frac{2x-1}{\sqrt{x}}\ dx\\&=\int\left(2x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)\ dx\\&=\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c\\&=\tfrac{4}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+c\end{aligned}$$

h
$$\begin{aligned}&\int\left(\frac{1}{x\sqrt{x}}-\frac{4}{x}\right)\ dx\\&=\int\left(x^{-\frac{3}{2}}-\frac{4}{x}\right)\ dx\\&=\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}-4\ln|x|+c\\&=-2x^{-\frac{1}{2}}-4\ln|x|+c\\&=-\frac{2}{\sqrt{x}}-4\ln|x|+c\end{aligned}$$

i
$$\begin{aligned}&\int(x+1)^3\,dx\\&=\int(x^3+3x^2+3x+1)\ dx\\&=\tfrac{1}{4}x^4+x^3+\tfrac{3}{2}x^2+x+c\end{aligned}$$

2 **a**
$$\begin{aligned}\frac{dy}{dx}&=(1-2x)^2\\\therefore\quad y&=\int(1-2x)^2\ dx\\&=\int\left(1-4x+4x^2\right)\ dx\\&=x-\frac{4x^2}{2}+\frac{4x^3}{3}+c\\&=x-2x^2+\tfrac{4}{3}x^3+c\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx}&=\sqrt{x}-\frac{2}{\sqrt{x}}\\&=x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\\\therefore\quad y&=\int\left(x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)\ dx\\&=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}+c\\&=\tfrac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+c\end{aligned}$$

c
$$\begin{aligned}\frac{dy}{dx}&=\frac{x^2+2x-5}{x^2}\\&=1+2x^{-1}-5x^{-2}\\\therefore\quad y&=\int\left(1+2x^{-1}-5x^{-2}\right)\ dx\\&=x+2\ln|x|-\frac{5x^{-1}}{-1}+c\\&=x+2\ln|x|+\frac{5}{x}+c\end{aligned}$$

3 **a**
$$\begin{aligned}f'(x)&=x^3-5x+3\\\therefore\quad f(x)&=\int(x^3-5x+3)\ dx\\&=\frac{x^4}{4}-\frac{5x^2}{2}+3x+c\end{aligned}$$

b
$$\begin{aligned}f'(x)&=2\sqrt{x}(1-3x)\\&=2x^{\frac{1}{2}}-6x^{\frac{3}{2}}\\\therefore\quad f(x)&=\int\left(2x^{\frac{1}{2}}-6x^{\frac{3}{2}}\right)\ dx\\&=\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{6x^{\frac{5}{2}}}{\frac{5}{2}}+c\\&=\tfrac{4}{3}x^{\frac{3}{2}}-\tfrac{12}{5}x^{\frac{5}{2}}+c\end{aligned}$$

c
$$\begin{aligned}f'(x)&=3e^x-\frac{4}{x}\\\therefore\quad f(x)&=\int\left(3e^x-\frac{4}{x}\right)\ dx\\&=3e^x-4\ln|x|+c\end{aligned}$$

4 **a** $f'(x) = 2x - 1$ and $f(0) = 3$

$$\begin{aligned} f(x) &= \int (2x-1)\,dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c \end{aligned}$$

$$\begin{aligned} \text{But}\quad f(0) &= 3 \\ \therefore\quad 0 - 0 + c &= 3 \\ \text{i.e.,}\quad c &= 3 \\ \therefore\quad f(x) &= x^2 - x + 3 \end{aligned}$$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

$$\begin{aligned} f(x) &= \int (3x^2 + 2x)\,dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c \end{aligned}$$

$$\begin{aligned} \text{But}\quad f(2) &= 5 \\ \therefore\quad 8 + 4 + c &= 5 \\ \text{i.e.,}\quad c &= -7 \\ \therefore\quad f(x) &= x^3 + x^2 - 7 \end{aligned}$$

c $f'(x) = e^x + \dfrac{1}{\sqrt{x}}$ and $f(1) = 1$

$$\begin{aligned} f'(x) &= e^x + x^{-\frac{1}{2}} \\ f(x) &= \int \left(e^x + x^{-\frac{1}{2}}\right)\,dx \\ &= e^x + 2x^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \text{But}\quad f(1) &= 1 \\ \therefore\quad e^1 + 2 + c &= 1 \\ \text{i.e.,}\quad c &= -1 - e \\ \therefore\quad f(x) &= e^x + 2\sqrt{x} - 1 - e \end{aligned}$$

d $f'(x) = x - \dfrac{2}{\sqrt{x}}$ and $f(1) = 2$

$$\begin{aligned} f'(x) &= x - 2x^{-\frac{1}{2}} \\ f(x) &= \int \left(x - 2x^{-\frac{1}{2}}\right)\,dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \tfrac{1}{2}x^2 - 4\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \text{But}\quad f(1) &= 2 \\ \therefore\quad \tfrac{1}{2} - 4 + c &= 2 \\ \text{i.e.,}\quad c &= \tfrac{11}{2} \\ \therefore\quad f(x) &= \tfrac{1}{2}x^2 - 4\sqrt{x} + \tfrac{11}{2} \end{aligned}$$

5 **a** $f''(x) = 2x + 1 \qquad f'(1) = 3 \qquad f(2) = 7$

Now $f'(x) = \dfrac{2x^2}{2} + x + c = x^2 + x + c$

But $f'(1) = 3$

$$\begin{aligned} \therefore\quad 1 + 1 + c &= 3 \\ \therefore\quad c &= 1 \\ \therefore\quad f'(x) &= x^2 + x + 1 \\ \therefore\quad f(x) &= \frac{x^3}{3} + \frac{x^2}{2} + x + k \end{aligned}$$

and $f(2) = 7$

$$\begin{aligned} \therefore\quad \tfrac{8}{3} + 2 + 2 + k &= 7 \\ k &= 7 - 4 - \tfrac{8}{3} \\ \therefore\quad k &= \tfrac{1}{3} \\ \therefore\quad f(x) &= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + \tfrac{1}{3} \end{aligned}$$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}} \qquad f'(1) = 12 \qquad f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\therefore\quad f'(x) = \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$$

But $f'(1) = 12 \quad \therefore \quad 10 + 6 + c = 12$ and so $c = -4$

$$\therefore\quad f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$$

$$\therefore\quad f(x) = \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$$

But $f(0) = 5 \quad \therefore \quad k = 5 \qquad \therefore \quad f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f''(x) = 2x$ $(1, 0)$ and $(0, 5)$ lie on the curve

Now $f'(x) = \dfrac{2x^2}{2} + c = x^2 + c$ and $f(x) = \dfrac{x^3}{3} + cx + k$

But $f(0) = 5$ $\therefore$ $0 + 0 + k = 5$ and so $k = 5$

and $f(1) = 0$ $\therefore$ $\frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$\therefore$ $f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 26F

1 **a** $\int (2x+5)^3\,dx$

$= \frac{1}{2} \times \dfrac{(2x+5)^4}{4} + c$

$= \frac{1}{8}(2x+5)^4 + c$

b $\displaystyle\int \frac{1}{(3-2x)^2}\,dx$

$= \int (3-2x)^{-2}\,dx$

$= \frac{1}{-2} \times \dfrac{(3-2x)^{-1}}{-1} + c$

$= \dfrac{1}{2(3-2x)} + c$

c $\displaystyle\int \frac{4}{(2x-1)^4}\,dx$

$= \int 4\,(2x-1)^{-4}\,dx$

$= 4(\frac{1}{2}) \times \dfrac{(2x-1)^{-3}}{-3} + c$

$= -\frac{2}{3}(2x-1)^{-3} + c$

d $\int (4x-3)^7\,dx$

$= \frac{1}{4} \times \dfrac{(4x-3)^8}{8} + c$

$= \frac{1}{32}(4x-3)^8 + c$

e $\int \sqrt{3x-4}\,dx$

$= \int (3x-4)^{\frac{1}{2}}\,dx$

$= \frac{1}{3} \times \dfrac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

f $\displaystyle\int \frac{10}{\sqrt{1-5x}}\,dx$

$= \int 10(1-5x)^{-\frac{1}{2}}\,dx$

$= 10(\frac{1}{-5}) \times \dfrac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -4(1-5x)^{\frac{1}{2}} + c$

g $\int 3(1-x)^4\,dx$

$= 3\int (1-x)^4\,dx$

$= 3(\frac{1}{-1}) \times \dfrac{(1-x)^5}{5} + c$

$= -\frac{3}{5}(1-x)^5 + c$

h $\displaystyle\int \frac{4}{\sqrt{3-4x}}\,dx$

$= \int 4(3-4x)^{-\frac{1}{2}}\,dx$

$= 4(\frac{1}{-4}) \times \dfrac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -2\sqrt{3-4x} + c$

i $\int \sqrt[3]{2x-1}\,dx$

$= \int (2x-1)^{\frac{1}{3}}\,dx$

$= \frac{1}{2} \times \dfrac{(2x-1)^{\frac{4}{3}}}{\frac{4}{3}} + c$

$= \frac{3}{8}(2x-1)^{\frac{4}{3}} + c$

2 **a** $\dfrac{dy}{dx} = \sqrt{2x-7}$

$= (2x-7)^{\frac{1}{2}}$

$\therefore$ $y = \frac{1}{2} \times \dfrac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$

But $y = 11$ when $x = 8$

$\therefore$ $\frac{1}{3}(9)^{\frac{3}{2}} + c = 11$

$\frac{1}{3}(27) + c = 11$

$9 + c = 11$ and so $c = 2$

$\therefore$ $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$

b $f(x)$ has tangent-slope function $\dfrac{4}{\sqrt{1-x}}$

i.e., $f'(x) = \dfrac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$f(x) = 4(\frac{1}{-1}) \times \dfrac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -8\sqrt{1-x} + c$

But $y = 11$ when $x = -3$

$\therefore$ $-8\sqrt{1-(-3)} + c = 11$

$\therefore$ $-8\sqrt{4} + c = 11$

$\therefore$ $-16 + c = 11$ and $c = 5$

$\therefore$ $f(x) = 5 - 8\sqrt{1-x}$

When $x = -8$, $y = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19$, $\therefore$ point is $(-8, -19)$.

3 **a** $\int 3(2x-1)^2\,dx$
$= 3\int (2x-1)^2\,dx$
$= 3\left(\frac{1}{2}\right)\frac{(2x-1)^3}{3}+c$
$= \frac{1}{2}(2x-1)^3+c$

b $\int (x^2-x)^2\,dx$
$= \int \left(x^4-2x^3+x^2\right)dx$
$= \frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^3}{3}+c$
$= \frac{1}{5}x^5-\frac{1}{2}x^4+\frac{1}{3}x^3+c$

c $\int (1-3x)^3\,dx$
$= \left(\frac{1}{-3}\right)\frac{(1-3x)^4}{4}+c$
$= -\frac{1}{12}(1-3x)^4+c$

d $\int (1-x^2)^2\,dx$
$= \int (1-2x^2+x^4)\,dx$
$= x-\frac{2}{3}x^3+\frac{1}{5}x^5+c$

e $\int 4\sqrt{5-x}\,dx$
$= 4\int (5-x)^{\frac{1}{2}}\,dx$
$= 4\left(\frac{1}{-1}\right)\frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}}+c$
$= -4\left(\frac{2}{3}\right)(5-x)^{\frac{3}{2}}+c$
$= -\frac{8}{3}(5-x)^{\frac{3}{2}}+c$

f $\int (x^2+1)^3\,dx$
$= \int (x^6+3x^4+3x^2+1)\,dx$
$= \frac{x^7}{7}+\frac{3x^5}{5}+\frac{3x^3}{3}+x+c$
$= \frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x+c$

4 **a** $\int \left(2e^x+5e^{2x}\right)dx$
$= 2e^x+5(\frac{1}{2})e^{2x}+c$
$= 2e^x+\frac{5}{2}e^{2x}+c$

b $\int \left(x^2-2e^{-3x}\right)dx$
$= \frac{x^3}{3}-2(\frac{1}{-3})e^{-3x}+c$
$= \frac{1}{3}x^3+\frac{2}{3}e^{-3x}+c$

c $\int \left(\sqrt{x}+4e^{2x}-e^{-x}\right)dx$
$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}+4(\frac{1}{2})e^{2x}-(-1)e^{-x}+c$
$= \frac{2}{3}x^{\frac{3}{2}}+2e^{2x}+e^{-x}+c$

d $\int \frac{1}{2x-1}\,dx$
$= \frac{1}{2}\ln|2x-1|+c$

e $\int \frac{5}{1-3x}\,dx$
$= 5\int \frac{1}{1-3x}\,dx$
$= 5(\frac{1}{-3})\ln|1-3x|+c$
$= -\frac{5}{3}\ln|1-3x|+c$

f $\int \left(e^{-x}-\frac{4}{2x+1}\right)dx$
$= \frac{1}{-1}e^{-x}-4(\frac{1}{2})\ln|2x+1|+c$
$= -e^{-x}-2\ln|2x+1|+c$

g $\int (e^x+e^{-x})^2\,dx$
$= \int \left(e^{2x}+2+e^{-2x}\right)dx$
$= \frac{1}{2}e^{2x}+2x+(\frac{1}{-2})e^{-2x}+c$
$= \frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+c$

h $\int (e^{-x}+2)^2\,dx$
$= \int \left(e^{-2x}+4e^{-x}+4\right)dx$
$= \frac{1}{-2}e^{-2x}+4(\frac{1}{-1})e^{-x}+4x+c$
$= -\frac{1}{2}e^{-2x}-4e^{-x}+4x+c$

i $\int \left(x-\frac{5}{1-x}\right)dx = \frac{x^2}{2}-5(\frac{1}{-1})\ln|1-x|+c$
$= \frac{1}{2}x^2+5\ln|1-x|+c$

5 **a** $\frac{dy}{dx} = (1-e^x)^2$
$= 1-2e^x+e^{2x}$
$\therefore\ y = x-2e^x+\frac{1}{2}e^{2x}+c$

b $\frac{dy}{dx} = 1-2x+\frac{3}{x+2}$
$\therefore\ y = x-\frac{2x^2}{2}+3\ln|x+2|+c$
$= x-x^2+3\ln|x+2|+c$

c $\frac{dy}{dx} = e^{-2x}+\frac{4}{2x-1}$ $\quad\therefore\ y = \frac{1}{-2}e^{-2x}+4(\frac{1}{2})\ln|2x-1|+c$
$= -\frac{1}{2}e^{-2x}+2\ln|2x-1|+c$

6 Differentiating Tracy's answer gives

$$\frac{d}{dx}\left(\tfrac{1}{4}\ln|4x|+c\right)$$
$$=\tfrac{1}{4}\left(\frac{1}{4x}\right)\times 4+0$$
$$=\frac{1}{4x}$$

The derivative of Nadine's answer is

$$\frac{d}{dx}(\tfrac{1}{4}\ln|x|+c)$$
$$=\tfrac{1}{4}\left(\frac{1}{x}\right)+0$$
$$=\frac{1}{4x}$$

$\therefore$ both answers give the correct derivative and both are correct.

Note that this result occurs because $\log 4x=\log 4+\log x$ $\therefore$ their answers differ by a constant ($\log 4$) which is just part of the constant c.

7 **a**

$$f'(x)=2e^{-2x} \qquad f(0)=3$$
$$\therefore\ f(x)=2(\tfrac{1}{-2})e^{-2x}+c$$
$$=-e^{-2x}+c$$

But $f(0)=3$

$$\therefore\ -e^0+c=3$$
$$\therefore\ c=4$$
$$\therefore\ f(x)=-e^{-2x}+4$$

b

$$f'(x)=2x-\frac{2}{1-x} \qquad f(-1)=3$$
$$\therefore\ f(x)=\frac{2x^2}{2}-\tfrac{2}{-1}\ln|1-x|+c$$
$$=x^2+2\ln|1-x|+c$$

But $f(-1)=3$

$$\therefore\ 1+2\ln|2|+c=3$$
$$\therefore\ c=2-2\ln 2$$
$$\therefore\ f(x)=x^2+2\ln|1-x|+2-2\ln 2$$

c

$$f'(x)=\sqrt{x}+\tfrac{1}{2}e^{-4x}$$
$$\therefore\ f'(x)=x^{\frac{1}{2}}+\tfrac{1}{2}e^{-4x}$$
$$\text{and}\quad f(x)=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\tfrac{1}{2}(\tfrac{1}{-4})e^{-4x}+c$$
$$=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{8}e^{-4x}+c$$

But $f(1)=0$

$$\therefore\ \tfrac{2}{3}-\tfrac{1}{8}e^{-4}+c=0$$
$$\therefore\ c=\tfrac{1}{8}e^{-4}-\tfrac{2}{3}$$
$$\therefore\ f(x)=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{8}e^{-4x}+\tfrac{1}{8}e^{-4}-\tfrac{2}{3}$$

8

$$\frac{3}{x+2}-\frac{1}{x-2}$$
$$=\frac{3(x-2)-1(x+2)}{(x+2)(x-2)}$$
$$=\frac{3x-6-x-2}{x^2-4}$$
$$=\frac{2x-8}{x^2-4}$$

$$\therefore\ \int\frac{2x-8}{x^2-4}\,dx$$
$$=\int\left(\frac{3}{x+2}-\frac{1}{x-2}\right)dx$$
$$=3\ln|x+2|-\ln|x-2|+c$$

9

$$\frac{1}{2x-1}-\frac{1}{2x+1}$$
$$=\frac{1(2x+1)-1(2x-1)}{(2x-1)(2x+1)}$$
$$=\frac{2x+1-2x+1}{(2x-1)(2x+1)}$$
$$=\frac{2}{4x^2-1}$$

$$\therefore\ \int\frac{2}{4x^2-1}\,dx$$
$$=\int\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)dx$$
$$=\tfrac{1}{2}\ln|2x-1|-\tfrac{1}{2}\ln|2x+1|+c$$

EXERCISE 26G

1 **a** Let $u = x^3 + 1$, $\therefore \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \quad & \int 3x^2(x^3+1)^4\ dx\\ &= \int u^4\ \frac{du}{dx}\,dx\\ &= \int u^4\ du\\ &= \tfrac{1}{5}u^5 + c\\ &= \tfrac{1}{5}(x^3+1)^5 + c\end{aligned}$$

b Let $u = x^2 + 3$ $\therefore \dfrac{du}{dx} = 2x$

$$\begin{aligned}\therefore \quad & \int \frac{2x}{\sqrt{x^2+3}}\ dx\\ &= \int \left(\left(x^2+3\right)^{-\frac{1}{2}} \times 2x\right)\ dx\\ &= \int u^{-\frac{1}{2}}\ \frac{du}{dx}\,dx\\ &= \int u^{-\frac{1}{2}}\ du\\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= 2\sqrt{u} + c\\ &= 2\sqrt{x^2+3} + c\end{aligned}$$

c Let $u = x^3 + x$ $\therefore \dfrac{du}{dx} = 3x^2 + 1$

$$\begin{aligned}\therefore \quad & \int \sqrt{x^3+x}\,(3x^2+1)\ dx\\ &= \int \sqrt{u}\ \frac{du}{dx}\,dx\\ &= \int u^{\frac{1}{2}}\ du\\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c\\ &= \tfrac{2}{3}(x^3+x)^{\frac{3}{2}} + c\end{aligned}$$

d Let $u = 2 + x^4$ $\therefore \dfrac{du}{dx} = 4x^3$

$$\begin{aligned}\therefore \quad & \int 4x^3(2+x^4)^3\ dx\\ &= \int u^3\ \frac{du}{dx}\,dx\\ &= \int u^3\ du\\ &= \frac{u^4}{4} + c\\ &= \tfrac{1}{4}(2+x^4)^4 + c\end{aligned}$$

e Let $u = x^3 + 2x + 1$ $\therefore \dfrac{du}{dx} = 3x^2 + 2$

$$\begin{aligned}\therefore \quad & \int (x^3+2x+1)^4(3x^2+2)\ dx\\ &= \int u^4\ \frac{du}{dx}\,dx\\ &= \int u^4\ du\\ &= \frac{u^5}{5} + c\\ &= \tfrac{1}{5}(x^3+2x+1)^5 + c\end{aligned}$$

f Let $u = 3x^3 - 1$ $\therefore \dfrac{du}{dx} = 9x^2$

$$\begin{aligned}\therefore \quad & \int \frac{x^2}{(3x^3-1)^4}\ dx\\ &= \int (3x^3-1)^{-4} \times x^2)\ dx\\ &= \int u^{-4}\left(\tfrac{1}{9}\,\frac{du}{dx}\right)\ dx\\ &= \tfrac{1}{9}\int u^{-4}\ du\\ &= \tfrac{1}{9}\,\frac{u^{-3}}{-3} + c\\ &= -\frac{1}{27(3x^3-1)^3} + c\end{aligned}$$

g Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore \quad & \int \frac{x}{(1-x^2)^5}\, dx \\ &= \int \left((1-x^2)^{-5} \times x\right) dx \\ &= \int u^{-5}\left(-\tfrac{1}{2}\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\int u^{-5}\, du \\ &= -\tfrac{1}{2}\,\frac{u^{-4}}{-4} + c \\ &= \frac{1}{8(1-x^2)^4} + c\end{aligned}$$

h Let $u = x^2 + 4x - 3 \quad \therefore \quad \dfrac{du}{dx} = 2x + 4$

$$\begin{aligned}\therefore \quad & \int \frac{x+2}{(x^2+4x-3)^2}\, dx \\ &= \int (x^2+4x-3)^{-2}(x+2))\, dx \\ &= \int u^{-2}\left(\tfrac{1}{2}\frac{du}{dx}\right) dx \\ &= \tfrac{1}{2}\int u^{-2}\, du \\ &= \tfrac{1}{2}\,\frac{u^{-1}}{-1} + c \\ &= \frac{-1}{2(x^2+4x-3)} + c\end{aligned}$$

i Let $u = x^2 + x \quad \therefore \quad \dfrac{du}{dx} = 2x + 1$

$$\begin{aligned}\therefore \quad & \int x^4(x+1)^4(2x+1)\, dx \\ &= \int (x^2+x)^4(2x+1)\, dx \quad \{\text{as } a^4b^4 = (ab)^4\} \\ &= \int u^4 \frac{du}{dx}\, dx = \int u^4\, du \\ &= \tfrac{1}{5}u^5 + c \\ &= \tfrac{1}{5}(x^2+x)^5 + c\end{aligned}$$

2 **a** Let $u = 1 - 2x \quad \therefore \quad \dfrac{du}{dx} = -2$

$$\begin{aligned}\therefore \quad & \int -2e^{1-2x}\, dx \\ &= \int e^u \frac{du}{dx}\, dx \\ &= \int e^u\, du \\ &= e^u + c \\ &= e^{1-2x} + c\end{aligned}$$

b Let $u = x^2 \quad \therefore \quad \dfrac{du}{dx} = 2x$

$$\begin{aligned}\therefore \quad & \int 2xe^{x^2}\, dx \\ &= \int e^u \frac{du}{dx}\, dx \\ &= \int e^u\, du \\ &= e^u + c \\ &= e^{x^2} + c\end{aligned}$$

c Let $u = x^3 + 1 \quad \therefore \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}\therefore \quad & \int x^2e^{x^3+1}\, dx \\ &= \int e^u\left(\tfrac{1}{3}\frac{du}{dx}\right) dx \\ &= \tfrac{1}{3}\int e^u\, du \\ &= \tfrac{1}{3}e^u + c \\ &= \tfrac{1}{3}e^{x^3+1} + c\end{aligned}$$

d Let $u = \sqrt{x} \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$$\begin{aligned}\therefore \quad & \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx \\ &= \int e^u\left(2\frac{du}{dx}\right) dx \\ &= 2\int e^u\, du \\ &= 2e^u + c \\ &= 2e^{\sqrt{x}} + c\end{aligned}$$

e Let $u = x - x^2 \quad \therefore \quad \dfrac{du}{dx} = 1 - 2x$

$$\begin{aligned}\therefore \quad & \int (2x-1)e^{x-x^2}\, dx \\ &= \int e^u\left(-\frac{du}{dx}\right) dx \\ &= \int -e^u\, du \\ &= -e^u + c \\ &= -e^{x-x^2} + c\end{aligned}$$

f Let $u = \dfrac{x-1}{x} = 1 - x^{-1} \quad \therefore \quad \dfrac{du}{dx} = 0 - (-1)x^{-2} = \dfrac{1}{x^2}$

$$\begin{aligned}\therefore \quad \int \frac{e^{\frac{x-1}{x}}}{x^2}\, dx &= \int e^u \frac{du}{dx}\, dx \\ &= \int e^u\, du \\ &= e^u + c \\ &= e^{\frac{x-1}{x}} + c\end{aligned}$$

3 **a** Let $u = x^2 + 1 \quad \therefore \quad \dfrac{du}{dx} = 2x$

$$\begin{aligned}\therefore \quad & \int \frac{2x}{x^2+1}\, dx \\ &= \int \frac{1}{x^2+1}\,(2x)\, dx \\ &= \int \frac{1}{u}\,\frac{du}{dx}\, dx \\ &= \int \frac{1}{u}\, du \\ &= \ln|u| + c \\ &= \ln\left|x^2+1\right| + c \\ &= \ln(x^2+1) + c \quad \text{as} \quad x^2+1>0\end{aligned}$$

b Let $u = 2 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore \quad & \int \frac{x}{2-x^2}\, dx \\ &= \int \left(\frac{1}{2-x^2} \times x\right) dx \\ &= \int \frac{1}{u}\left(-\tfrac{1}{2}\,\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\int \frac{1}{u}\, du \\ &= -\tfrac{1}{2}\ln|u| + c \\ &= -\tfrac{1}{2}\ln\left|2-x^2\right| + c\end{aligned}$$

c Let $u = x^2 - 3x \quad \therefore \quad \dfrac{du}{dx} = 2x - 3$

$$\begin{aligned}\therefore \quad & \int \frac{2x-3}{x^2-3x}\, dx \\ &= \int \frac{1}{x^2-3x}\,(2x-3)\, dx \\ &= \int \frac{1}{u}\,\frac{du}{dx}\, dx \\ &= \int \frac{1}{u}\, du \\ &= \ln|u| + c \\ &= \ln\left|x^2-3x\right| + c\end{aligned}$$

d Let $u = x^3 - x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore \quad & \int \frac{6x^2-2}{x^3-x}\, dx \\ &= \int \frac{1}{x^3-3x}\,(6x^2-2)\, dx \\ &= \int \frac{1}{u}\left(2\,\frac{du}{dx}\right) dx \\ &= 2\int \frac{1}{u}\, du \\ &= 2\ln|u| + c \\ &= 2\ln\left|x^3-x\right| + c\end{aligned}$$

e Let $u = 5x - x^2 \quad \therefore \quad \dfrac{du}{dx} = 5 - 2x$

$$\begin{aligned}\therefore \quad & \int \frac{4x-10}{5x-x^2}\, dx \\ &= \int \frac{1}{5x-x^2}\,(4x-10)\, dx \\ &= \int \frac{1}{u}\left(-2\,\frac{du}{dx}\right) dx \\ &= -2\int \frac{1}{u}\, dx \\ &= -2\ln|u| + c \\ &= -2\ln\left|5x-x^2\right| + c\end{aligned}$$

f Let $u = x^3 - 3x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 3$

$$\begin{aligned}\therefore \quad & \int \frac{1-x^2}{x^3-3x}\, dx \\ &= \int \frac{1}{x^3-3x}\,(1-x^2)\, dx \\ &= \int \frac{1}{u}\left(-\tfrac{1}{3}\,\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{3}\int \frac{1}{u}\, du \\ &= -\tfrac{1}{3}\ln|u| + c \\ &= -\tfrac{1}{3}\ln\left|x^3-3x\right| + c\end{aligned}$$

4 **a** $f(x) = \int x^2(3-x^3)^2\, dx = \displaystyle\int u^2\left(-\tfrac{1}{3}\,\frac{du}{dx}\right) dx \quad \left\{\text{letting} \quad u = 3 - x^3 \quad \therefore \quad \frac{du}{dx} = -3x^2\right\}$

$$\begin{aligned}&= -\tfrac{1}{3}\int u^2\, du \\ &= -\tfrac{1}{3} \times \frac{u^3}{3} + c \\ &= -\tfrac{1}{9}(3-x^3)^3 + c\end{aligned}$$

b Let $u = x^2 - 2 \quad \therefore \quad \dfrac{du}{dx} = 2x$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{3x}{x^2-2}\, dx \\ &= \int \frac{1}{x^2-2}(3x)\, dx \\ &= \int \frac{1}{u}\left(\tfrac{3}{2}\frac{du}{dx}\right) dx \\ &= \tfrac{3}{2}\int \frac{1}{u}\, du \\ &= \tfrac{3}{2}\ln|u| + c \\ &= \tfrac{3}{2}\ln\left|x^2-2\right| + c\end{aligned}$$

c Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore f(x) &= \int x\sqrt{1-x^2}\, dx \\ &= \int \sqrt{u}\left(-\tfrac{1}{2}\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= -\tfrac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= -\tfrac{1}{3}u^{\frac{3}{2}} + c \\ &= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c\end{aligned}$$

d Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore \quad f(x) &= \int xe^{1-x^2}\, dx \\ &= \int e^u\left(-\tfrac{1}{2}\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\int e^u\, du \\ &= -\tfrac{1}{2}e^u + c \\ &= -\tfrac{1}{2}e^{1-x^2} + c\end{aligned}$$

e Let $u = x^3 - x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{1-3x^2}{x^3-x}\, dx \\ &= \int \frac{1}{u}\left(-\frac{du}{dx}\right) dx \\ &= -\int \frac{1}{u}\, du \\ &= -\ln|u| + c \\ &= -\ln\left|x^3-x\right| + c\end{aligned}$$

f Let $u = \ln x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{(\ln x)^3}{x}\, dx \\ &= \int u^3\, \frac{du}{dx}\, dx \\ &= \int u^3\, du \\ &= \frac{u^4}{4} + c \\ &= \tfrac{1}{4}(\ln x)^4 + c\end{aligned}$$

g Let $u = x^3 + 2x^2 - 1 \quad \therefore \quad \dfrac{du}{dx} = 3x^2 + 4x$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{4x+3x^2}{x^3+2x^2-1}\, dx \\ &= \int \frac{1}{u}\, \frac{du}{dx}\, dx \\ &= \int \frac{1}{u}\, du \\ &= \ln|u| + c \\ &= \ln\left|x^3+2x^2-1\right| + c\end{aligned}$$

h Let $u = \ln x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{4}{x\ln x}\, dx \\ &= \int u^{-1}\left(4\frac{du}{dx}\right) dx \\ &= 4\int \frac{1}{u}\, du \\ &= 4\ln|u| + c \\ &= 4\ln|\ln x| + c\end{aligned}$$

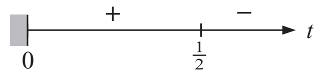
i Let $u = \ln x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore f(x) &= \int \frac{1}{x(\ln x)^2}\, dx \\ &= \int u^{-2}\, \frac{du}{dx}\, dx \\ &= \int u^{-2}\, du \\ &= \frac{u^{-1}}{-1} + c \\ &= -\frac{1}{\ln x} + c\end{aligned}$$

EXERCISE 26H

1 $v(t) = 1 - 2t \text{ cms}^{-1}, \quad t \geqslant 0$

$v(t) = s'(t) = 1 - 2t$ which has sign diagram:



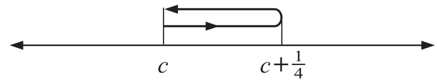
$\therefore$ a direction reversal occurs at $t = \frac{1}{2}$.

Now $s(t) = \int (1 - 2t)\, dt = t - \dfrac{2t^2}{2} + c = t - t^2 + c$

$\therefore \quad s(0) = c$

and $s(\frac{1}{2}) = \frac{1}{4} + c$ $\qquad \therefore$ motion diagram is:

and $s(1) = c$



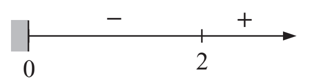
$\therefore$ total distance travelled $= (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c)$

$= \frac{1}{2}$ cm

2 $v(t) = t^2 - t - 2 \text{ cms}^{-1}, \quad t \geqslant 0$

$v(t) = s'(t) = t^2 - t - 2$

$= (t-2)(t+1)$ which has sign diagram:



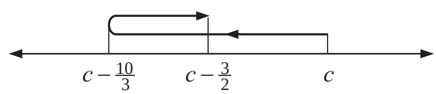
$\therefore$ a direction reversal occurs at $t = 2$.

Now $s(t) = \int (t^2 - t - 2)\, dt = \dfrac{t^3}{3} - \dfrac{t^2}{2} - 2t + c$

$s(0) = c$

$s(2) = c - \frac{10}{3}$ $\qquad \therefore$ motion diagram is:

$s(3) = c - \frac{3}{2}$



$\therefore$ total distance travelled $= \left(c - [c - \frac{10}{3}]\right) + \left(c - \frac{3}{2} - [c - \frac{10}{3}]\right)$

$= \frac{10}{3} - \frac{3}{2} + \frac{10}{3}$

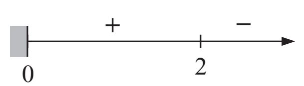
$= \frac{31}{6}$

$= 5\frac{1}{6}$ cm

3 $x'(t) = 16t - 4t^3 \text{ units s}^{-1}, \quad t \geqslant 0$

$= 4t(4 - t^2)$

$= 4t(2+t)(2-t)$ which has sign diagram:



$\therefore$ a direction reversal occurs at $t = 2$.

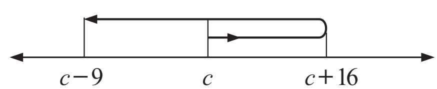
Now $x(t) = \int (16t - 4t^3)\, dt = 8t^2 - t^4 + c$

a $0 \leqslant t \leqslant 3$ $\qquad \therefore$ motion diagram is:

$x(0) = c$

$x(2) = 32 - 16 + c = c + 16$

$x(3) = 72 - 81 + c = c - 9$

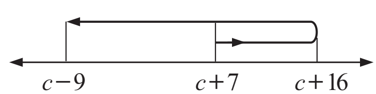


$\therefore$ total distance travelled $= (c + 16 - c) + (c + 16 - [c - 9])$

$= 41$ units

b $1 \leqslant t \leqslant 3$

$x(1) = 7 + c = c + 7$ $\qquad \therefore$ motion diagram is:



$\therefore$ total distance travelled $= (c + 16 - [c + 7]) + (c + 16 - [c - 9])$

$= 34$ units

4 $v(t) = 50 - 10e^{-0.5t}$ ms^{-1} $\quad t \geqslant 0$

a $v(0) = 50 - \dfrac{10}{e^0} = 50 - 10 = 40$ ms^{-1}

b $v(3) = 50 - \dfrac{10}{e^{1.5}} \doteqdot 47.77$ ms^{-1}

c The velocity reaches 45 ms^{-1} when

$$\begin{aligned} 45 &= 50 - 10e^{-0.5t} \\ \therefore\ 10e^{-\frac{t}{2}} &= 5 \\ \therefore\ e^{-\frac{t}{2}} &= \tfrac{1}{2} \\ \therefore\ e^{\frac{t}{2}} &= 2 \\ \therefore\ \frac{t}{2} &= \ln 2 \\ \therefore\ t &= 2\ln 2 \doteqdot 1.386 \text{ sec} \end{aligned}$$

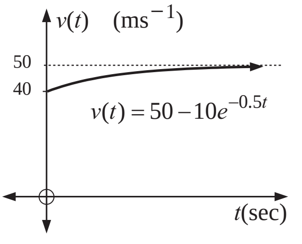
d $v(t) = 50 - \dfrac{10}{e^{\frac{t}{2}}}$, $\quad \therefore$ as $\quad t \to \infty$, $\quad \dfrac{10}{e^{\frac{t}{2}}} \to {}^{+}0$ $\quad$ and so $\quad v(t) \to 50$ ms^{-1} $\quad$ (below)

e
$$\begin{aligned} a(t) &= v'(t) \\ &= -10e^{-0.5t}(-0.5) \\ &= 5e^{-0.5t} \text{ ms}^{-2} \\ &= \frac{5}{e^{0.5t}} \text{ ms}^{-2} \end{aligned}$$

$\therefore\quad a(t) > 0 \quad$ for all $t \quad \{e^x > 0 \ $ for all $x\}$

i.e., $\quad$ the acceleration is always positive

f



g $s(t) = \int \left(50 - 10e^{-0.5t}\right) dt = 50t - 10\left(\frac{1}{-0.5}\right) e^{-0.5t} + c = 50t \ + \ 20e^{-0.5t} + c$

$s(0) = 0 + 20 + c = 20 + c \quad$ and $\quad s(3) = 150 + 20e^{-1.5} + c$

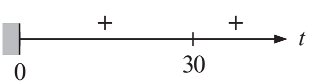
and from the graph in **f** it can be seen that the particle does not change direction,

$$\begin{aligned} \therefore\ \text{distance travelled} &= (150 + 20e^{-\frac{3}{2}} + c) - (20 + c) \\ &= 130 + 20e^{-\frac{3}{2}} \quad \text{or} \quad \doteqdot 134.5 \text{ m} \end{aligned}$$

5 $\quad a(t) = \dfrac{t}{10} - 3$ ms^{-2} $\quad \therefore \quad v(t) = \displaystyle\int \left(\frac{t}{10} - 3\right) dt = \frac{t^2}{20} - 3t + c$

But $\quad v(0) = 45 \quad \therefore \quad c = 45$

Now
$$\begin{aligned} v(t) &= \frac{t^2}{20} - 3t + 45 \\ &= \frac{t^2 - 60t + 900}{20} \\ &= \frac{(t-30)^2}{20} \end{aligned}$$
which has sign diagram:



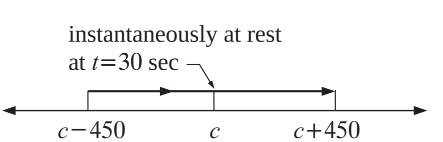
$\therefore\quad$ the train only stops and does not change direction at $t = 30$ seconds

Now
$$\begin{aligned} s(t) &= \int \frac{(t-30)^2}{20}\, dt \\ &= \tfrac{1}{20}\, \frac{(t-30)^3}{3} + c \\ &= \tfrac{1}{60}(t-30)^3 + c \end{aligned}$$

$\therefore\quad s(0) = -\frac{27\,000}{60} + c = c - 450$

and $\quad s(60) = \frac{27\,000}{60} + c = c + 450$,

the motion diagram is:



$\therefore\quad$ the total distance travelled $= c + 450 - (c - 450) = 900$ m

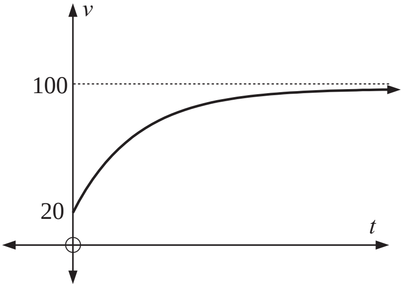
6 $a(t) = 4e^{-\frac{t}{20}}\ \text{ms}^{-2}$

$\therefore\ v(t) = \int 4e^{-\frac{t}{20}}\ dt = 4\left(\dfrac{1}{-\frac{1}{20}}\right)e^{-\frac{t}{20}} + c = -80e^{-\frac{t}{20}} + c$

Now $v(0) = 20\ \text{ms}^{-1}$ $\therefore$ $c = 100$

$\therefore\ v(t) = 100 - 80e^{-\frac{t}{20}}$

a as $t \to \infty$, $e^{-\frac{t}{20}} \to 0$ (above) $\therefore$ $v(t) \to 100$ (below)
i.e., the body approaches a limiting velocity of $100\ \text{ms}^{-1}$



b Now for all $t > 0$, $v(t) > 0$
$\therefore$ no change in direction occurs.

$s(t) = \int \left(100 - 80e^{-\frac{t}{20}}\right)\ dt$

$= 100t - 80\left(\dfrac{1}{-\frac{1}{20}}\right)e^{-\frac{t}{20}} + c$

$= 100t + 1600e^{-\frac{t}{20}} + c$

$s(0) = 0 + 1600 + c = 1600 + c$

$s(10) = 1000 + 1600e^{-\frac{1}{2}} + c$

$\therefore$ distance travelled $= (1000 + 1600e^{-\frac{1}{2}} + c) - (1600 + c) \doteq 370.4$ m
$\doteq 370.4$ m

EXERCISE 26I

1 **a** $\int_0^1 x^3\ dx$
$= \left[\dfrac{x^4}{4}\right]_0^1$
$= \frac{1}{4} - 0$
$= \frac{1}{4}$

b $\int_0^2 (x^2 - x)\,dx$
$= \left[\dfrac{x^3}{3} - \dfrac{x^2}{2}\right]_0^2$
$= (\frac{8}{3} - 2) - (0 - 0)$
$= \frac{2}{3}$

c $\int_0^1 e^x\ dx$
$= [e^x]_0^1$
$= e^1 - e^0$
$= e - 1$
$\doteq 1.718$

d $\displaystyle\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right)\ dx$
$= \int_1^4 \left(x - 3x^{-\frac{1}{2}}\right)\ dx$
$= \left[\dfrac{x^2}{2} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$
$= \left[\dfrac{x^2}{2} - 6\sqrt{x}\right]_1^4$
$= \left[\frac{16}{2} - 12\right] - (\frac{1}{2} - 6)$
$= 1\frac{1}{2}$

e $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\ dx$
$= \int_4^9 \left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\right)\ dx$
$= \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9$
$= \left[\frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_4^9$
$= \left[\frac{2}{3}(27) - 6(3)\right] - \left[\frac{2}{3}(8) - 6(2)\right]$
$= (18 - 18) - (\frac{16}{3} - 12)$
$= 6\frac{2}{3}$

f $\displaystyle\int_1^3 \frac{1}{x}\ dx$
$= [\ln|x|]_1^3$
$= \ln 3 - \ln 1$
$= \ln 3 - 0$
$= \ln 3$
$\doteq 1.099$

g $\int_1^2 (e^{-x}+1)^2\ dx$

$= \int_1^2 (e^{-2x}+2e^{-x}+1)\ dx$

$= \left[(\frac{1}{-2})e^{-2x} + 2(\frac{1}{-1})e^{-x} + x\right]_1^2$

$= \left[-\dfrac{e^{-2x}}{2} - 2e^{-x} + x\right]_1^2$

$= \left(-\dfrac{e^{-4}}{2} - 2e^{-2} + 2\right) - \left(-\dfrac{e^{-2}}{2} - 2e^{-1} + 1\right)$

$\doteqdot 1.524$

h $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\ dx$

$= \int_2^6 (2x-3)^{-\frac{1}{2}}\ dx$

$= \left[\frac{1}{2}\dfrac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^6$

$= \left[\sqrt{2x-3}\right]_2^6$

$= \sqrt{9} - \sqrt{1}$

$= 2$

i $\int_0^1 e^{1-x}\ dx$

$= \left[(\frac{1}{-1})e^{1-x}\right]_0^1$

$= \left(\dfrac{e^0}{-1}\right) - \left(\dfrac{e^1}{-1}\right)$

$= -1 + e$

$\doteqdot 1.718$

2 **a** In $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\ dx$

we let $u = x^2 + 2$

$\therefore\ \dfrac{du}{dx} = 2x$

and when $x = 1,\quad u = 3$

when $x = 2,\quad u = 6$

$\therefore\ \displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\ dx = \int_1^2 u^{-2}\left(\frac{1}{2}\frac{du}{dx}\right) dx$

$= \frac{1}{2}\int_3^6 u^{-2}\ du$

$= \frac{1}{2}\left[\dfrac{u^{-1}}{-1}\right]_3^6$

$= \frac{1}{2}\left[-\frac{1}{6} - (-\frac{1}{3})\right]$

$= \frac{1}{2}(\frac{1}{6})$

$= \frac{1}{12}$

b In $\int_0^1 x^2 e^{x^3+1}\ dx$

we let $u = x^3 + 1$

$\therefore\ \dfrac{du}{dx} = 3x^2$

and when $x = 0,\quad u = 1$

when $x = 1,\quad u = 2$

$\therefore\ \int_0^1 x^2 e^{x^3+1}\ dx = \int_0^1 e^u \left(\frac{1}{3}\dfrac{du}{dx}\right) dx$

$= \frac{1}{3}\int_1^2 e^u\ du$

$= \frac{1}{3}\left[e^u\right]_1^2$

$= \frac{1}{3}(e^2 - e) \qquad (\doteqdot 1.557)$

c In $\int_0^3 x\sqrt{x^2+16}\ dx$

we let $u = x^2 + 16$

$\therefore\ \dfrac{du}{dx} = 2x$

and when $x = 0,\quad u = 16$

when $x = 3,\quad u = 25$

$\therefore\ \int_0^3 x\sqrt{x^2+16}\ dx = \int_0^3 u^{\frac{1}{2}}(\frac{1}{2}\dfrac{du}{dx})\ dx$

$= \frac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\ du$

$= \frac{1}{2}\left[\dfrac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{16}^{25}$

$= \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25}$

$= \frac{1}{3}(125 - 64) = 20\frac{1}{3}$

d In $\int_1^2 xe^{-2x^2}\,dx$ we let $u = -2x^2$

$\therefore \quad \dfrac{du}{dx} = -4x$

and when $x=1,\quad u=-2$

when $x=2,\quad u=-8$

$$\begin{aligned}\therefore \int_1^2 xe^{-2x^2}\,dx &= \int_1^2 e^u\,(-\tfrac{1}{4}\,\frac{du}{dx})\,dx\\ &= -\tfrac{1}{4}\int_{-2}^{-8} e^u\,du\\ &= -\tfrac{1}{4}\,[e^u]_{-2}^{-8}\\ &= -\tfrac{1}{4}(e^{-8}-e^{-2})\\ &\doteqdot 0.0337\end{aligned}$$

e In $\displaystyle\int_2^3 \frac{x}{2-x^2}\,dx$ we let $u = 2-x^2$

$\therefore \quad \dfrac{du}{dx} = -2x$

and when $x=2,\quad u=-2$

when $x=3,\quad u=-7$

$$\begin{aligned}\therefore \int_2^3 \frac{x}{2-x^2} &= \int_2^3 \frac{1}{u}(-\tfrac{1}{2}\,\frac{du}{dx})\,dx\\ &= -\tfrac{1}{2}\int_{-2}^{-7}\frac{1}{u}\,du\\ &= -\tfrac{1}{2}\,[\ln|u|]_{-2}^{-7}\\ &= -\tfrac{1}{2}(\ln 7-\ln 2)\\ &= -\tfrac{1}{2}\ln(\tfrac{7}{2}) \qquad (\doteqdot -0.6264)\end{aligned}$$

f In $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$ we let $u = \ln x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

and when $x=1,\quad u=0$

when $x=2,\quad u=\ln 2$

$$\begin{aligned}\therefore \int_1^2 \frac{\ln x}{x}\,dx &= \int_1^2 u\,\frac{du}{dx}\,dx\\ &= \int_0^{\ln 2} u\,du\\ &= \left[\frac{u^2}{2}\right]_0^{\ln 2}\\ &= \frac{(\ln 2)^2}{2}-0\\ &\doteqdot 0.2402\end{aligned}$$

g In $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx$ we let $u = 1-x^3+x$

$\therefore \quad \dfrac{du}{dx} = -3x^2+1$ and when $x=0,\quad u=1$

when $x=1,\quad u=1$

$$\begin{aligned}\therefore \int_0^1 \frac{1-3x^2}{1-x^3+x}\,dx &= \int_0^1 \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= \int_1^1 \frac{1}{u}\,du\\ &= 0 \qquad \{\text{since limits of integration are the same}\}\end{aligned}$$

h In $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$ we let $u = x^3-x^2+2x$

$\therefore \quad \dfrac{du}{dx} = 3x^2-2x+2$ and when $x=2,\quad u=8$

when $x=4,\quad u=56$

$$\begin{aligned}\therefore \int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx &= \int_2^4 \frac{1}{u}(2\,\frac{du}{dx})\,dx\\ &= 2\int_8^{56}\frac{1}{u}\,du\\ &= 2\,[\ln|u|]_8^{56}\\ &= 2(\ln 56-\ln 8)\\ &= 2\ln 7 \qquad (\doteqdot 3.892)\end{aligned}$$

i In $\int_0^1 (x^2+2x)^n(x+1)\,dx$ we let $u = x^2+2x$

$$\therefore\quad \frac{du}{dx} = 2x+2 \quad\text{and}\quad \text{when } x=0,\ u=0$$
$$\text{when } x=1,\ u=3$$

$$\therefore\quad \int_0^1 (x^2+2x)^n(x+1)\,dx = \int_0^1 u^n\left(\tfrac{1}{2}\frac{du}{dx}\right)dx = \tfrac{1}{2}\int_0^3 u^n\,du$$

Now if $n \neq -1$, the integral $= \frac{1}{2}\left[\dfrac{u^{n+1}}{n+1}\right]_0^3 = \frac{1}{2}\left(\dfrac{3^{n+1}}{n+1}\right)$

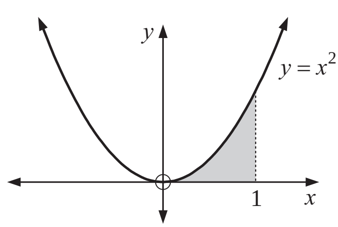
but if $n=-1$, the integral $= \frac{1}{2}\displaystyle\int_0^3 \frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]_0^3$ which is undefined as $\ln 0$ is not defined

3
$$\begin{aligned}\frac{3}{x+4} - \frac{2}{x-1} &= \frac{3(x-1)-2(x+4)}{(x+4)(x-1)}\\ &= \frac{3x-3-2x-8}{x^2+3x-4}\\ &= \frac{x-11}{x^2+3x-4}\end{aligned}$$

$$\begin{aligned}\therefore\quad \int_{-2}^{-1}\frac{x-11}{x^2+3x-4}\,dx &= \int_{-2}^{-1}\left(\frac{3}{x+4}-\frac{2}{x-1}\right)dx\\ &= \left[3\ln|x+4| - 2\ln|x-1|\right]_{-2}^{-1}\\ &= (3\ln 3 - 2\ln 2) - (3\ln 2 - 2\ln 3)\\ &= 3\ln 3 - 2\ln 2 - 3\ln 2 + 2\ln 3\\ &= 5\ln 3 - 5\ln 2\\ &= 5\ln\left(\tfrac{3}{2}\right)\end{aligned}$$

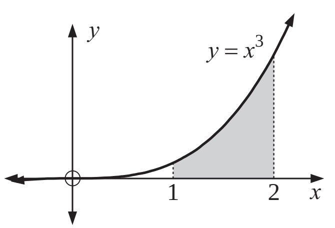
EXERCISE 26J

1 **a**



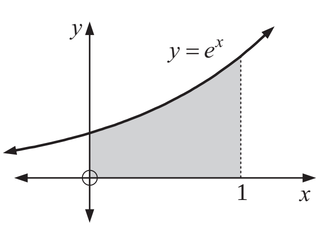
$$\begin{aligned}\text{Area} &= \int_0^1 x^2\,dx\\ &= \left[\frac{x^3}{3}\right]_0^1\\ &= \tfrac{1}{3} - 0\\ &= \tfrac{1}{3}\text{ units}^2\end{aligned}$$

b



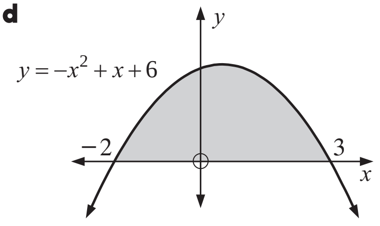
$$\begin{aligned}\text{Area} &= \int_1^2 x^3\,dx\\ &= \left[\frac{x^4}{4}\right]_1^2\\ &= \tfrac{16}{4} - \tfrac{1}{4}\\ &= 3\tfrac{3}{4}\text{ units}^2\end{aligned}$$

c



$$\begin{aligned}\text{Area} &= \int_0^1 e^x\,dx\\ &= \left[e^x\right]_0^1\\ &= e-1\\ &\doteqdot 1.718\text{ units}^2\end{aligned}$$

d



Cuts the x-axis at $y=0$ $\therefore$ $6+x-x^2=0$

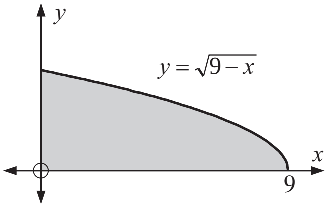
$\therefore$ $(3-x)(2+x)=0$

$\therefore$ $x=3$ or -2

i.e., x-intercepts are 3 and -2

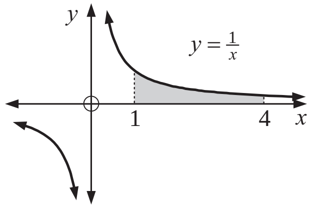
$$\begin{aligned}\text{Area} &= \int_{-2}^{3}(6+x-x^2)\,dx\\ &= \left[6x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-2}^{3}\\ &= (18+\tfrac{9}{2}-9)-(-12+2+\tfrac{8}{3})\\ &= 20\tfrac{5}{6}\text{ units}^2\end{aligned}$$

e



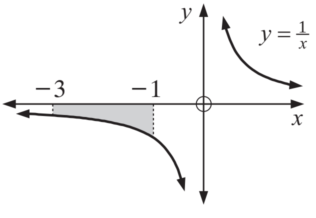
$$\begin{aligned}\text{Area} &= \int_0^9(9-x)^{\frac{1}{2}}\,dx\\ &= \left[\left(\frac{1}{-1}\right)\frac{(9-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9\\ &= -\tfrac{2}{3}\left[(9-x)^{\frac{3}{2}}\right]_0^9\\ &= -\tfrac{2}{3}[0-27]\\ &= 18\text{ units}^2\end{aligned}$$

f



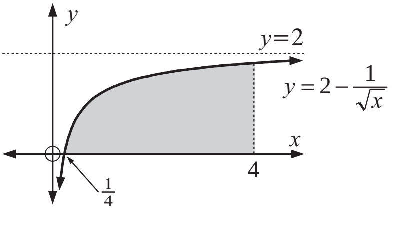
$$\begin{aligned}\text{Area} &= \int_1^4\frac{1}{x}\,dx\\ &= [\ln|x|]_1^4\\ &= \ln 4-\ln 1\\ &= \ln 4-0\\ &\doteqdot 1.386\text{ units}^2\end{aligned}$$

g As the curve lies below the x-axis, the integral will be negative, but we want to find the area, which is positive.



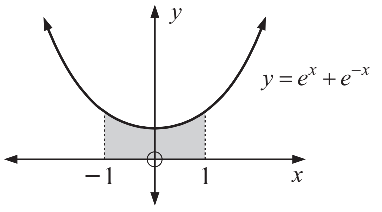
$$\begin{aligned}\text{Area} &= -\int_{-3}^{-1}\frac{1}{x}\,dx\\ &= -[\ln|x|]_{-3}^{-1}\\ &= -(\ln 1-\ln 3)\\ &= 0+\ln 3\\ &\doteqdot 1.099\text{ units}^2\end{aligned}$$

h



$$\begin{aligned}\text{Area} &= \int_{\frac{1}{4}}^{4}(2-x^{-\frac{1}{2}})\,dx\\ &= \left[2x-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\frac{1}{4}}^{4}\\ &= \left[2x-2\sqrt{x}\right]_{\frac{1}{4}}^{4}\\ &= (8-4)-(\tfrac{1}{2}-1)\\ &= 4\tfrac{1}{2}\text{ units}^2\end{aligned}$$

i



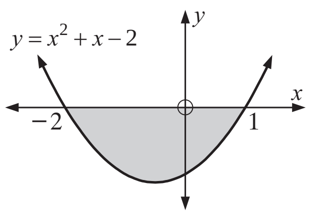
$$\begin{aligned}\text{Area } &= \int_{-1}^{1} (e^x + e^{-x})\ dx \\ &= \left[e^x - e^{-x}\right]_{-1}^{1} \\ &= (e - e^{-1}) - (e^{-1} - e) \\ &= 2e - \frac{2}{e} \\ &\doteqdot 4.701 \text{ units}^2\end{aligned}$$

2 **a** The curve cuts the x-axis when $y = 0$ $\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

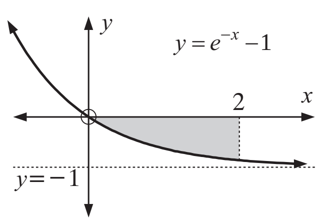
$\therefore\ x = -2$ or 1

i.e, x-intercepts are -2 and 1



$$\begin{aligned}\text{Area } &= \int_{-2}^{1} [0 - (x^2 + x - 2)]\ dx \\ &= \int_{-2}^{1} (-x^2 - x + 2)\ dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} \\ &= (-\tfrac{1}{3} - \tfrac{1}{2} + 2) - (\tfrac{8}{3} - 2 - 4) \\ &= 4\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b The curve cuts the x-axis at (0, 0).



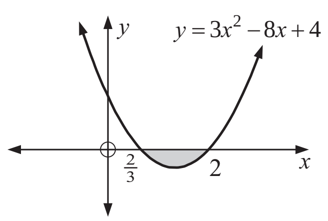
$$\begin{aligned}\text{Area } &= \int_{0}^{2} [0 - (e^{-x} - 1)]\ dx \\ &= \int_{0}^{2} (1 - e^{-x})\ dx \\ &= \left[x + e^{-x}\right]_{0}^{2} \\ &= \left(2 + \frac{1}{e^2}\right) - \left(0 + e^0\right) \\ &= 1 + \frac{1}{e^2} \qquad (\doteqdot 1.135 \text{ units}^2)\end{aligned}$$

c The curve cuts the x-axis when $y = 0$ $\therefore\ 3x^2 - 8x + 4 = 0$

$\therefore\ (3x-2)(x-2) = 0$

$\therefore\ x = 2$ or $\frac{2}{3}$

i.e, x-intercepts are 2 and $\frac{2}{3}$



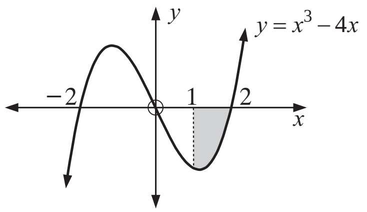
$$\begin{aligned}\text{Area } &= \int_{\frac{2}{3}}^{2} [0 - (3x^2 - 8x + 4)]\ dx \\ &= \int_{\frac{2}{3}}^{2} (-3x^2 + 8x - 4)\ dx \\ &= \left[-x^3 + 4x^2 - 4x\right]_{\frac{2}{3}}^{2} \\ &= (-8 + 16 - 8) - (-\tfrac{8}{27} + \tfrac{16}{3} - \tfrac{8}{3}) \\ &= 1\tfrac{5}{27} \text{ units}^2\end{aligned}$$

d The curve cuts the x-axis when $y = 0$ $\therefore\ x^3 - 4x = 0$

$\therefore\ x(x^2 - 4) = 0$

$\therefore\ x(x+2)(x-2) = 0$

i.e., x-intercepts are 0 and ± 2

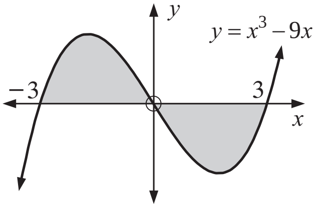


$$\begin{aligned}\text{Area} &= \int_1^2 [0 - f(x)]\, dx \\ &= \int_1^2 (-x^3 + 4x)\, dx \\ &= \left[-\frac{x^4}{4} + 2x^2\right]_1^2 \\ &= (-4+8) - (-\tfrac{1}{4} + 2) \\ &= 2\tfrac{1}{4} \text{ units}^2\end{aligned}$$

3 **a**

$$\begin{aligned} f(x) &= x^3 - 9x \\ &= x(x^2 - 9) \\ &= x(x+3)(x-3)\end{aligned}$$

$\therefore \quad y = f(x)$ cuts the x-axis at $0, \pm 3$

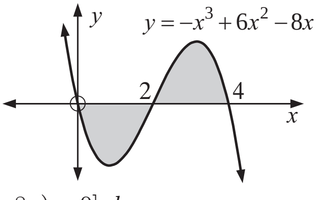


$$\begin{aligned}\text{Area} &= \int_{-3}^0 (x^3 - 9x)\, dx + \int_0^3 [0 - (x^3 - 9x)]\, dx \\ &= \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_{-3}^0 + \left[-\frac{x^4}{4} + \frac{9x^2}{2}\right]_0^3 \\ &= (0 - [\tfrac{81}{4} - \tfrac{81}{2}]) + ([-\tfrac{81}{4} + \tfrac{81}{2}] - 0) \\ &= 40\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b

$$\begin{aligned} f(x) &= -x(x-2)(x-4) \\ &= -x^3 + 6x^2 - 8x\end{aligned}$$

$\therefore \quad y = f(x)$ cuts the x-axis at 0, 2 and 4

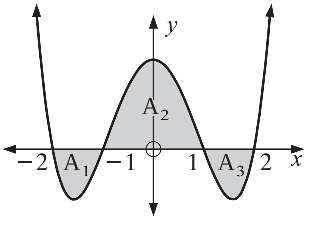


$$\begin{aligned}\text{Area} &= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)]\, dx + \int_2^4 [(-x^3 + 6x^2 - 8x) - 0]\, dx \\ &= \int_0^2 (x^3 - 6x^2 + 8x)\, dx + \int_2^4 (-x^3 + 6x^2 - 8x)\, dx \\ &= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2\right]_2^4 \\ &= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16]) \\ &= 8 \text{ units}^2\end{aligned}$$

c

$$\begin{aligned} f(x) &= x^4 - 5x^2 + 4 \\ &= (x^2 - 1)(x^2 - 4) \\ &= (x+1)(x-1)(x+2)(x-2)\end{aligned}$$

$\therefore \quad y = f(x)$ cuts the x-axis at $\pm 1, \pm 2$



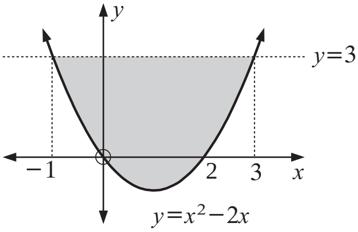
$$\begin{aligned} A_1 &= \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)]\, dx \\ &= \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\, dx \\ &= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} \\ &= (\tfrac{1}{5} - \tfrac{5}{3} + 4) - (\tfrac{32}{5} - \tfrac{40}{3} + 8) \\ &= \tfrac{22}{15} \text{ units}^2\end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-1}^{1} (x^4 - 5x^2 + 4)\, dx \\ &= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} \\ &= (\tfrac{1}{5} - \tfrac{5}{3} + 4) - (-\tfrac{1}{5} + \tfrac{5}{3} - 4) \\ &= \tfrac{76}{15} \text{ units}^2\end{aligned}$$

Now by symmetry, $A_3 = A_1$ $\quad \therefore \quad A = \tfrac{22}{15} + \tfrac{76}{15} + \tfrac{22}{15} = \tfrac{120}{15} = 8 \text{ units}^2$

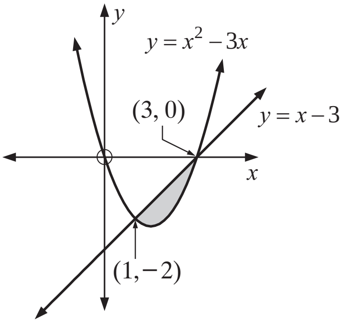
4 **a** $y = x^2 - 2x$ meets $y = 3$ where

$$\begin{aligned} x^2 - 2x &= 3 \\ \therefore \quad x^2 - 2x - 3 &= 0 \\ \therefore \quad (x-3)(x+1) &= 0 \\ \therefore \quad x = 3 \quad \text{and} \quad x &= -1 \end{aligned}$$



$$\begin{aligned} A &= \int_{-1}^{3} [3 - (x^2 - 2x)]\, dx \\ &= \int_{-1}^{3} (3 + 2x - x^2)\, dx \\ &= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} \\ &= (9 + 9 - 9) - (-3 + 1 + \tfrac{1}{3}) \\ &= 10\tfrac{2}{3} \text{ units}^2 \end{aligned}$$

b **i**



ii $y = x - 3$ meets $y = x^2 - 3x$

where $x = 3$ i.e., (3, 0)

and where $x = 1$ i.e., (1, −2)

Checking algebraically:

the graphs meet where $x - 3 = x^2 - 3x$

$$\begin{aligned} \therefore \quad x^2 - 3x - x + 3 &= 0 \\ \therefore \quad x^2 - 4x + 3 &= 0 \\ \therefore \quad (x-1)(x-3) &= 0 \\ \therefore \quad x &= 1 \text{ or } 3 \quad \checkmark \end{aligned}$$

iii

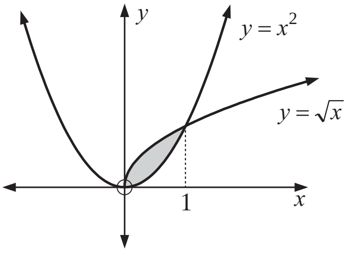
$$\begin{aligned} \text{Area} &= \int_{1}^{3} [(x-3) - (x^2 - 3x)]\, dx \\ &= \int_{1}^{3} (-3 + 4x - x^2)\, dx \\ &= \left[-3x + 2x^2 - \frac{x^3}{3}\right]_{1}^{3} \\ &= (-9 + 18 - 9) - (-3 + 2 - \tfrac{1}{3}) \\ &= 1\tfrac{1}{3} \text{ units}^2 \end{aligned}$$

c $y = \sqrt{x}$ meets $y = x^2$ where

$$\begin{aligned} \sqrt{x} &= x^2 \\ \therefore \quad x &= x^4 \\ \therefore \quad x^4 - x &= 0 \\ \therefore \quad x(x^3 - 1) &= 0 \\ \therefore \quad x(x-1)(x^2 + x + 1) &= 0 \\ \therefore \quad x &= 0 \text{ or } 1 \end{aligned}$$

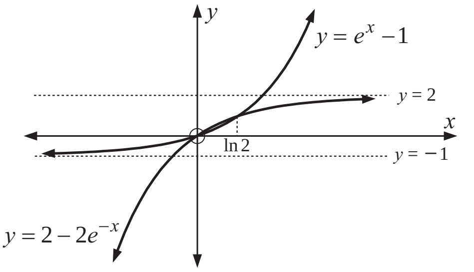
The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0.

$$\begin{aligned} \text{Area} &= \int_0^1 \left(\sqrt{x} - x^2\right) dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} - x^2\right) dx \\ &= \left[\tfrac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3} \text{ unit}^2 \end{aligned}$$



d $y = e^x - 1$ meets $y = 2 - 2e^{-x}$ where

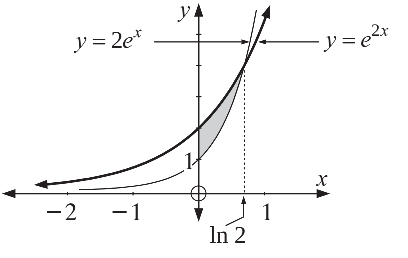
$$\begin{aligned} e^x - 1 &= 2 - 2e^{-x} \\ \therefore\quad e^{2x} - e^x &= 2e^x - 2 \quad \{\times\, e^x\} \\ \therefore\quad e^{2x} - 3e^x + 2 &= 0 \\ \therefore\quad (e^x - 1)(e^x - 2) &= 0 \\ \therefore\quad e^x &= 1 \text{ or } 2 \\ \therefore\quad x &= 0 \text{ or } \ln 2 \end{aligned}$$



$$\begin{aligned} A &= \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)]\, dx \\ &= \int_0^{\ln 2} (3 - e^x - 2e^{-x})\, dx \\ &= \left[3x - e^x + 2e^{-x}\right]_0^{\ln 2} \\ &= (3\ln 2 - 2 + 1) - (0 - 1 + 2) \\ &= 3\ln 2 - 2 \\ &\doteqdot 0.0794 \text{ units}^2 \end{aligned}$$

e $y = 2e^x$ meets $y = e^{2x}$ where

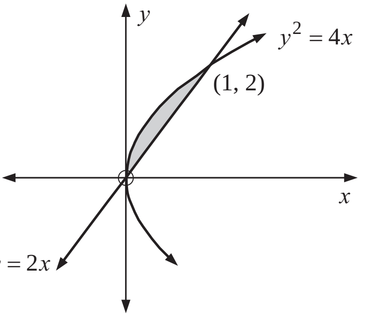
$$\begin{aligned} 2e^x &= e^{2x} \\ \therefore\quad e^{2x} - 2e^x &= 0 \\ \therefore\quad e^x(e^x - 2) &= 0 \\ \therefore\quad e^x &= 2 \qquad \{\text{as } e^x > 0 \text{ for all } x\} \\ \therefore\quad x &= \ln 2 \end{aligned}$$



$$\begin{aligned} A &= \int_0^{\ln 2} (2e^x - e^{2x})\, dx \\ &= \left[2e^x - \tfrac{1}{2}e^{2x}\right]_0^{\ln 2} \\ &= (4 - 2) - (2 - \tfrac{1}{2}) \\ &= \tfrac{1}{2} \text{ unit}^2 \end{aligned}$$

5 $y = 2x$ meets $y^2 = 4x$ where

$$\begin{aligned} (2x)^2 &= 4x \\ \therefore\quad 4x^2 &= 4x \\ \therefore\quad 4x^2 - 4x &= 0 \\ \therefore\quad 4x(x - 1) &= 0 \\ \therefore\quad x &= 0 \text{ or } 1 \end{aligned}$$



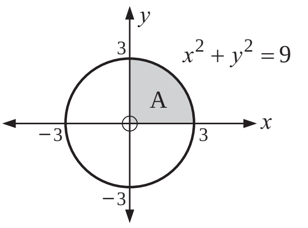
The upper part of $y^2 = 4x$ is $y = \sqrt{4x}$ i.e., $y = 2\sqrt{x}$

$$\begin{aligned} \text{Area} &= \int_0^1 (2\sqrt{x} - 2x)\, dx \\ &= \int_0^1 (2x^{\frac{1}{2}} - 2x)\, dx \\ &= \left[\tfrac{4}{3}x^{\frac{3}{2}} - x^2\right]_0^1 \\ &= \tfrac{4}{3} - 1 \\ &= \tfrac{1}{3} \text{ unit}^2 \end{aligned}$$

6 **a** Now $x^2 + y^2 = 9$

$\therefore \quad y^2 = 9 - x^2$

$y = \pm\sqrt{9 - x^2}$



In the upper half of the circle all y-values are $\geqslant 0$

$\therefore \quad y = +\sqrt{9 - x^2}$ is the required equation.

b Now the shaded area above is A where $A = \int_0^3 \sqrt{9 - x^2}\, dx$

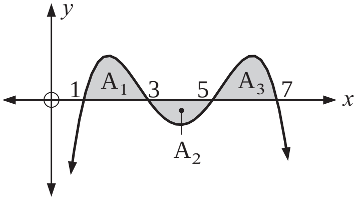
This is a quarter of the area of a circle radius 3 units $\therefore \quad A = \frac{1}{4}(\pi \times 3^2)$

$= \frac{9}{4}\pi$

$\doteqdot 7.069$ units^2

c The answer checks using technology.

7



a $\int_1^7 f(x)\, dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leqslant x \leqslant 7$.
But $f(x)$ is not positive for $3 \leqslant x \leqslant 5$,
so $\int_1^7 f(x)\, dx$ will give us

(Area enclosed in $1 \leqslant x \leqslant 3$)
− (Area enclosed in $3 \leqslant x \leqslant 5$)
+ (Area enclosed in $5 \leqslant x \leqslant 7$)

which is NOT the shaded area.

b shaded area $= \int_1^3 f(x)\, dx + \int_3^5 [0 - f(x)]\, dx + \int_5^7 f(x)\, dx$

$= \int_1^3 f(x)\, dx - \int_3^5 f(x)\, dx + \int_5^7 f(x)\, dx$

8 $$\text{Area} = \int_1^k \frac{1}{1+2x}\, dx = 0.2 \text{ units}^2$$

$\therefore \quad \left[\frac{1}{2}\ln|1+2x|\right]_1^k = 0.2$

$\therefore \quad [\ln|1+2x|]_1^k = 0.4$

$\therefore \quad \ln|1+2k| - \ln 3 = 0.4$

$\therefore \quad \ln(1+2k) - \ln 3 = 0.4 \qquad \{\text{as } k > 0, \text{then } 1+2k > 0 \text{ also}\}$

$\therefore \quad \ln\left(\frac{1+2k}{3}\right) = 0.4$

$\therefore \quad \frac{1+2k}{3} = e^{0.4}$

$\therefore \quad 1 + 2k = 3e^{0.4}$

$\therefore \quad k = \frac{3e^{0.4} - 1}{2} \doteqdot 1.7377$

9 $\text{Area} = \int_0^b \sqrt{x}\, dx$

$\therefore \quad \int_0^b x^{\frac{1}{2}}\, dx = 1$

$\therefore \quad \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^b = 1$

$\therefore \quad \frac{2}{3}b\sqrt{b} - 0 = 1$

$\therefore \quad b\sqrt{b} = \frac{3}{2}$

$\therefore \quad b^{\frac{3}{2}} = 1.5$

$\therefore \quad b = (1.5)^{\frac{2}{3}} \doteqdot 1.3104$

10 $y = x^2$ meets $y = k$ where $x^2 = k$ $\therefore$ $x = \pm\sqrt{k}$

Now area $= \int_0^{\sqrt{k}} (k - x^2)\, dx$ $\therefore$ $\int_0^{\sqrt{k}} (k - x^2) dx = 2.4$

$$\therefore \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = 2.4$$

$$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} - 0 = 2.4$$

$$\therefore \frac{2k\sqrt{k}}{3} = 2.4$$

$$\therefore k^{\frac{3}{2}} = 3.6$$

$$\therefore k = (3.6)^{\frac{2}{3}} \doteqdot 2.3489$$

11 By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(3a)$ units^2.

$$\therefore \int_0^a (x^2 + 2) dx = \tfrac{3}{2}a$$

$$\therefore \left[\frac{x^3}{3} + 2x\right]_0^a = \tfrac{3}{2}a$$

$$\therefore \frac{a^3}{3} + 2a - 0 = \tfrac{3}{2}a$$

$$\therefore 2a^3 + 12a = 9a \qquad \{\times \text{ by } 6\}$$

$$\therefore 2a^3 + 3a = 0$$

$$\therefore a(2a^2 + 3) = 0$$

$$\therefore a = 0 \qquad \{a^2 \geqslant 0 \text{ for all } a, \ \therefore\ 2a^2 + 3 > 0 \text{ for all } a\}$$

EXERCISE 26K

1 The marginal cost is $C'(x)$ and $C'(x) = 3.15 + 0.004x$ \$/gadget

$$\therefore C(x) = \int (3.15 + 0.004x)\, dx$$
$$= 3.15x + 0.002x^2 + c$$

But $C(0) = 450$ $\therefore$ $c = 450$

$$\therefore C(x) = 3.15x + 0.002x^2 + 450 \text{ dollars}$$
$$C(800) = 3.15(800) + 0.002(800)^2 + 450$$
$$= \$4250$$

$\therefore$ total cost is \$4250

2 The marginal cost is $C'(x)$ and $C'(x) = 10 - \dfrac{4}{\sqrt{x+1}}$ \$/item

$$\therefore C(x) = \int \left(10 - 4(x+1)^{-\frac{1}{2}}\right) dx$$
$$= 10x - 8(x+1)^{\frac{1}{2}} + c$$
$$= 10x - 8\sqrt{x+1} + c$$

But $C(0) = 200$ $\therefore$ $0 - 8 + c = 200$

$\therefore$ $c = 208$

$$\therefore C(x) = 10x - 8\sqrt{x+1} + 208 \text{ dollars}$$
$$C(100) = 10(100) - 8\sqrt{101} + 208$$
$$\doteqdot \$1127.60$$

$\therefore$ total cost is \$1127.60

3 The marginal cost is $\frac{dC}{dx}$ and $\frac{dC}{dx} = 25x - 4x^{0.8} + 0.0024x^2$ million dollars/aeroplane

$$\begin{aligned}
\therefore \quad C(x) &= \int \left(25x - 4x^{0.8} + 0.0024x^2\right)\, dx \\
&= \frac{25x^2}{2} - \tfrac{4}{1.8}x^{1.8} + 0.0008x^3 + c \\
&= 12.5x^2 - \tfrac{20}{9}x^{1.8} + 0.0008x^3 + c
\end{aligned}$$

But $C(0) = 275 \quad \therefore \quad c = 275$

$$\begin{aligned}
\therefore \quad C(x) &= 12.5x^2 - \tfrac{20}{9}x^{1.8} + 0.0008x^3 + 275 \text{ million dollars} \\
C(20) &= 12.5(400) - \tfrac{20}{9}(20^{1.8}) + 0.0008(8000) + 275 \\
&= \$4793.2 \text{ million} \qquad \text{i.e., total cost is } \$4793.2 \text{ million}
\end{aligned}$$

4 $\frac{dS}{dx} = 10 + \frac{3}{x} + x^2$ for $2 \leqslant x \leqslant 10$

$\therefore \quad S(x) = 10x + 3\ln|x| + \frac{x^3}{3} + c \qquad \therefore \quad S(3) \doteqdot 42.30 + c$

and $S(6) \doteqdot 137.38 + c$

$$\begin{aligned}
\therefore \quad \text{total change in sales from 3 to 6} &= S(6) - S(3) \\
&\doteqdot 95.08 \text{ units}
\end{aligned}$$

5 **a** The marginal profit is $P'(x)$ and $P'(x) = 15 - 0.03x$ \$/plate

$$\begin{aligned}
\therefore \quad P(x) &= \int (15 - 0.03x)\, dx \\
&= 15x - 0.015x^2 + c
\end{aligned}$$

But $P(0) = -650 \quad \therefore \quad c = -650$

$\therefore \quad P(x) = 15x - 0.015x^2 - 650$ dollars

b The maximum profit occurs when $P'(x) = 0$ i.e., when

$$\begin{aligned}
15 - 0.03x &= 0 \\
\therefore \quad 0.03x &= 15 \\
\therefore \quad x &= \frac{15}{0.03} \\
\text{i.e.,} \quad x &= 500
\end{aligned}$$

and $P''(x) = -0.03 < 0 \quad \therefore$ profit is at a maximum when $x = 500$

$$\begin{aligned}
\therefore \quad \text{maximum profit} &= P(500) \\
&= 15(500) - 0.015(500)^2 - 650 \\
&= \$3100
\end{aligned}$$

c In order for a profit to be made $P(x)$ must be greater than 0

i.e., $15x - 0.015x^2 - 650 > 0$

Using technology to graph $P(x)$ and find x intercepts we get $x_1 = 45.39$ and $x_2 = 954.6$

$\therefore \quad 46 \leqslant x \leqslant 954 \qquad$ {We cannot produce part plates.}

6 $E(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories/day

$$\begin{aligned}
\text{Total energy needs over the first week} &= \int_0^7 E(t)\, dt \\
&= \int_0^7 [350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)]\, dt \\
&= \left[\tfrac{1}{0.15} \times \frac{350(80 + 0.15t)^{1.8}}{1.8} - 9600t - 9t^2\right]_0^7 \\
&\doteqdot 14\,400 \text{ calories}
\end{aligned}$$

7 $\dfrac{dT}{dx} = \dfrac{-20}{x^{0.63}} = -20x^{-0.63} \quad \therefore \quad T = \int -20x^{-0.63}\, dx$

$$= \frac{-20x^{0.37}}{0.37} + c$$

But when $x = 3$, $T = 100$, $\therefore \quad \dfrac{-20(3^{0.37})}{0.37} + c = 100$

$$\therefore \quad c = 100 + \frac{20(3^{0.37})}{0.37} \doteqdot 181.1639$$

$$\text{i.e.,} \quad T = \frac{-20x^{0.37}}{0.37} + 181.1639$$

and when $x = 6$, $T = -104.8925 + 181.1639$

$$\doteqdot 76.27$$

$\therefore$ the outer surface temperature is 76.27^oC

8 **a** $\dfrac{d^2y}{dx^2} = -\frac{1}{10}(1-x)^2$ and $\dfrac{dy}{dx} = \displaystyle\int \frac{d^2y}{dx^2}\, dx$

$$= \int -\tfrac{1}{10}(1-x)^2\, dx$$

$$= -\tfrac{1}{10}(\tfrac{1}{-1}) \times \frac{(1-x)^3}{3} + c$$

$$= \tfrac{1}{30}(1-x)^3 + c$$

But when $x = 0$, $\dfrac{dy}{dx} = 0$ {horizontal tangent}

$\therefore \quad \frac{1}{30}(1-0)^3 + c = 0$ and so $c = -\frac{1}{30}$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}$$

$$\therefore \quad y = \int [\tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}]\, dx$$

$$= \tfrac{1}{30}\,(\tfrac{1}{-1})\,\frac{(1-x)^4}{4} - \tfrac{1}{30}x + d$$

$$= -\frac{(1-x)^4}{120} - \frac{x}{30} + d$$

Also, when $x = 0$, $y = 0$

$\therefore \quad -\frac{1}{120} - 0 + d = 0 \quad \therefore \quad d = \frac{1}{120}$

and so $y = \dfrac{1}{120} - \dfrac{(1-x)^4}{120} - \dfrac{x}{30}$

b Maximum deflection occurs at the right hand end where $x \doteqdot 1$

and at $x \doteqdot 1$, $y \doteqdot \frac{1}{120} - 0 - \frac{1}{30} \doteqdot -0.025$ m

i.e., maximum deflection is about 2.5 cm.

9 **a** $\dfrac{d^2y}{dx^2} = \frac{1}{100}\left(2x - \dfrac{x^2}{2}\right) = \frac{1}{50}x - \frac{1}{200}x^2$

$\therefore \quad \dfrac{dy}{dx} = \int (\frac{1}{50}x - \frac{1}{200}x^2)\, dx = \frac{1}{100}x^2 - \frac{1}{600}x^3 + c$

The sag, $y = \int (\frac{1}{100}x^2 - \frac{1}{600}x^3 + c)\, dx$

i.e., $y = \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx + d$

Now when $x = 0$, $y = 0$ $\therefore$ $0 - 0 + 0 + d = 0$

$$\therefore \quad d = 0$$

$$\therefore \quad y = \tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 + cx$$

Also when $x = 4$, $y = 0$ $\therefore$ $\tfrac{1}{300}\left(4^3\right) - \tfrac{1}{2400}\left(4^4\right) + 4c = 0$

$$\therefore \quad 4c = \tfrac{1}{2400}\left(4^4\right) - \tfrac{1}{300}\left(4^3\right)$$

$$\therefore \quad c = \tfrac{1}{2400}\left(4^3\right) - \tfrac{1}{300}\left(4^2\right)$$

$$\therefore \quad c = -\tfrac{2}{75}$$

$$\therefore \quad y = \left(\tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 - \tfrac{2}{75}x\right) \text{ m}$$

b The maximum sag occurs when $\dfrac{dy}{dx} = 0$ i.e., $\tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 - \tfrac{2}{75} = 0$

$$6x^2 - x^3 - 16 = 0$$

Using technology the three solutions are: $x = -1.464$, 2 or 5.4641

But the maximum lies between 0 and 4, $\therefore$ maximum sag occurs when $x = 2$.

When $x = 2$, $y = \tfrac{1}{300}\left(2^3\right) - \tfrac{1}{2400}\left(2^4\right) - \tfrac{2}{75}(2)$

$\doteqdot -0.033\,33$ m

$\doteqdot -3.333$ cm $\therefore$ the maximum sag is $\doteqdot 3.33$ cm

c When 1 m from P, i.e., $x = 3$ m,

$y = \tfrac{1}{300}\left(3^3\right) - \tfrac{1}{2400}\left(3^4\right) - \tfrac{2}{75}(3)$

$= -0.023\,75$ m

$= -2.375$ cm $\therefore$ the sag is 2.375 cm

d When 1 m from P, i.e., $x = 3$ m, $\dfrac{dy}{dx} = \tfrac{1}{100}\left(3^2\right) - \tfrac{1}{600}\left(3^3\right) - \tfrac{2}{75} \doteqdot 0.0183$

i.e., the angle θ, that the plank makes with the horizontal is such that $\tan\theta \doteqdot 0.0183$

i.e., $\theta = \tan^{-1}(0.0183) \doteqdot 1.05^o$

10 The cost per unit volume V is $\tfrac{1}{2}x^2 + 4$ \$/m^3

$\therefore$ $\dfrac{dC}{dV} = (\tfrac{1}{2}x^2 + 4)$ \$/m^3 and as $V = \pi r^2 x$, $\dfrac{dV}{dx} = \pi r^2$

Now $\dfrac{dC}{dx} = \dfrac{dC}{dV}\dfrac{dV}{dx}$ {Chain rule}

$$\therefore \quad \frac{dC}{dx} = (\tfrac{1}{2}x^2 + 4) \times \pi r^2$$

$\therefore$ $\dfrac{dC}{dx} = \pi r^2\left(\tfrac{1}{2}x^2 + 4\right)$ and the cost C, of digging a well $= \displaystyle\int \frac{dC}{dx}\,dx$

$$= \int [\pi r^2(\tfrac{1}{2}x^2 + 4)]\,dx$$

$$= \pi r^2\left(\frac{x^3}{6} + 4x\right) + c$$

$\therefore$ cost of digging a well h metres deep $= \pi r^2\left(\dfrac{h^3}{6} + 4h\right) + c$

Now if the initial cost $= C_0$, $\pi r^2(\tfrac{0}{6} + 0) + c = C_0$

$$\therefore \quad c = C_0$$

$$\therefore \quad C(h) = \pi r^2\left(\frac{h^3 + 24h}{6}\right) + C_0$$

11 **a** The yield Y per unit area A is proportional to $\dfrac{1}{\sqrt{x+4}}$.

i.e., $\dfrac{dY}{dA} \propto \dfrac{1}{\sqrt{x+4}}$

$\therefore \quad \dfrac{dY}{dA} = k\left(\dfrac{1}{\sqrt{x+4}}\right) = \dfrac{k}{\sqrt{x+4}}$ where k is a constant

b Now the shaded area $A =$ length $\times$ width

$\therefore \quad A = (4-2p)x$

$\therefore \quad \dfrac{dA}{dx} = 4 - 2p$

Now $\dfrac{dY}{dx} = = \dfrac{dY}{dA}\,\dfrac{dA}{dx}$ {Chain rule}

$\therefore \quad \dfrac{dY}{dx} \quad \dfrac{k}{\sqrt{x+4}} \times (4-2p)$

i.e., $\dfrac{dY}{dx} = \dfrac{k(4-2p)}{\sqrt{x+4}}$

c $\dfrac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.

$\therefore$ total yield $= Y \;=\; \displaystyle\int_0^p \frac{dY}{dx}\,dx$

$\qquad = \displaystyle\int_0^p \frac{k(4-2p)}{\sqrt{x+4}}\,dx$

d $\quad Y = k(4-2p)\int_0^p (x+4)^{-\frac{1}{2}}\,dx$ {using **c**}

$\qquad = k(4-2p) \times \left[\dfrac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^p$

$\qquad = 2k(4-2p)\left[\sqrt{x+4}\right]_0^p$

$\qquad = 4k(2-p)\left[\sqrt{p+4} - \sqrt{4}\right]$

$\therefore \quad Y = 4k(2-p)\left(\sqrt{p+4} - 2\right)$

e For yield to be a maximum we need to maximise Y. Using technology to graph Y and find the maximum, we find that the maximum occurs when $p \doteqdot 0.9735$ km

i.e., orchard is 0.9735 km $\times$ 2.053 km

REVIEW SET 26A

1 **a** $\displaystyle\int \frac{4}{\sqrt{x}}\,dx = 4\int x^{-\frac{1}{2}}\,dx$

$\qquad = 4\,\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$\qquad = 8\sqrt{x} + c$

b $\displaystyle\int \frac{3}{1-2x}\,dx = 3\int \frac{1}{1-2x}\,dx$

$\qquad = 3(\tfrac{1}{-2})\ln|1-2x| + c$

$\qquad = -\tfrac{3}{2}\ln|1-2x| + c$

c $\displaystyle\int xe^{1-x^2}\,dx = \int e^u\left(-\tfrac{1}{2}\frac{du}{dx}\right)du$ {letting $u = 1-x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$}

$\qquad = -\tfrac{1}{2}\int e^u\,du$

$\qquad = -\tfrac{1}{2}e^u + c$

$\qquad = -\tfrac{1}{2}e^{1-x^2} + c$

2 **a** $\int_{-5}^{-1} \sqrt{1-3x}\; dx$

$= \int_{-5}^{-1} (1-3x)^{\frac{1}{2}}\; dx$

$= \left[\frac{1}{-3} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1}$

$= -\frac{2}{9}\left[(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1}$

$= -\frac{2}{9}\left[4^{\frac{3}{2}} - 16^{\frac{3}{2}}\right]$

$= -\frac{2}{9}[8-64]$

$= 12\frac{4}{9}$

b In $\displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\; dx$

we let $u = x^3+2$

$\therefore \dfrac{du}{dx} = 3x^2$ and when $x=0,\ u=2$

when $x=1,\ u=3$

$\therefore \displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\; dx \int_0^1 \frac{1}{u^3}\left(\frac{4}{3}\frac{du}{dx}\right) dx$

$= \frac{4}{3}\int_2^3 u^{-3}\, du$

$= \frac{4}{3}\left[\dfrac{u^{-2}}{-2}\right]_2^3$

$= \frac{4}{3}\left[-\dfrac{1}{2u^2}\right]_2^3$

$= \frac{4}{3}\left[(-\frac{1}{18}) - (-\frac{1}{8})\right]$

$= \frac{5}{54}$

3 $y = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}} \times 2x$

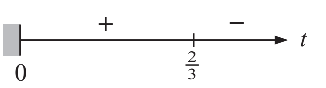
$= \dfrac{x}{(x^2-4)^{\frac{1}{2}}}$

$= \dfrac{x}{\sqrt{x^2-4}}$

$\therefore \displaystyle\int \frac{x}{\sqrt{x^2-4}}\; dx = \sqrt{x^2-4}+c$

4 $v(t) = 2t-3t^2$

$= t(2-3t)$ which has sign diagram:



Now $s(t) = \int (2t-3t^2)\; dt$

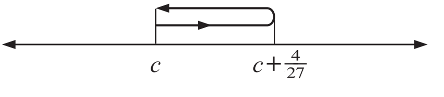
$s(t) = t^2 - t^3 + c$ metres

and so $s(0) = c$

$s(\frac{2}{3}) = \frac{4}{9} - \frac{8}{27} + c = c + \frac{4}{27}$

$s(1) = 1 - 1 + c = c$

with motion diagram:



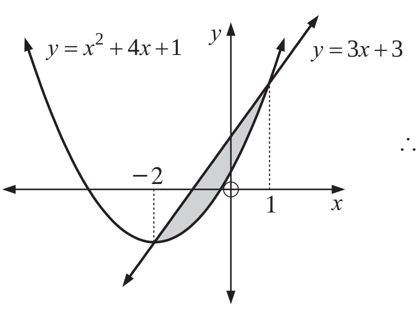
$\therefore$ total distance travelled $= (c+\frac{4}{27}-c) + (c+\frac{4}{27}-c) = \frac{8}{27}$ m $\doteqdot 29.6$ cm

5 $y = x^2+4x+1$ meets $y = 3x+3$ where $x^2+4x+1 = 3x+3$

$\therefore x^2+x-2 = 0$

$\therefore (x+2)(x-1) = 0$

$\therefore x = -2$ or 1



$\therefore$ area $= \int_{-2}^{1}[(3x+3)-(x^2+4x+1)]\; dx$

$= \int_{-2}^{1}(-x^2-x+2)\; dx$

$= \left[-\dfrac{x^3}{3} - \dfrac{x^2}{2} + 2x\right]_{-2}^{1}$

$= \left(-\frac{1}{3}-\frac{1}{2}+2\right) - \left(\frac{8}{3}-2-4\right)$

$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4$

$= 4\frac{1}{2}$ units^2

6 $$\frac{dI}{dt} = -\frac{100}{t^2} \qquad t \geqslant 0.2 \text{ seconds}$$

$$\therefore \quad I(t) = \int \frac{dI}{dt}\,dt = \int -100t^{-2}\,dt = 100t^{-1} + c$$

Now $I(2) = 150$ amps

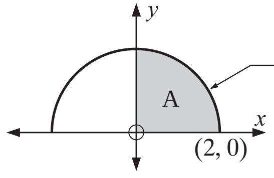
$$\therefore \quad \frac{100}{2} + c = 150 \quad \text{and so} \quad c = 100$$

$$\therefore \quad I(t) = \left(\frac{100}{t} + 100\right) \text{ amps}$$

a $I(20) = \dfrac{100}{20} + 100$
$= 105$ amps

b as $t \to +\infty$
$I(t) \to 100$ amps (above)

7



$y = \sqrt{4 - x^2}$ is a semi-circle above the x-axis with centre 0 and radius 2

Now $\int_0^2 \sqrt{4 - x^2}\,dx$ = shaded area = $\frac{1}{4}$ of the area of a circle of radius 2 units
$= \frac{1}{4}\pi(2^2)$
$= \pi$ units^2

8 $\int_{-1}^{3} f(x)\,dx$ gives us the correct area only if $f(x)$ is positive on the interval $-1 \leqslant x \leqslant 3$.

But $f(x)$ is not positive for $1 \leqslant x \leqslant 3$ so $\int_{-1}^{3} f(x)\,dx$ does not provide the correct answer.

The shaded area below the x-axis is given by $\int_1^3 [0 - f(x)]\,dx = -\int_1^3 f(x)\,dx$

9 **a** $f(x) = \dfrac{x}{1 + x^2}$

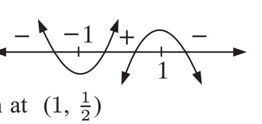
$$\therefore \quad f'(x) = \frac{1(1 + x^2) - x(2x)}{(1 + x^2)^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{1 + x^2 - 2x^2}{(1 + x^2)^2}$$

$$= \frac{1 - x^2}{(1 + x^2)^2}$$

$$= \frac{(1 + x)(1 - x)}{(1 + x^2)^2}$$

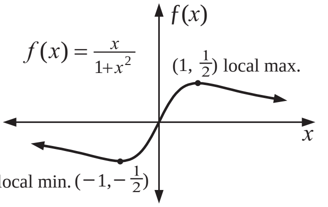
which has sign diagram:



$\therefore$ there is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(1, \frac{1}{2})$

b as $x \to \infty$ $f(x) \to 0$ (above) and
as $x \to -\infty$ $f(x) \to 0$ (below)

c



d Area $= \displaystyle\int_{-2}^{0} \left[0 - \frac{x}{1+x^2}\right] dx$

$= \displaystyle\int_{-2}^{0} \frac{-x}{1+x^2}\, dx$

Let $u = 1 + x^2 \quad \therefore \quad \dfrac{du}{dx} = 2x$

and when $x = 0, \quad u = 1$

when $x = -2, \quad u = 5$

$\therefore \displaystyle\int_{-2}^{0} \frac{-x}{1+x^2}\, dx = \int_{-2}^{0} \frac{1}{u}\,(-\tfrac{1}{2}\frac{du}{dx})\, dx$

$= -\frac{1}{2}\displaystyle\int_{5}^{1} \frac{1}{u}\, du$

$= -\frac{1}{2}\left[\ln|u|\right]_5^1$

$= -\frac{1}{2}(0 - \ln 5)$

$= \frac{1}{2}\ln 5$ units^2 $\quad (\doteqdot 0.8047$ units$^2)$

10 Shaded area $= \displaystyle\int_{-4}^{-1} [g(x) - f(x)]\, dx + \int_{-1}^{5} [f(x) - g(x)]\, dx$

11 The coordinates of B are $(2,\ 4+k)$

$\therefore$ area rectangle OABC $= 2 \times (4+k)$

$= 8 + 2k$

$\therefore$ since the two shaded regions are equal in area, each area is $4+k$ units^2.

$\therefore \int_0^2 (x^2 + k)\, dx = 4 + k$

$\therefore \left[\dfrac{x^3}{3} + kx\right]_0^2 = 4 + k$

$\therefore \frac{8}{3} + 2k = 4 + k$

$\therefore k = 4 - \frac{8}{3}$

$\therefore k = \frac{4}{3}$

REVIEW SET 26B

1 **a** $\int (2e^{-x} - \dfrac{1}{x} + 3)\, dx$

$= (\frac{1}{-1})2e^{-x} - \ln|x| + 3x + c$

$= -2e^{-x} - \ln|x| + 3x + c$

b $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$= \displaystyle\int \left(x - 2 + \frac{1}{x}\right) dx$

$= \dfrac{x^2}{2} - 2x + \ln|x| + c$

2 **a** $\int_1^2 (x^2-1)^2\, dx$

$= \int_1^2 (x^4 - 2x^2 + 1)\, dx$

$= \left[\dfrac{x^5}{5} - \dfrac{2x^3}{3} + x\right]_1^2$

$= \left[\frac{32}{5} - \frac{16}{3} + 2\right] - \left[\frac{1}{5} - \frac{2}{3} + 1\right]$

$= \frac{31}{5} - \frac{14}{3} + 1$

$= 2\frac{8}{15}$

b In $\int_1^2 x(x^2-1)^2\, dx$

We let $u = x^2 - 1 \quad \therefore \quad \dfrac{du}{dx} = 2x$

and when $x = 1, \quad u = 0$

when $x = 2, \quad u = 3$

$\therefore \int_1^2 x(x^2-1)^2\, dx = \int_0^3 u^2(\frac{1}{2}\dfrac{du}{dx}\, dx)$

$= \frac{1}{2}\displaystyle\int_0^3 u^2\, du$

$= \frac{1}{2}\left[\dfrac{u^3}{3}\right]_0^3$

$= \frac{1}{2}\left(\frac{27}{3} - 0\right)$

$= 4\frac{1}{2}$

3 $f(x) = (3x^2 + x)^3$

$\therefore f'(x) = 3(3x^2+x)^2(6x+1)$

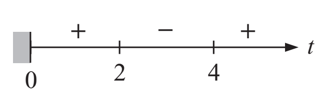
$\therefore \int 3(3x^2+x)^2(6x+1)\, dx = (3x^2+x)^3 + c_1$

$\therefore 3\int (3x^2+x)^2(6x+1)\, dx = (3x^2+x)^3 + c_1$

$\therefore \int (3x^2+x)^2(6x+1)\, dx = \frac{1}{3}(3x^2+x)^3 + c$

4 **a** $v(t) = t^2 - 6t + 8$ ms^{-1}, $\quad t \geqslant 0$

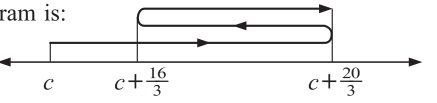
$= (t-4)(t-2)$ which has sign diagram:



b Now $s(t) = \int (t^2 - 6t + 8)\, dt$
$= \dfrac{t^3}{3} - 3t^2 + 8t + c$

and so $s(0) = c$
$s(2) = c + 6\frac{2}{3}$
$s(4) = c + 5\frac{1}{3}$
$s(5) = c + 6\frac{2}{3}$

the motion diagram is:

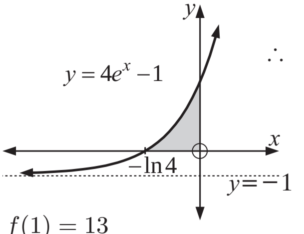


The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$
$= 9\frac{1}{3}$ m

5 $y = 4e^x - 1$ cuts the x-axis
when $4e^x - 1 = 0$
$\therefore\ e^x = \frac{1}{4}$
$\therefore\ x = \ln\frac{1}{4}$
$= -\ln 4$



$\therefore$ area $= \int_{-\ln 4}^{0} (4e^x - 1)\, dx$
$= [4e^x - x]_{-\ln 4}^{0}$
$= [4 - 0] - [1 + \ln 4]$
$= 3 - \ln 4$ $(\doteqdot 1.614 \text{ units}^2)$

6 $y = f(x)$ $f(0) = -1$ $f(1) = 13$
$f''(x) = 18x + 10$
$\therefore\ f'(x) = 9x^2 + 10x + c$
and $f(x) = 3x^3 + 5x^2 + cx + d$
But $f(0) = -1$ $\therefore\ d = -1$
$\therefore\ f(x) = 3x^3 + 5x^2 + ax - 1$

But $f(1) = 13$
$\therefore\ 3 + 5 + a - 1 = 13$
$\therefore\ a + 7 = 13$
$\therefore\ a = 6$
$\therefore\ f(x) = 3x^3 + 5x^2 + 6x - 1$

7 The area between $x = 0$ and $x = a$ is 2 units^2.

$\therefore\ \int_0^a e^x\, dx = 2$
$\therefore\ [e^x]_0^a = 2$
$\therefore\ e^a - e^0 = 2$
$\therefore\ e^a = 3$
$\therefore\ a = \ln 3$

The area between $x = a = \ln 3$ and $x = b$ is 2 units^2.

$\therefore\ \int_{\ln 3}^{b} e^x\, dx = 2$
$\therefore\ [e^x]_{\ln 3}^{b} = 2$
$\therefore\ e^b - e^{\ln 3} = 2$
$\therefore\ e^b - 3 = 2$
$\therefore\ e^b = 5$
$\therefore\ b = \ln 5$

8 $\dfrac{4x-3}{2x+1} = \dfrac{2(2x+1)-5}{2x+1} = 2 + \dfrac{-5}{2x+1}$ $\therefore\ A = 2 \quad B = -5$

$$\therefore\ \int_0^2 \frac{4x-3}{2x+1}\, dx = \int_0^2 \left(2 - 5\left(\frac{1}{2x+1}\right)\right) dx$$
$$= \left[2x - 5\left(\tfrac{1}{2}\right)\ln|2x+1|\right]_0^2$$
$$= [4 - \tfrac{5}{2}\ln 5] - [0 - \tfrac{5}{2}\ln 1]$$
$$= 4 - \tfrac{5}{2}\ln 5$$
$$\doteqdot -0.0236$$

9 The shaded area $= \int_0^2 ax(x-2)\ dx = 4$ units^2

$$\therefore \quad \int_0^2 (ax^2 - 2ax)\ dx = 4$$
$$\therefore \quad \left[\frac{ax^3}{3} - ax^2\right]_0^2 = 4$$
$$\therefore \quad \left[\frac{8a}{3} - 4a\right] - 0 = 4$$
$$\therefore \quad \frac{8a}{3} - \frac{12a}{3} = 4$$
$$\therefore \quad -\frac{4a}{3} = 4$$
$$\therefore \quad a = -3$$
$$\therefore \quad y = -3x(x-2)$$

Suppose A has coordinates $(k,\ -3k(k-2))$

$\therefore$ slope OA $= \dfrac{-3k(k-2)-0}{k-0} = -3(k-2)$ $\quad\therefore$ equation of OA is $y = -3(k-2)x$

Now if OA divides the shaded area into equal areas,

$$\int_0^k [-3x(x-2) - (-3(k-2)x)]\ dx = 2$$
$$\therefore \quad \int_0^k (-3x^2 + 6x + 3kx - 6x)dx = 2$$
$$\therefore \quad \int_0^k (-3x^2 + 3kx)\ dx = 2$$
$$\therefore \quad \left[-x^3 + \frac{3kx^2}{2}\right]_0^k = 2$$
$$\therefore \quad -k^3 + \frac{3k^3}{2} = 2$$
$$\therefore \quad \frac{k^3}{2} = 2$$
$$\therefore \quad k^3 = 4$$
$$\therefore \quad k = \sqrt[3]{4}$$

$\therefore$ the x-coordinate of A is $\sqrt[3]{4}$

10 $f(x) = \dfrac{3x-5}{(x-2)^2}$

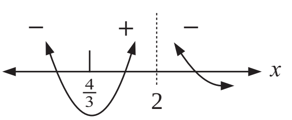
a $f(x)$ cuts the y-axis when $x = 0$ i.e., $y = -\frac{5}{4}$
$f(x)$ cuts the x-axis when $f(x) = 0$ $\therefore$ $3x - 5 = 0$ and so $x = \frac{5}{3}$
$\therefore$ the x-intercept is $\frac{5}{3}$ and the y-intercept is $-\frac{5}{4}$

b There is one vertical asymptote, $x = 2$

c $\dfrac{3x-5}{(x-2)^2}$ is a quotient with $u = 3x - 5$ and $v = (x-2)^2$
$\therefore$ $u' = 3$ and $v' = 2(x-2)^1$

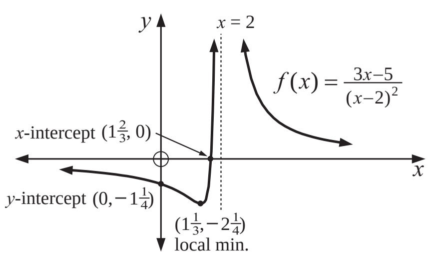
$$\therefore \quad f'(x) = \frac{3(x-2)^2 - 2(3x-5)(x-2)}{(x-2)^4} \quad \{\text{quotient rule}\}$$
$$= \frac{(x-2)\,[3(x-2) - 2(3x-5)]}{(x-2)^4}$$
$$= \frac{3x - 6 - 6x + 10}{(x-2)^3}$$
$$= \frac{4-3x}{(x-2)^3}$$

which has sign diagram:



$\therefore$ it has a local minimum at $(\frac{4}{3}, -\frac{9}{4})$

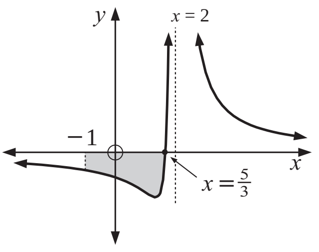
d



e
$$\frac{3x-5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$
$$\therefore \quad \frac{3x-5}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$$
$$\therefore \quad \frac{3x-5}{(x-2)^2} = \frac{Ax + (B-2A)}{(x-2)^2}$$

Equating coefficients, $A = 3$
and $B - 2A = -5$
$\therefore$ $B = 2A - 5 = 6 - 5 = 1$

f



Area

$$= -\int_{-1}^{\frac{5}{3}} f(x)\,dx$$

$$= -\int_{-1}^{\frac{5}{3}} \frac{3x-5}{(x-2)^2}\,dx$$

$$= -\int_{-1}^{\frac{5}{3}} \left(\frac{3}{x-2} + \frac{1}{(x-2)^2}\right) dx$$

$$= -\int_{-1}^{\frac{5}{3}} \frac{3}{x-2}\,dx - \int_{-1}^{\frac{5}{3}} (x-2)^{-2}\,dx$$

$$= -\left[3\ln|x-2|\right]_{-1}^{\frac{5}{3}} - \left[-\frac{1}{x-2}\right]_{-1}^{\frac{5}{3}}$$

$$= -3\ln\tfrac{1}{3} + 3\ln 3 + \tfrac{1}{(-\frac{1}{3})} - \tfrac{1}{(-3)}$$

$$= 3\ln 3 + 3\ln 3 - 3 + \tfrac{1}{3}$$

$$= 6\ln 3 - \tfrac{8}{3} \doteqdot 3.925 \text{ units}^2$$

11 The line $y = mx + c$ passes through $(-1, 0)$. $\therefore$ $0 = -m + c$ and so $c = m$

$\therefore$ the line is $y = cx + c$

The curve and the line meet when $cx + c = -x^2 + 2x + 3$

$\therefore$ $x^2 + (c-2)x + (c-3) = 0$

$\therefore$ $(x+1)(x + [c-3]) = 0$ {as we know $x = -1$ is a solution}

$\therefore$ $x = -1$ or $3 - c$

If we let $a = 3 - c$, then the enclosed area $= \int_{-1}^{a} [(-x^2 + 2x + 3) - (3-a)(x+1)]\,dx$

$$= \int_{-1}^{a} [-x^2 + (a-1)x + a]\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{(a-1)x^2}{2} + ax\right]_{-1}^{a}$$

$$= \left(-\frac{a^3}{3} + \frac{(a-1)a^2}{2} + a^2\right) - \left(\tfrac{1}{3} + \frac{(a-1)}{2} - a\right)$$

$$= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3 - \tfrac{1}{2}a^2 + a^2 - \tfrac{1}{3} - \tfrac{1}{2}a + \tfrac{1}{2} + a$$

$$= \tfrac{1}{6}a^3 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a + \tfrac{1}{6}$$

But this area is 4.5 units^2, $\therefore$ $\tfrac{1}{6}a^3 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a + \tfrac{1}{6} = \tfrac{9}{2}$

$\therefore$ $a^3 + 3a^2 + 3a + 1 = 27$

$\therefore$ $a^3 + 3a^2 + 3a - 26 = 0$

$\therefore$ $(a-2)(a^2 + 5a + 13) = 0$ where the quadratic has $\Delta < 0$, $\therefore$ no real solutions

$\therefore$ $a = 2$ and so $c = 1$ $(= m)$

From technology, the only real solution to this equation is $c = 1$.

So $c = 1$, $m = 1$, and the line has equation $y = x + 1$.

REVIEW SET 26C

1 **a** $\dfrac{dy}{dx} = (x^2 - 1)^2$

$\therefore$ $y = \int (x^2-1)^2\,dx$

$= \int (x^4 - 2x^2 + 1)\,dx$

$= \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x + c$

b $\dfrac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$ $\therefore$ $y = \int (400 - 20e^{-\frac{x}{2}})\,dx$

$= 400x - \dfrac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c$

$= 400x + 40e^{-\frac{x}{2}} + c$

2 **a** $\displaystyle\int_{-2}^{0} \frac{4}{2x-1}\,dx$

$\displaystyle = 4\int_{-2}^{0} \frac{1}{2x-1}\,dx$

$= 4\left[\left(\frac{1}{2}\right)\ln|2x-1|\right]_{-2}^{0}$

$= 2\left[\ln|2x-1|\right]_{-2}^{0}$

$= 2\left[\ln|-1| - \ln|-5|\right]$

$= 2[0 - \ln 5]$

$= -2\ln 5$

$\doteqdot -3.219$

b In $\displaystyle\int_{0}^{1} \frac{10x}{\sqrt{3x^2+1}}\,dx$

we let $u = 3x^2+1, \quad \therefore \quad \dfrac{du}{dx} = 6x$

When $x = 0,\ u = 1$. When $x = 1,\ u = 4$.

$$\therefore \int_{0}^{1} \frac{10x}{\sqrt{3x^2+1}}\,dx = \int_{0}^{1} u^{-\frac{1}{2}}\left(\frac{5}{3}\frac{du}{dx}\right)dx$$

$$= \tfrac{5}{3}\int_{1}^{4} u^{-\frac{1}{2}}\,du$$

$$= \tfrac{5}{3}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_{1}^{4}$$

$$= \tfrac{10}{3}\left(\sqrt{4}-\sqrt{1}\right)$$

$$= \tfrac{10}{3}$$

3 $y = (3x^2+1)^{\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = \frac{1}{2}\left(3x^2+1\right)^{-\frac{1}{2}}\,6x$

$= \dfrac{3x}{\sqrt{3x^2+1}}$

$\therefore \quad \displaystyle\int \frac{3x}{\sqrt{3x^2+1}}\,dx = \sqrt{3x^2+1} + c_1$

$\displaystyle 3\int \frac{x}{\sqrt{3x^2+1}}\,dx = \sqrt{3x^2+1} + c_1$

$\therefore \quad \displaystyle\int \frac{x}{\sqrt{3x^2+1}}\,dx = \tfrac{1}{3}\sqrt{3x^2+1} + c$

4 $a(t) = 6t - 30 \text{ cms}^{-2}$

$\therefore \quad v(t) = \int(6t-30)\,dt = 3t^2 - 30t + c$

But $v(0) = 27 \quad \therefore \quad c = 27$

$\therefore \quad v(t) = 3t^2 - 30t + 27 \text{ cms}^{-1}$

$\therefore \quad s(t) = \int(3t^2 - 30t + 27)\,dt = t^3 - 15t^2 + 27t + d$

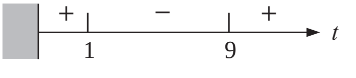
But $s(0) = 0 \quad \therefore \quad d = 0$

$\therefore \quad s(t) = t^3 - 15t^2 + 27t$

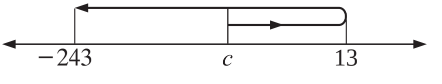
Now $v(t) = 3t^2 - 30t + 27$

$= 3(t^2 - 10t + 9)$

$= 3(t-1)(t-9)$ which has sign diagram:



The particle comes to rest for the second time at $t = 9$ seconds.

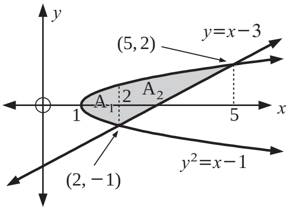


and $s(0) = 0$ cm

$s(1) = 13$ cm

$s(9) = -243$ cm $\qquad \therefore \quad D = 13 + 13 + 243 = 269$ cm

5 **a**



$y^2 = x - 1$ meets $y = x - 3$ where

$x - 1 = (x-3)^2$

$\therefore \quad x - 1 = x^2 - 6x + 9$

$\therefore \quad x^2 - 7x + 10 = 0$

$\therefore \quad (x-5)(x-2) = 0$

$\therefore \quad x = 2$ or $x = 5$

$\therefore$ at $(5, 2)$ and $(2, -1)$

b Area $= A_1 + A_2$

$$= 2\int_1^2 (x-1)^{\frac{1}{2}}\,dx + \int_2^5 [(x-1)^{\frac{1}{2}} - (x-3)]\,dx$$

$$= 2\left[\tfrac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 + \left[\tfrac{2}{3}(x-1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x\right]_2^5$$

$$= 2\left[\tfrac{2}{3} - 0\right] + \left[(\tfrac{2}{3}(8) - \tfrac{25}{2} + 15) - (\tfrac{2}{3} - 2 + 6)\right]$$

$$= 4\tfrac{1}{2} \text{ units}^2$$

6 $y = k$ meets $y = x^2$ where $x^2 = k$ $\therefore$ $x = \pm\sqrt{k}$

By symmetry, $\int_0^{\sqrt{k}} (k - x^2)\,dx = \frac{1}{2} \times 5\frac{1}{3} = \frac{1}{2} \times \frac{16}{3}$

$$\therefore \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = \tfrac{8}{3}$$

$$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} = \tfrac{8}{3}$$

$$\therefore \tfrac{2}{3}k\sqrt{k} = \tfrac{8}{3}$$

$$\therefore k\sqrt{k} = 4$$

$$\therefore k^{\frac{3}{2}} = 4$$

$$\therefore k = 4^{\frac{2}{3}} = \sqrt[3]{16}$$

7

$$f'(x) = 2\sqrt{x} + \frac{a}{\sqrt{x}} = 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$$

$$\therefore f(x) = \tfrac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$$

Now $f(0) = 2$ $\therefore$ $c = 2$ $\therefore$ $f(x) = \dfrac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$

Also $f(1) = 4$ $\therefore$ $\frac{4}{3} + 2a + 2 = 4$ $\therefore$ $2a = \frac{2}{3}$ $\therefore$ $a = \frac{1}{3}$

Now $f'(x) = 2\sqrt{x} + \dfrac{1}{3\sqrt{x}} = \dfrac{6x+1}{3\sqrt{x}} > 0$ since $f(x)$ is only defined for $x > 0$

and for $x > 0$, $6x + 1$ and $3\sqrt{x}$ are > 0

$\therefore$ the function has no stationary point as $f'(x) \neq 0$ for any value of x.

8

$$\frac{-2x}{4-x^2} = \frac{A}{x+2} + \frac{B}{2-x} \quad \text{for all } x$$

$$\therefore \frac{-2x}{4-x^2} = \frac{A(2-x) + B(x+2)}{(x+2)(2-x)}$$

$$\therefore \frac{-2x}{4-x^2} = \frac{2A - Ax + Bx + 2B}{4-x^2}$$

$$\therefore \frac{0-2x}{4-x^2} = \frac{(2A+2B) + x(-A+B)}{4-x^2}$$

Equating coefficients, $-A + B = -2$ and $2A + 2B = 0$, i.e., $B = -A$

$$\therefore -A + -A = -2$$

$$\therefore -2A = -2 \quad \therefore A = 1 \text{ and } B = -1$$

$$\therefore \int_3^4 -\frac{2x}{4-x^2}\,dx = \int_3^4 \left(\frac{1}{x+2} + \frac{1}{x-2}\right) dx \quad \left\{\text{as } \frac{-1}{2-x} = \frac{1}{x-2}\right\}$$

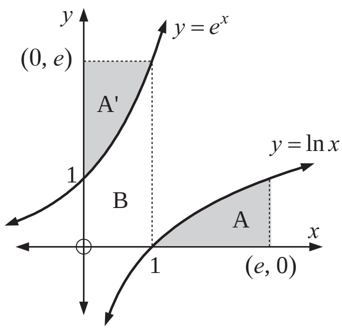
$$= [\ln|x+2| + \ln|x-2|]_3^4$$

$$= (\ln 6 + \ln 2) - (\ln 5 + \ln 1)$$

$$= \ln 12 - \ln 5$$

$$= \ln(\tfrac{12}{5})$$

9



$y = e^x$ and $y = \ln x$ are inverse functions, i.e., they are symmetrical about $y = x$

$\therefore$ area A $=$ area A$'$

but area A$'$ $+$ area B $=$ area of rectangle

$\therefore$ area A $+$ area B $= e \times 1 = e$

and as area A $= \int_1^e \ln x \; dx$

and area B $= \int_0^1 e^x \; dx$

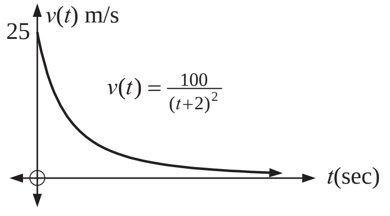
then $\int_1^e \ln x \; dx + \int_0^1 e^x \; dx = e$

10 $v(t) = \dfrac{100}{(t+2)^2} = 100(t+2)^{-2}$ ms^{-1}

a At $t = 0$, $v(0) = \dfrac{100}{2^2} = 25$ ms^{-1} At $t = 3$, $v(3) = \dfrac{100}{5^2} = 4$ ms^{-1}

b as $t \to +\infty$, $v(t) \to 0$ ms^{-1} (above)

c



d As $v(t)$ is always positive, the boat is always travelling forwards.

$$\begin{aligned} s(t) &= \int v(t) \; dt \\ &= \int 100(t+2)^{-2} \; dt \\ &= -100(t+2)^{-1} + c \\ &= \frac{-100}{t+2} + c \end{aligned}$$

$\therefore$ $s(0) = c - 50$ m

$\therefore$ when the boat has travelled 30 m,

$s(t) = c - 20$

$$\therefore \; c - 20 = \frac{-100}{t+2} + c$$

$$\therefore \; \frac{-100}{t+2} = -20$$

$$\therefore \; t + 2 = 5$$

$$\therefore \; t = 3 \text{ seconds}$$

e

$$\begin{aligned} a(t) &= v'(t) \\ &= -200(t+2)^{-3} \text{ ms}^{-2} \\ &= \frac{-200}{(t+2)^3} \end{aligned}$$

f

$$\begin{aligned} \frac{dv}{dt} &= \frac{-200}{(t+2)^3} \\ &= -\tfrac{1}{5} \frac{1000}{(t+2)^3} \\ &= -\tfrac{1}{5} \left(\frac{100}{(t+2)^2} \right)^{\frac{3}{2}} \\ &= -\tfrac{1}{5} \, v^{\frac{3}{2}} \end{aligned}$$

$$\therefore \; \frac{dv}{dt} = -kv^{\frac{3}{2}} \quad \text{where} \quad k = \tfrac{1}{5}$$

11

$$\begin{aligned} \frac{d}{dx}(\ln x)^2 &= 2(\ln x)^1 \left(\frac{1}{x}\right) \\ &= \frac{2 \ln x}{x} \end{aligned}$$

$$\therefore \; \int \frac{2 \ln x}{x} \, dx = (\ln x)^2 + c_1$$

$$\therefore \; \int \frac{\ln x}{x} \, dx = \tfrac{1}{2} (\ln x)^2 + c$$

Chapter 27
TRIGONOMETRIC INTEGRATION

EXERCISE 27A

1 **a** $\int (3\sin x - 2)\ dx$
$= -3\cos x - 2x + c$

b $\int (4x - 2\cos x)\ dx$
$= 2x^2 - 2\sin x + c$

c $\int (2\sqrt{x} + \frac{4}{\cos^2 x})\ dx$
$= \int (2x^{\frac{1}{2}} + 4\frac{1}{\cos^2 x})\ dx$
$= \frac{4}{3}x^{\frac{3}{2}} + 4\tan x + c$

d $\int (\frac{1}{\cos^2 x} + 2\sin x)\ dx$
$= \tan x - 2\cos x + c$

e $\int \left(\frac{x}{2} - \frac{1}{\cos^2 x}\right)\ dx$
$= \frac{x^2}{4} - \tan x + c$

f $\int (\sin x - 2\cos x + e^x)\ dx$
$= -\cos x - 2\sin x + e^x + c$

2 **a** $\int (\sqrt{x} + \frac{1}{2}\cos x)\ dx$
$= \int (x^{\frac{1}{2}} + \frac{1}{2}\cos x)\ dx$
$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}\sin x + c$

b $\int (\theta - \sin\theta)\ d\theta$
$= \frac{\theta^2}{2} + \cos\theta + c$

c $\int (t\sqrt{t} + \frac{2}{\cos^2 t})\ dt$
$= \int (t^{\frac{3}{2}} + 2\sec^2 t)\ dt$
$= \frac{2}{5}t^{\frac{5}{2}} + 2\tan t + c$

d $\int (2e^t - 4\sin t)\ dt$
$= 2e^t + 4\cos t + c$

e $\int \left(3\cos t - \frac{1}{t}\right)\ dt$
$= 3\sin t - \ln|t| + c$

f $\int (3 - \frac{2}{\theta} + \frac{1}{\cos^2\theta})\ d\theta$
$= 3\theta - 2\ln|\theta| + \tan\theta + c$

3 **a** $\frac{d}{dx}(e^x\sin x) = e^x\sin x + e^x\cos x$ $\quad\therefore \int e^x(\sin x + \cos x)\ dx = \int (e^x\sin x + e^x\cos x)\ dx$
$= e^x\sin x + c$

b $\frac{d}{dx}\left(e^{-x}\sin x\right) = -e^{-x}\sin x + e^{-x}\cos x$
$= \frac{\cos x - \sin x}{e^x}$ $\quad\therefore \int \frac{\cos x - \sin x}{e^x}\ dx = e^{-x}\sin x + c$

c $\frac{d}{dx}(x\cos x) = \cos x + x(-\sin x)$
$= \cos x - x\sin x$

$\therefore \int (\cos x - x\sin x)\ dx = x\cos x + c_1$
$\therefore \int \cos x\ dx - \int x\sin x\ dx = x\cos x + c_1$
$\therefore \sin x - \int x\sin x\ dx = x\cos x + c_1$
$\therefore \int x\sin x\ dx = -x\cos x + \sin x + c$

d $\frac{1}{\cos x} = (\cos x)^{-1}$ $\quad\therefore \frac{d}{dx}(\frac{1}{\cos x}) = -(\cos x)^{-2}(-\sin x)$
$= \frac{\sin x}{\cos^2 x}$
$= \frac{\frac{\sin x}{\cos x}}{\cos x}$
$= \frac{\tan x}{\cos x}$ $\quad\therefore \int \frac{\tan x}{\sin x}\ dx = \frac{1}{\cos x} + c$

4 **a**
$$f'(x) = x^2 - 4\cos x$$
$$\therefore \quad f(x) = \int (x^2 - 4\cos x)\ dx$$
$$= \frac{x^3}{3} - 4\sin x + c$$
But $f(0) = 3$
$$\therefore \quad 0 - 4\sin(0) + c = 3$$
$$\therefore \quad c = 3$$
$$\therefore \quad f(x) = \frac{x^3}{3} - 4\sin x + 3$$

b
$$f'(x) = 2\cos x - 3\sin x$$
$$\therefore \quad f(x) = \int (2\cos x - 3\sin x)\ dx$$
$$= 2\sin x + 3\cos x + c$$
But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$
$$\therefore \quad 2\sin\tfrac{\pi}{4} + 3\cos\tfrac{\pi}{4} + c = \tfrac{1}{\sqrt{2}}$$
$$\therefore \quad 2(\tfrac{1}{\sqrt{2}}) + 3(\tfrac{1}{\sqrt{2}}) + c = \tfrac{1}{\sqrt{2}}$$
$$\therefore \quad c = -\tfrac{4}{\sqrt{2}}$$
$$\therefore \quad c = -2\sqrt{2}$$
$$\therefore \quad f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$$

c
$$f'(x) = \sqrt{x} - \frac{2}{\cos^2 x}$$
$$\therefore \quad f(x) = \int (x^{\frac{1}{2}} - \frac{2}{\cos^2 x})\ dx$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x + c$$
But $f(\pi) = 0$
$$\therefore \quad \tfrac{2}{3}\pi^{\frac{3}{2}} - 2\tan\pi + c = 0$$
$$\therefore \quad c = -\tfrac{2}{3}\pi^{\frac{3}{2}}$$
$$\therefore \quad f(x) = \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x - \tfrac{2}{3}\pi^{\frac{3}{2}}$$

EXERCISE 27B

1 **a** $\int \sin(3x)\ dx$
$= -\frac{1}{3}\cos(3x) + c$

b $\int 2\cos(4x)\ dx$
$= 2 \times \frac{1}{4}\sin(4x) + c$
$= \frac{1}{2}\sin(4x) + c$

c $\displaystyle\int \frac{1}{\cos^2(2x)}\ dx$
$= \frac{1}{2}\tan(2x) + c$

d $\int 3\cos\left(\frac{x}{2}\right)\ dx$
$= 6\sin\left(\frac{x}{2}\right) + c$

e $\int (3\sin(2x) - e^{-x})\ dx$
$= -\frac{3}{2}\cos(2x) + e^{-x} + c$

f $\displaystyle\int \left[e^{2x} - \frac{2}{\sec^2\left(\frac{x}{2}\right)}\right]\ dx$
$= \frac{1}{2}e^{2x} - 2 \times 2\tan\left(\frac{x}{2}\right) + c$
$= \frac{1}{2}e^{2x} - 4\tan\left(\frac{x}{2}\right) + c$

g $\int 2\sin(2x + \frac{\pi}{6})\ dx$
$= -\frac{2}{2}\cos(2x + \frac{\pi}{6}) + c$
$= -\cos(2x + \frac{\pi}{6}) + c$

h $\int -3\cos(\frac{\pi}{4} - x)\ dx$
$= -3(-\sin(\frac{\pi}{4} - x)) + c$
$= 3\sin(\frac{\pi}{4} - x) + c$

i $\displaystyle\int \frac{4}{\cos^2\left(\frac{\pi}{3} - 2x\right)}\ dx$
$= 4 \times (-\frac{1}{2})\tan(\frac{\pi}{3} - 2x) + c$
$= -2\tan(\frac{\pi}{3} - 2x) + c$

j $\int \cos(2x) + \sin(2x)\, dx$
$= \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + c$

k $\int 2\sin(3x) + 5\cos(4x)\, dx$
$= -\frac{2}{3}\cos(3x) + \frac{5}{4}\sin(4x) + c$

l $\int \frac{1}{2}\cos(8x) - 3\sin x\, dx$
$= \frac{1}{2}(\frac{1}{8})\sin(8x) + 3\cos x + c$
$= \frac{1}{16}\sin(8x) + 3\cos x + c$

2 **a** $\int \cos^2 x \, dx$
$= \int (\frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

b $\int \sin^2 x \, dx$
$= \int (\frac{1}{2} - \frac{1}{2}\cos(2x)) \, dx$
$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

c $\int (1 + \cos^2(2x)) \, dx$
$= \int (1 + \frac{1}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \int (\frac{3}{2} + \frac{1}{2}\cos(4x)) \, dx$
$= \frac{3}{2}x + \frac{1}{8}\sin(4x) + c$

d $\int (3 - \sin^2(3x)) \, dx$
$= \int (3 - (\frac{1}{2} - \frac{1}{2}\cos(6x))) \, dx$
$= \int (\frac{5}{2} + \frac{1}{2}\cos(6x)) \, dx$
$= \frac{5}{2}x + \frac{1}{12}\sin(6x) + c$

e $\int \frac{1}{2}\cos^2(4x) \, dx$
$= \int \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\cos(8x)) \, dx$
$= \int (\frac{1}{4} + \frac{1}{4}\cos(8x)) \, dx$
$= \frac{1}{4}x + \frac{1}{32}\sin(8x) + c$

f $\int (1 + \cos x)^2 \, dx$
$= \int (1 + 2\cos x + \cos^2 x) \, dx$
$= \int (1 + 2\cos x + \frac{1}{2} + \frac{1}{2}\cos(2x)) \, dx$
$= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin(2x) + c$

3 $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ $\therefore$
$$\begin{aligned}\cos^4\theta &= (\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta))^2\\ &= \tfrac{1}{4} + \tfrac{1}{4}\cos^2(2\theta) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{4} + \tfrac{1}{4}(\tfrac{1}{2} + \tfrac{1}{2}\cos(4\theta)) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta) + \tfrac{3}{8} \quad \text{as required}\end{aligned}$$

$\therefore \int \cos^4 x \, dx = \int (\frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}) \, dx = \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$

4 **a** Consider $\int \sin^4 x \cos x \, dx$

Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$\therefore \int \sin^4 x \cos x \, dx$
$= \int u^4 \frac{du}{dx} \, dx$
$= \int u^4 \, du$
$= \frac{u^5}{5} + c$
$= \frac{1}{5}\sin^5 x + c$

b Consider $\int \frac{\sin x}{\sqrt{\cos x}} \, dx$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$\therefore \int \frac{\sin x}{\sqrt{\cos x}} \, dx$
$= \int u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx$
$= \int -u^{-\frac{1}{2}} \, du$
$= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$ or $-2\sqrt{\cos x} + c$

c Consider $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$\therefore \int \frac{\sin x}{\cos x} \, dx$
$= \int \frac{1}{u}\left(-\frac{du}{dx}\right) dx$
$= \int -\frac{1}{u} \, du$
$= -\ln|u| + c$
$= -\ln|\cos x| + c$

d Consider $\int \sqrt{\sin x}\cos x \, dx$

Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$\therefore \int \sqrt{\sin x}\cos x \, dx$
$= \int u^{\frac{1}{2}} \frac{du}{dx} \, dx$
$= \int u^{\frac{1}{2}} \, du$
$= \frac{2}{3}u^{\frac{3}{2}} + c$
$= \frac{2}{3}(\sin x)^{\frac{3}{2}} + c$

e Consider $\displaystyle\int \frac{\cos x}{(2+\sin x)^2}\,dx$

Let $u = 2+\sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \frac{\cos x}{(2+\sin x)^2}\,dx$$
$$= \int u^{-2}\,\frac{du}{dx}\,dx$$
$$= \int u^{-2}\,du$$
$$= -u^{-1} + c$$
$$= \frac{-1}{2+\sin x} + c$$

f Consider $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\therefore \int \frac{\sin x}{\cos^3 x}\,dx$$
$$= \int u^{-3}\left(-\frac{du}{dx}\right)dx$$
$$= \int -u^{-3}\,du$$
$$= \frac{-u^{-2}}{-2} + c$$
$$= \tfrac{1}{2}u^{-2} + c$$
$$= \tfrac{1}{2}(\cos x)^{-2} + c$$
$$= \frac{1}{2\cos^2 x} + c$$

g Consider $\displaystyle\int \frac{\sin x}{1-\cos x}\,dx$

Let $u = 1-\cos x$, $\dfrac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1-\cos x}\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + c$$
$$= \ln|1-\cos x| + c$$

h Consider $\displaystyle\int \frac{\cos(2x)}{\sin(2x)-3}\,dx$

Let $u = \sin(2x)-3$, $\dfrac{du}{dx} = 2\cos(2x)$

$$\therefore \cos(2x) = \tfrac{1}{2}\frac{du}{dx}$$

$$\therefore \int \frac{\cos(2x)}{\sin(2x)-3}\,dx$$
$$= \int \frac{1}{u}\left(\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{2}\ln|u| + c$$
$$= \tfrac{1}{2}\ln|\sin(2x)-3| + c$$

i Consider $\displaystyle\int \frac{\cos(3x)}{\sin(3x)}\,dx$

Let $u = \sin(3x)$, $\dfrac{du}{dx} = 3\cos(3x)$

$$\therefore \cos(3x) = \tfrac{1}{3}\frac{du}{dx}$$

$$\therefore \int \frac{\cos(3x)}{\sin(3x)}\,dx$$
$$= \int \frac{1}{u}\left(\tfrac{1}{3}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{3}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{3}\ln|u| + c$$
$$= \tfrac{1}{3}\ln|\sin(3x)| + c$$

5 **a** $\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$
$= \int (1-\sin^2 x)\cos x\, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$\therefore \int \cos^3 x\, dx = \int (1-u^2)\dfrac{du}{dx}\,dx$
$= \int (1-u^2)\,du$
$= u - \dfrac{u^3}{3} + c$
$= \sin x - \dfrac{\sin^3 x}{3} + c$
$= \sin x - \frac{1}{3}\sin^3 x + c$

b $\int \sin^5 x\, dx$
$= \int \sin^4 x \sin x\, dx$
$= \int (1-\cos^2 x)^2 \sin x\, dx$
$= \int (1-2\cos^2 x + \cos^4 x)\sin x\, dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$\therefore \int \sin^5 x\, dx = \int (1-2u^2+u^4)\left(-\dfrac{du}{dx}\right)dx$
$= \int (-1+2u^2-u^4)\,du$
$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c$
$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

c $\int \sin^4 x \cos^3 x\, dx$
$= \int \sin^4 x \cos^2 x \cos x\, dx$
$= \int \sin^4 x(1-\sin^2 x)\cos x\, dx$
$= \int (\sin^4 x - \sin^6 x)\cos x\, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$\therefore \int \sin^4 x \cos^3 x\, dx = \int (u^4-u^6)\dfrac{du}{dx}\,dx$
$= \int (u^4-u^6)\,du$
$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c$
$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

6 **a** Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$\therefore f(x) = \int \sin x e^{\cos x}\, dx$
$= \int e^{\cos x}\sin x\, dx$
$= \int e^u\left(-\dfrac{du}{dx}\right)dx$
$= -\int e^u\, du$
$= -e^u + c$
$= -e^{\cos x} + c$

b Let $u = \sin(2x)$, $\dfrac{du}{dx} = 2\cos(2x)$

$\therefore f(x) = \int \sin^3(2x)\cos(2x)\, dx$
$= \int u^3\left(\frac{1}{2}\dfrac{du}{dx}\right)dx$
$= \frac{1}{2}\int u^3\, du$
$= \frac{1}{2}\times\dfrac{u^4}{4} + c$
$= \frac{1}{8}\sin^4(2x) + c$

c Let $u = \sin x - \cos x$,

$\dfrac{du}{dx} = \cos x + \sin x$

$\therefore f(x) = \displaystyle\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$
$= \displaystyle\int \frac{1}{u}\frac{du}{dx}\,dx$
$= \displaystyle\int \frac{1}{u}\,du$
$= \ln|u| + c$
$= \ln|\sin x - \cos x| + c$

d Let $u = \tan x$

$\therefore \dfrac{du}{dx} = \dfrac{1}{\cos^2 x}$

So, $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx$
$= \displaystyle\int e^u\left(\frac{du}{dx}\right)dx$
$= \displaystyle\int e^u\,du$
$= e^u + c$
$= e^{\tan x} + c$

EXERCISE 27C

1 **a** $\int_0^{\frac{\pi}{6}} \cos x \, dx$
$= [\sin x]_0^{\frac{\pi}{6}}$
$= \sin \frac{\pi}{6} - \sin 0$
$= \frac{1}{2} - 0$
$= \frac{1}{2}$

b $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx$
$= [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$
$= -\cos \frac{\pi}{2} + \cos \frac{\pi}{3}$
$= 0 + \frac{1}{2}$
$= \frac{1}{2}$

c $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx$
$= [\tan x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$
$= \tan \frac{\pi}{3} - \tan \frac{\pi}{4}$
$= \sqrt{3} - 1$

d $\int_0^{\frac{\pi}{6}} \sin(3x) \, dx$
$= \left[-\frac{1}{3} \cos (3x)\right]_0^{\frac{\pi}{6}}$
$= -\frac{1}{3}[\cos \frac{\pi}{2} - \cos 0]$
$= -\frac{1}{3}[0 - 1]$
$= \frac{1}{3}$

e $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$
$= \int_0^{\frac{\pi}{4}} (\frac{1}{2} + \frac{1}{2} \cos(2x)) \, dx$
$= \left[\frac{x}{2} + \frac{1}{4} \sin(2x)\right]_0^{\frac{\pi}{4}}$
$= [\frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2}] - 0$
$= \frac{\pi}{8} + \frac{1}{4}$

f $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$
$= \int_0^{\frac{\pi}{2}} (\frac{1}{2} - \frac{1}{2} \cos (2x)) \, dx$
$= \left[\frac{x}{2} - \frac{1}{4} \sin (2x)\right]_0^{\frac{\pi}{2}}$
$= [\frac{\pi}{4} - \frac{1}{4} \sin \pi] - 0$
$= \frac{\pi}{4}$

2 **a** Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$
When $x = \frac{\pi}{3}$, $u = \cos \frac{\pi}{3} = \frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} \, dx$
$= \int_1^{\frac{1}{2}} u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx$
$= \int_{\frac{1}{2}}^1 u^{-\frac{1}{2}} \, du$
$= \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^1$
$= 2\sqrt{1} - 2\sqrt{\frac{1}{2}}$
$= 2 - \sqrt{2}$

b Let $u = \sin x$, $\frac{du}{dx} = \cos x$

When $x = 0$, $u = \sin 0 = 0$
When $x = \frac{\pi}{6}$, $u = \sin \frac{\pi}{6} = \frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx$
$= \int_0^{\frac{1}{2}} u^2 \frac{du}{dx} \, dx$
$= \int_0^{\frac{1}{2}} u^2 \, du$
$= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}}$
$= \frac{1}{3}(\frac{1}{2})^3 - 0$ or $\frac{1}{24}$

c Let $u = \cos x$ $\therefore \frac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$
When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \int_0^{\frac{\pi}{4}} \tan x \, dx$
$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$
$= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx}\right) dx$
$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} \, du$
$= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1$
$= \ln 1 - \ln \frac{1}{\sqrt{2}}$
$= \ln \sqrt{2}$
$= \frac{1}{2} \ln 2$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x}\,dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\,du$$
$$= [\ln|u|]_{\frac{1}{2}}^{1}$$
$$= \ln 1 - \ln\tfrac{1}{2}$$
$$= 0 + \ln 2 \qquad \{\ln\left(\frac{1}{a}\right) = -\ln a\}$$
$$= \ln 2$$

e Let $u = 1 - \sin x$, $\dfrac{du}{dx} = -\cos x$

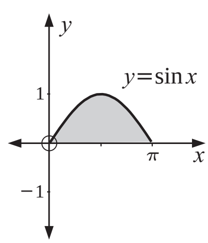
When $x = 0$, $u = 1 - \sin 0 = 1$

When $x = \frac{\pi}{6}$, $u = 1 - \sin\frac{\pi}{6} = \frac{1}{2}$

$$\therefore \int_{0}^{\frac{\pi}{6}} \frac{\cos x}{1 - \sin x}\,dx$$
$$= \int_{1}^{\frac{1}{2}} \frac{1}{u}\left(-\frac{du}{dx}\right)dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\,du$$
$$= [\ln|u|]_{\frac{1}{2}}^{1}$$
$$= \ln 1 - \ln\tfrac{1}{2}$$
$$= 0 + \ln 2$$
$$= \ln 2$$

EXERCISE 27D

1 **a**

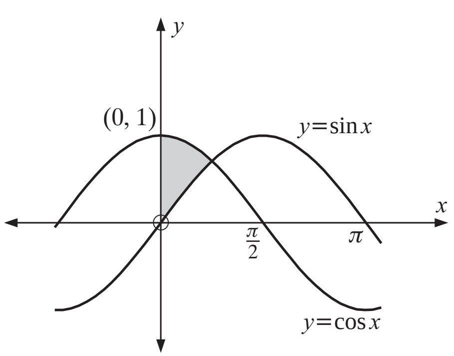


$$A = \int_0^{\pi} \sin x\,dx$$
$$= [-\cos x]_0^{\pi}$$
$$= [-\cos\pi + \cos 0]$$
$$= -(-1) + 1$$
$$= 2 \text{ units}^2$$

b We first note that $\sin^2 x \geqslant 0$ always, so the function never drops below the x-axis.

$$\therefore\ A = \int_0^{\pi} \sin^2 x\,dx$$
$$= \int_0^{\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)dx$$
$$= \left[\frac{x}{2} - \tfrac{1}{4}\sin(2x)\right]_0^{\pi}$$
$$= \left[\tfrac{\pi}{2} - \tfrac{1}{4}\sin(2\pi)\right] - \left[0 - \tfrac{1}{4}\sin 0\right]$$
$$= \tfrac{\pi}{2} \text{ units}^2$$

2



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\therefore\ A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$
$$= \left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - (\sin 0 + \cos 0)$$
$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - (0 + 1)$$
$$= \tfrac{2}{\sqrt{2}} - 1$$
$$= (\sqrt{2} - 1) \text{ units}^2$$

3 **a** $y = \tan x$

A has y-coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}]$.

$\therefore \quad \tan x = 1$

$\therefore \quad x = \frac{\pi}{4}$

i.e., A is at $(\frac{\pi}{4}, 1)$

b $\tan x = \dfrac{\sin x}{\cos x}$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned}
\therefore \quad \text{area} &= \int_0^{\frac{\pi}{4}} \tan x \; dx \\
&= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx \\
&= \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u}\; du \\
&= [\ln |u|]_{\frac{1}{\sqrt{2}}}^{1} \\
&= \ln 1 - \ln \tfrac{1}{\sqrt{2}} \\
&= \ln\sqrt{2} \quad \text{or} \quad \tfrac{1}{2}\ln 2 \text{ units}^2
\end{aligned}$$

4 **a** $y = \sin(2x)$ is the curve C_1

$y = \sin x$ is the curve C_2

b The curves meet when $\sin(2x) = \sin x$

$$\begin{aligned}
\therefore \quad 2\sin x\cos x - \sin x &= 0 \\
\therefore \quad \sin x(2\cos x - 1) &= 0 \\
\text{i.e., } \sin x = 0 \text{ or } \cos x &= \tfrac{1}{2} \\
\therefore \quad x \text{ coordinate of A} &= \tfrac{\pi}{3} \\
\therefore \quad \text{A is at } &(\tfrac{\pi}{3}, \tfrac{\sqrt{3}}{2})
\end{aligned}$$

c
$$\begin{aligned}
\text{Area} &= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x)\; dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x))\; dx \\
&= \left[-\tfrac{1}{2}\cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \tfrac{1}{2}\cos(2x)\right]_{\frac{\pi}{3}}^{\pi} \\
&= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right) + \left(-\cos\pi + \tfrac{1}{2}\cos 2\pi\right) \\
&\quad - \left(-\cos\tfrac{\pi}{3} + \tfrac{1}{2}\cos\tfrac{2\pi}{3}\right) \\
&= (\tfrac{1}{4} + \tfrac{1}{2}) - (-\tfrac{1}{2} + 1) + (1 + \tfrac{1}{2}) - (-\tfrac{1}{2} - \tfrac{1}{4}) \\
&= 2\tfrac{1}{2} \text{ units}^2
\end{aligned}$$

5 **a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1

b When $x = 0$, $y = \cos 0 = 1$. $\quad \therefore$ A is at $(0, 1)$

When $x = \frac{\pi}{4}$, $y = \cos(\frac{\pi}{2}) = 0$. $\quad \therefore$ B is at $(\frac{\pi}{4}, 0)$

Also, $\cos^2(\frac{\pi}{2}) = 0$, $\quad \therefore$ C is at $(\frac{\pi}{2}, 0)$

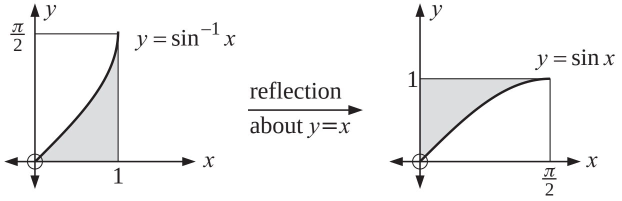
When $x = \frac{3\pi}{4}$, $y = \cos(\frac{3\pi}{2}) = 0$, $\quad \therefore$ D is at $(\frac{3\pi}{4}, 0)$

$\cos(2\pi) = \cos^2\pi = 1$ $\therefore$ $y = \cos(2x)$ and $y = \cos^2 x = 1$ when $x = \pi$ $\quad \therefore$ E is at $(\pi, 1)$

c
$$\begin{aligned}
A &= \int_0^{\pi} (\cos^2 x - \cos(2x))\; dx \\
&= \int_0^{\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) - \cos(2x)\; dx \\
&= \int_0^{\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\; dx \\
&= \left[\frac{x}{2} - \tfrac{1}{4}\sin(2x)\right]_0^{\pi} = (\tfrac{\pi}{2} - 0) - (0 - 0) = \tfrac{\pi}{2} \text{ units}^2
\end{aligned}$$

6 $y = \sin^{-1} x$ is the reflection of the function $y = \sin x$ in the line $y = x$.

$\therefore$ since $\sin^{-1} 1 = \frac{\pi}{2}$, the shaded areas in the figures below are equal.



$$\therefore \int_0^1 \sin^{-1} x \, dx = \text{area of rectangle} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$
$$= \tfrac{\pi}{2} \times 1 - [-\cos x]_0^{\frac{\pi}{2}}$$
$$= \tfrac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$$
$$= (\tfrac{\pi}{2} - 1) \text{ units}^2$$

REVIEW SET 27A

1 **a** $\int \sin^7 x \cos x \, dx$

$= \int u^7 \left(\frac{du}{dx}\right) dx$ on letting $u = \sin x$

$= \int u^7 \, du$

$= \dfrac{u^8}{8} + c$

$= \frac{1}{8} \sin^8 x + c$

b Consider $\int \tan(2x) \, dx = \displaystyle\int \frac{\sin(2x)}{\cos(2x)} \, dx$

Let $u = \cos(2x)$, $\dfrac{du}{dx} = -2\sin(2x)$

$\therefore \sin(2x) = -\frac{1}{2} \dfrac{du}{dx}$

$$\therefore \int \tan(2x) \, dx = \int \frac{1}{u} \left(-\tfrac{1}{2} \frac{du}{dx}\right) dx$$
$$= -\tfrac{1}{2} \int \frac{1}{u} \, du$$
$$= -\tfrac{1}{2} \ln|u| + c$$
$$= -\tfrac{1}{2} \ln|\cos(2x)| + c$$

c Consider $\int e^{\sin x} \cos x \, dx$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int e^{\sin x} \cos x \, dx = \int e^u \frac{du}{dx} \, dx$$
$$= \int e^u \, du$$
$$= e^u + c$$
$$= e^{\sin x} + c$$

2 If $y = x \tan x$ then

$$\frac{dy}{dx} = 1 \tan x + x \left(\frac{1}{\cos^2 x}\right)$$
$$= \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}$$

$$\therefore \int \frac{\sin x}{\cos x} \, dx + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$$

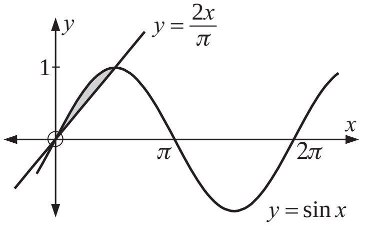
$$\therefore \int \frac{1}{u} \left(-\frac{du}{dx}\right) dx + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$$

{letting $u = \cos x$, $\therefore \dfrac{du}{dx} = -\sin x$}

$$\therefore -\int \frac{1}{u} \, du + \int \frac{x}{\cos^2 x} \, dx = x \tan x + c_1$$

$$\therefore \int \frac{x}{\cos^2 x} \, dx = x \tan x + \ln|u| + c$$
$$= x \tan x + \ln|\cos x| + c$$

3 Using technology, the graphs meet when $x=0$ and $x=\frac{\pi}{2}$. $\therefore$ enclosed area



$$\begin{aligned}&= \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2x}{\pi}\right) dx \\ &= \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}} \\ &= (0-\tfrac{\pi}{4}) - (-1-0) \\ &= (1-\tfrac{\pi}{4}) \text{ units}^2\end{aligned}$$

4 **a**
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx &= \int_0^{\frac{\pi}{3}} \left(\tfrac{1}{2}+\tfrac{1}{2}\cos x\right) dx \\ &= \left[\frac{x}{2}+\tfrac{1}{2}\sin x\right]_0^{\frac{\pi}{3}} \\ &= \tfrac{\pi}{6}+\tfrac{1}{2}\sin\tfrac{\pi}{3} - 0 - \tfrac{1}{2}\sin 0 \\ &= \tfrac{\pi}{6}+\tfrac{\sqrt{3}}{4}\end{aligned}$$

b $\int_0^{\frac{\pi}{4}} \tan x\, dx = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\, dx$

Let $u=\cos x$, $\dfrac{du}{dx} = -\sin x$

When $x=0$, $u=\cos 0 = 1$

When $x=\frac{\pi}{4}$, $u=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{4}} \tan x\, dx &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx \\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\, du \\ &= [\ln|u|]_{\frac{1}{\sqrt{2}}}^1 \\ &= \ln 1 - \ln\tfrac{1}{\sqrt{2}} \\ &= \ln\sqrt{2} \\ &= \tfrac{1}{2}\ln 2\end{aligned}$$

5
$$\begin{aligned}y &= \ln\left(\frac{1}{\cos x}\right) \\ &= -\ln(\cos x) \\ \therefore \frac{dy}{dx} &= -\frac{1}{\cos x}\times(-\sin x) \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \\ \therefore \int \tan x\, dx &= \ln\left(\frac{1}{\cos x}\right)+c \\ &= -\ln(\cos x)+c \\ &\left\{\text{since } \frac{1}{\cos x}>0\right\}\end{aligned}$$

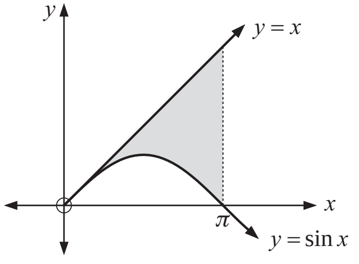
6 Let $u=\sin\theta$ $\therefore$ $\dfrac{du}{d\theta}=\cos\theta$

$u(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$u(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$\begin{aligned}&\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta}\, d\theta \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\frac{du}{d\theta}\, d\theta \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\, du \\ &= [\ln|u|]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln\left(\tfrac{1}{2}\right) \\ &= 0+\ln 2 \\ &= \ln 2\end{aligned}$$

7



Required area

= area of Δ − area under sine curve

$= \frac{1}{2}\pi \times \pi - \int_0^{\pi} \sin x \, dx$

$= \dfrac{\pi^2}{2} - [-\cos x]_0^{\pi}$

$= \dfrac{\pi^2}{2} - [-\cos \pi + \cos 0]$

$= \dfrac{\pi^2}{2} - 2$ units^2

REVIEW SET 27B

1 **a** $\displaystyle\int 4\sin^2\left(\frac{x}{2}\right) dx$

$\displaystyle = \int 2\left(2\sin^2\left(\tfrac{x}{2}\right)\right) dx$

Now $\cos(2\theta) = 1 - 2\sin^2\theta$

$\therefore \; 2\sin^2\theta = 1 - \cos(2\theta)$

$\displaystyle\therefore \int 4\sin^2\left(\frac{x}{2}\right) dx = 2\int (1-\cos x)\, dx$

$= 2[x - \sin x] + c$

$= 2x - 2\sin x + c$

b $\displaystyle\int (2-\cos x)^2 \, dx$

$\displaystyle = \int \left(4 - 4\cos x + \cos^2 x\right) dx$

$\displaystyle = \int 4 - 4\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) dx$

$= \frac{9}{2}x - 4\sin x + \frac{1}{2}\left(\frac{1}{2}\right)\sin(2x)\, dx$

$= \frac{9}{2}x - 4\sin x + \frac{1}{4}\sin(2x) + c$

2 If $u = \cos x$, $\dfrac{du}{dx} = -\sin x$ $\quad \therefore \displaystyle\int \frac{\sin x}{\cos^4 x} dx$

$\displaystyle = \int u^{-4}\left(-\frac{du}{dx}\right) dx$

$\displaystyle = -\int u^{-4}\, du$

$= \dfrac{u^{-3}}{3} + c$

$= \dfrac{1}{3\cos^3 x} + c$

3 $\dfrac{d}{dx}(\sin(x^2))$

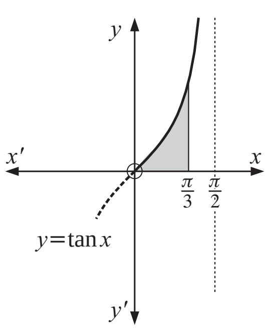
$= 2x\cos(x^2)$

$\therefore \int x\cos(x^2)\, dx$

$= \frac{1}{2}\int 2x\cos(x^2)\, dx$

$= \frac{1}{2}\sin(x^2) + c$

4



Now $\tan x \geqslant 0$ for $0 \leqslant x \leqslant \frac{\pi}{3}$

$\therefore \; A = \int_0^{\frac{\pi}{3}} \tan x \, dx$ $\qquad$ Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$\displaystyle = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$ $\qquad$ When $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

When $x = 0$, $u = \cos 0 = 1$

$\displaystyle = \int_1^{\frac{1}{2}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx$

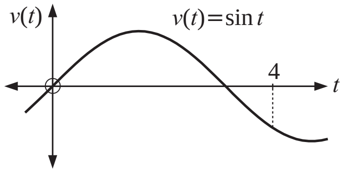
$\displaystyle = \int_{\frac{1}{2}}^{1} \frac{1}{u}\, du$

$= [\ln|u|]_{\frac{1}{2}}^{1}$

$= \ln 1 - \ln\frac{1}{2}$

$= \ln 2$ units^2

5



Since $v(t) = \sin t \text{ ms}^{-1}$, $v(t) = 0$ when $t = \pi$ seconds.

We may therefore see from the graph that the particle reverses its direction at time $t = \pi$.

$\therefore$ the distance travelled is
$$\begin{aligned} D &= \int_0^{\pi} \sin t \; dt + \int_{\pi}^{4} (-\sin t) \; dt \\ &= [-\cos t]_0^{\pi} + [\cos t]_{\pi}^{4} \\ &= -\cos \pi + \cos 0 + \cos 4 - \cos \pi \\ &= 1 + 1 + \cos 4 + 1 \\ &= 3 + \cos 4 \\ &\doteqdot 2.35 \text{ m} \end{aligned}$$

6 **a**
$$\begin{aligned} &\int_0^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right) \; dx \\ &= \int_0^{\frac{\pi}{6}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos x\right) \; dx \\ &= \left[\frac{x}{2} - \tfrac{1}{2}\sin x\right]_0^{\frac{\pi}{6}} \\ &= \left(\tfrac{\pi}{12} - \tfrac{1}{4}\right) - (0 - 0) \\ &= \tfrac{\pi}{12} - \tfrac{1}{4} \end{aligned}$$

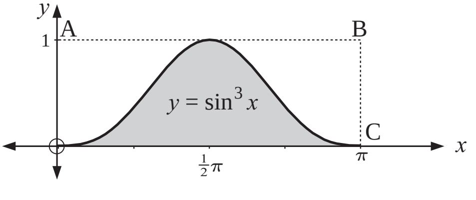
b Let $u = \tan x$, $\dfrac{du}{dx} = \dfrac{1}{\cos^2 x}$

When $x = \frac{\pi}{4}$, $u = \tan \frac{\pi}{4} = 1$

When $x = \frac{\pi}{3}$, $u = \tan \frac{\pi}{3} = \sqrt{3}$

$$\begin{aligned} \therefore \quad &\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x \tan x} \; dx \\ &= \int_1^{\sqrt{3}} \frac{1}{u} \; \frac{du}{dx} \; dx \\ &= \int_1^{\sqrt{3}} \frac{1}{u} \; du \\ &= [\ln |u|]_1^{\sqrt{3}} \\ &= \ln \sqrt{3} - \ln 1 \\ &= \tfrac{1}{2} \ln 3 \end{aligned}$$

7 Consider the graph of
$y = \sin^3 x$, $0 \leqslant x \leqslant \pi$



Now $\int_0^{\pi} \sin^3 x \; dx =$ shaded area

But the shaded area $<$ area of rectangle ABCO

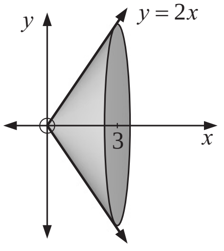
i.e., $\int_0^{\pi} \sin^3 x \; dx < \pi$

$\therefore \quad \int_0^{\pi} \sin^3 x \; dx < 4$

Chapter 28
VOLUMES OF REVOLUTION

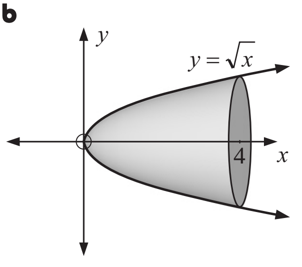
EXERCISE 28A.1

1 **a**



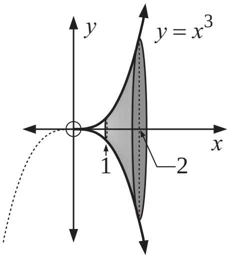
Volume
$$= \pi \int_0^3 (2x)^2\, dx$$
$$= 4\pi \int_0^3 x^2\, dx$$
$$= 4\pi \left[\frac{x^3}{3}\right]_0^3$$
$$= 4\pi(9-0)$$
$$= 36\pi \text{ units}^3$$

b



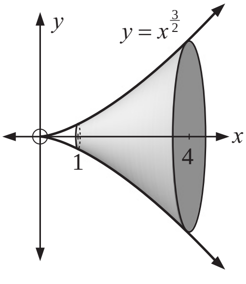
Volume
$$= \pi \int_0^4 (\sqrt{x})^2\, dx$$
$$= \pi \int_0^4 x\, dx$$
$$= \pi \left[\frac{x^2}{2}\right]_0^4$$
$$= \pi(8-0)$$
$$= 8\pi \text{ units}^3$$

c



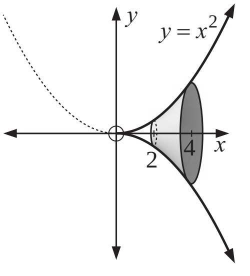
Volume
$$= \pi \int_1^2 (x^3)^2\, dx$$
$$= \pi \int_1^2 x^6\, dx$$
$$= \pi \left[\frac{x^7}{7}\right]_1^2$$
$$= \pi\left(\tfrac{128}{7} - \tfrac{1}{7}\right)$$
$$= \tfrac{127\pi}{7} \text{ units}^3$$

d



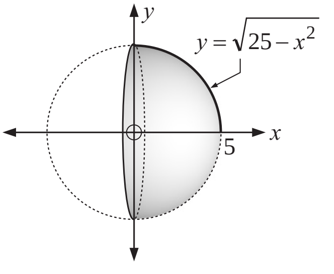
Volume
$$= \pi \int_1^4 \left(x^{\frac{3}{2}}\right)^2 dx$$
$$= \pi \int_1^4 x^3\, dx$$
$$= \pi \left[\frac{x^4}{4}\right]_1^4$$
$$= \pi\left(\tfrac{256}{4} - \tfrac{1}{4}\right)$$
$$= \tfrac{255\pi}{4} \text{ units}^3$$

e



Volume
$$= \pi \int_2^4 (x^2)^2\, dx$$
$$= \pi \int_2^4 x^4\, dx$$
$$= \pi \left[\frac{x^5}{5}\right]_2^4$$
$$= \pi\left[\tfrac{1024}{5} - \tfrac{32}{5}\right]$$
$$= \tfrac{992\pi}{5} \text{ units}^3$$

f



Volume
$$= \pi \int_0^5 (25 - x^2)\, dx$$
$$= \pi \left[25x - \frac{x^3}{3}\right]_0^5$$
$$= \pi\left[\left(125 - \tfrac{125}{3}\right) - 0\right]$$
$$= \pi\left(\tfrac{2}{3}\right) 125$$
$$= \tfrac{250\pi}{3} \text{ units}^3$$

2 **a**
$$\begin{aligned} V &= \pi \int_0^6 \left(\frac{x}{2}+4\right)^2 dx \\ &= \pi \int_0^6 \left(\tfrac{1}{4}x^2+4x+16\right) dx \\ &= \pi \left[\frac{x^3}{12}+\frac{4x^2}{2}+16x\right]_0^6 \\ &= \pi(18+72+96)-0 \\ &= 186\pi \text{ units}^3 \end{aligned}$$

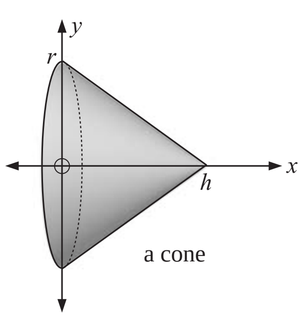
b
$$\begin{aligned} V &= \pi \int_1^2 (x^2+3)^2\, dx \\ &= \pi \int_1^2 \left(x^4+6x^2+9\right) dx \\ &= \pi \left[\frac{x^5}{5}+\frac{6x^3}{3}+9x\right]_1^2 \\ &= \pi\left(\left\{\tfrac{32}{5}+16+18\right\}-\left\{\tfrac{1}{5}+2+9\right\}\right) \\ &= \pi\left(\tfrac{146}{5}\right) \\ &= \tfrac{146\pi}{5} \text{ units}^3 \end{aligned}$$

c
$$\begin{aligned} V &= \pi \int_0^4 (e^x)^2\, dx \\ &= \pi \int_0^4 e^{2x}\, dx \\ &= \pi \left[\tfrac{1}{2}e^{2x}\right]_0^4 \\ &= \pi\left(\tfrac{1}{2}e^8-\tfrac{1}{2}\right) \\ &= \tfrac{\pi}{2}(e^8-1) \text{ units}^3 \end{aligned}$$

3 **a**
$$\begin{aligned} \text{Volume} &= \pi \int_5^8 y^2\, dx \\ &= \pi \int_5^8 (64-x^2)dx \\ &= \pi \left[64x-\frac{x^3}{3}\right]_5^8 \\ &= \pi\left(\left\{512-\tfrac{512}{3}\right\}-\left\{320-\tfrac{125}{3}\right\}\right) \\ &= 63\pi \text{ units}^3 \end{aligned}$$

b 63π cm^3
$\doteqdot 198$ cm^3

4 **a**



a cone

b
$$\begin{aligned} \text{slope} &= \frac{r-0}{0-h} \\ &= -\frac{r}{h} \end{aligned}$$

$\therefore$ equation is $y = -\dfrac{r}{h}x+r$

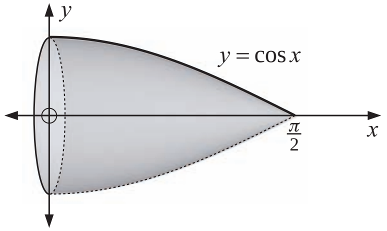
c
$$\begin{aligned} V &= \pi \int_0^h \left(\frac{-r}{h}x+r\right)^2 dx \\ &= \pi r^2 \int_0^h \left(-\frac{x}{h}+1\right)^2 dx \\ &= \pi r^2 \int_0^h \left(\frac{x^2}{h^2}-\frac{2x}{h}+1\right) dx \\ &= \pi r^2 \left[\frac{x^3}{3h^2}-\frac{2x^2}{2h}+x\right]_0^h \\ &= \pi r^2 \left(\left\{\frac{h}{3}-h+h\right\}-0\right) \\ &= \tfrac{1}{3}\pi r^2 h \text{ units}^3 \end{aligned}$$

5 **a** a sphere of radius r

b
$$\begin{aligned} V &= \pi \int_{-r}^{r} y^2\,dx \\ &= 2\pi \int_0^r (r^2 - x^2)\,dx \\ &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi \left(r^3 - \frac{r^3}{3} - 0 \right) \\ &= 2\pi \times \tfrac{2}{3} r^3 \\ &= \tfrac{4}{3}\pi r^3 \text{ units}^3 \end{aligned}$$

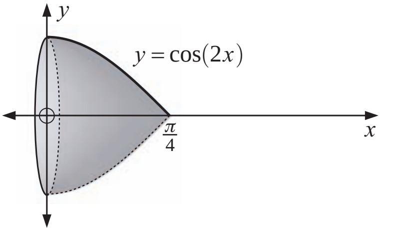
EXERCISE 28A.2

1 **a**



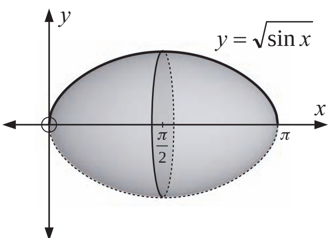
$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2\,dx \\ &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x\,dx \\ &= \pi \int_0^{\frac{\pi}{2}} \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\,dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x) \right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi - 0 \right] \\ &= \tfrac{\pi^2}{4} \text{ units}^3 \end{aligned}$$

b



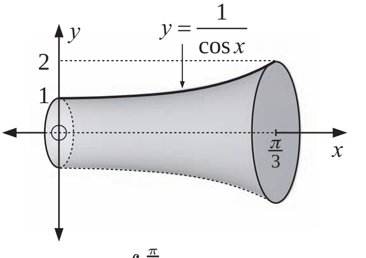
$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x)\,dx \\ &= \pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x) \right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x) \right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi - 0 \right] \\ &= \tfrac{\pi^2}{8} \text{ units}^3 \end{aligned}$$

c



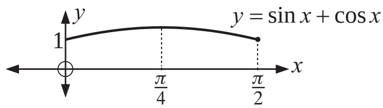
$$\begin{aligned} \text{Volume} &= \pi \int_0^{\pi} \sin x\,dx \\ &= \pi \left[-\cos x \right]_0^{\pi} \\ &= \pi \left[-\cos\pi - -\cos 0 \right] \\ &= \pi(2) \\ &= 2\pi \text{ units}^3 \end{aligned}$$

d



$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx \\ &= \pi \left[\tan x \right]_0^{\frac{\pi}{3}} \\ &= \pi \left(\tan\tfrac{\pi}{3} - \tan 0 \right) \\ &= \pi \times (\sqrt{3} - 0) \\ &= \pi\sqrt{3} \text{ units}^3 \end{aligned}$$

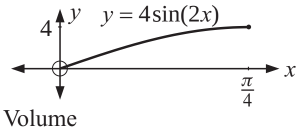
2 **a**



b Volume

$$= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 \, dx$$
$$= \pi \int_0^{\frac{\pi}{4}} \left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right) dx$$
$$= \pi \int_0^{\frac{\pi}{4}} (1 + \sin(2x)) \, dx$$
$$= \pi \left[x - \tfrac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$$
$$= \pi \left(\left\{\tfrac{\pi}{4} - \tfrac{1}{2}\cos\left(\tfrac{\pi}{2}\right)\right\} - \left\{0 - \tfrac{1}{2}(1)\right\}\right)$$
$$= \pi \left(\tfrac{\pi}{4} + \tfrac{1}{2}\right) \text{ units}^3$$

3 **a**



b Volume

$$= \pi \int_0^{\frac{\pi}{4}} (4\sin(2x))^2 \, dx$$
$$= 16\pi \int_0^{\frac{\pi}{4}} \sin^2(2x) \, dx$$
$$= 16\pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(4x)\right) dx$$
$$= 16\pi \left[\frac{x}{2} - \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}}$$
$$= 16\pi \left\{\tfrac{\pi}{8} - \tfrac{1}{8}\sin\pi - \left(0 - \tfrac{1}{8}\sin 0\right)\right\}$$
$$= 2\pi^2 \text{ units}^3$$

EXERCISE 28B

1 **a** The graphs meet where

$$4 - x^2 = 3$$
$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$

$\therefore$ A is $(-1, 3)$
and B is $(1, 3)$

b
$$V = 2\pi \int_0^1 \left((4 - x^2)^2 - 3^2\right) dx$$
$$= 2\pi \int_0^1 \left(16 - 8x^2 + x^4 - 9\right) dx$$
$$= 2\pi \int_0^1 \left(x^4 - 8x^2 + 7\right) dx$$
$$= 2\pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x\right]_0^1$$
$$= 2\pi \left\{\tfrac{1}{5} - \tfrac{8}{3} + 7 - 0\right\}$$
$$= \tfrac{136\pi}{15} \text{ units}^3$$

2 **a** The graphs meet where

$$e^{\frac{x}{2}} = e$$
$$\therefore \quad e^{\frac{x}{2}} = e^1$$
$$\therefore \quad \frac{x}{2} = 1$$
$$\therefore \quad x = 2$$

i.e., at A$(2, e)$

b
$$V = \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}}\right)^2\right) dx$$
$$= \pi \int_0^2 \left(e^2 - e^x\right) dx$$
$$= \pi \left[e^2 x - e^x\right]_0^2$$
$$= \pi \left[2e^2 - e^2 - (0 - 1)\right]$$
$$= \pi \left[e^2 + 1\right] \text{ units}^3$$

3 **a** The graphs meet where

$$x = \frac{1}{x}$$
$$\therefore \quad x^2 = 1$$
$$\therefore \quad x = \pm 1$$
$$\therefore \quad x = 1 \quad \{\text{as } x > 0\}$$

$\therefore$ A is $(1, 1)$

b
$$V = \pi \int_1^2 \left(x^2 - \left(\frac{1}{x}\right)^2\right) dx$$
$$= \pi \int_1^2 \left(x^2 - x^{-2}\right) dx$$
$$= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1}\right]_1^2$$
$$= \pi \left\{\left(\tfrac{8}{3} + \tfrac{1}{2}\right) - \left(\tfrac{1}{3} + 1\right)\right\}$$
$$= \tfrac{11\pi}{6} \text{ units}^3$$

4 **a** The curves meet where

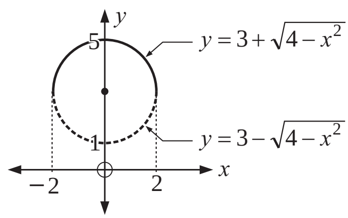
$$\sqrt{x-4} = 1$$
$$\therefore \quad x - 4 = 1$$
$$\therefore \quad x = 5$$

$\therefore$ A is (5, 1)

b
$$\begin{aligned} V &= \pi \int_5^8 \left(\left(\sqrt{x-4}\right)^2 - 1^2 \right) dx \\ &= \pi \int_5^8 (x - 4 - 1)\ dx \\ &= \pi \int_5^8 (x-5)\ dx \\ &= \pi \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= \pi \left\{ (32 - 40) - \left(\tfrac{25}{2} - 25\right) \right\} \\ &= \pi\left(\tfrac{9}{2}\right) = \tfrac{9\pi}{2} \text{ units}^3 \end{aligned}$$

5 **a**
$$\begin{aligned} x^2 + (y-3)^2 &= 4 \\ \therefore \quad (y-3)^2 &= 4 - x^2 \\ \therefore \quad y - 3 &= \pm\sqrt{4 - x^2} \\ \therefore \quad y &= 3 \pm \sqrt{4-x^2} \end{aligned}$$

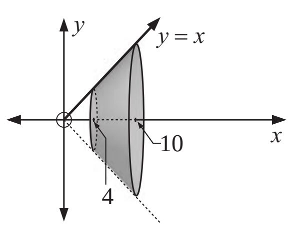
b



c
$$\begin{aligned} V &= \pi \int_{-2}^2 \left\{ \left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2 \right\} dx \\ &= 2\pi \int_0^2 \left\{ \left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2 \right\} dx \\ &= 2\pi \int_0^2 \left\{ \left(9 + 6\sqrt{4-x^2} + 4 - x^2\right) - \left(9 - 6\sqrt{4-x^2} + 4 - x^2\right) \right\} dx \\ &= 2\pi \int_0^2 12\sqrt{4-x^2}\, dx \\ &= 24\pi \int_0^2 \sqrt{4-x^2}\, dx \\ &= 24\pi^2 \text{ units}^3 \quad (\doteqdot 236.9 \text{ units}^3) \qquad \{\text{using technology}\} \end{aligned}$$

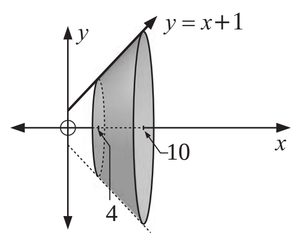
REVIEW SET 28

1 **a**



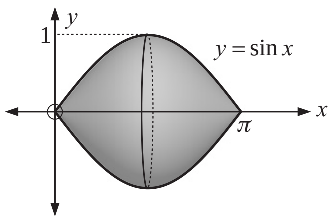
$$\begin{aligned} V &= \pi \int_4^{10} x^2\, dx \\ &= \pi \left[\frac{x^3}{3} \right]_4^{10} \\ &= \pi \left(\tfrac{1000}{3} - \tfrac{64}{3} \right) \\ &= \tfrac{936\pi}{3} \\ &= 312\pi \text{ units}^3 \end{aligned}$$

b



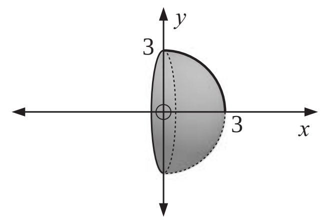
$$\begin{aligned} V &= \pi \int_4^{10} (x+1)^2\, dx \\ &= \pi \left[\left(\tfrac{1}{1}\right) \frac{(x+1)^3}{3} \right]_4^{10} \\ &= \pi \left(\tfrac{11^3}{3} - \tfrac{5^3}{3} \right) \\ &= \tfrac{1206\pi}{3} = 402\pi \text{ units}^3 \end{aligned}$$

c



$$\begin{aligned} V &= \pi \int_0^{\pi} \sin^2 x\, dx \\ &= \pi \int_0^{\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\pi} \\ &= \pi \left[\tfrac{1}{2}\pi - \tfrac{1}{4}\sin 2\pi - 0\right] \\ &= \tfrac{\pi^2}{2} \text{ units}^3 \end{aligned}$$

d



$$\begin{aligned} V &= \pi \int_0^{3} (9 - x^2)\, dx \\ &= \pi \left[9x - \frac{x^3}{3}\right]_0^3 \\ &= \pi \left\{27 - \tfrac{27}{3} - 0\right\} \\ &= 18\pi \text{ units}^3 \end{aligned}$$

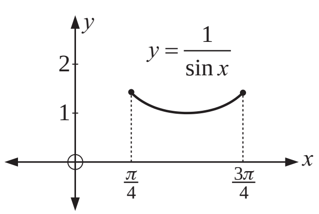
2 **a** $y = \cos(2x)$ meets the x-axis where

$$\begin{aligned} 2x &= \tfrac{\pi}{2} \\ \therefore \quad x &= \tfrac{\pi}{4} \end{aligned}$$

so,
$$\begin{aligned} V &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x)\, dx \\ &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\ &= \pi \left\{\left(\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi\right) - \left(\tfrac{\pi}{32} + \tfrac{1}{8}\sin\left(\tfrac{\pi}{4}\right)\right)\right\} \\ &= \pi \left\{\tfrac{\pi}{8} - \tfrac{\pi}{32} - \tfrac{1}{8}\left(\tfrac{1}{\sqrt{2}}\right)\right\} \\ &= \pi \left\{\tfrac{3\pi}{32} - \tfrac{1}{8\sqrt{2}}\right\} \text{ units}^3 \end{aligned}$$

b
$$\begin{aligned} V &= \pi \int_0^2 (e^{-x} + 4)^2\, dx \\ &= \pi \int_0^2 \left(e^{-2x} + 8e^{-x} + 16\right) dx \\ &= \pi \left[\tfrac{1}{-2}e^{-2x} + \tfrac{8}{-1}e^{-x} + 16x\right]_0^2 \\ &= \pi \left\{\left(-\tfrac{1}{2}e^{-4} - 8e^{-2} + 32\right) - \left(-\tfrac{1}{2} - 8\right)\right\} \\ &= \pi \left\{\frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2}\right\} \text{ units}^3 \qquad (\doteqdot 123.8 \text{ units}^3) \end{aligned}$$

3



$$\begin{aligned} V &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{\sin x}\right)^2 dx \\ &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x)^{-2}\, dx \\ &= \pi \times 2 \qquad \{\text{using technology}\} \\ &= 2\pi \text{ units}^3 \end{aligned}$$

Note: This is not one of the functions we can integrate in this course, so technology is used.

4 They meet where $x^2 = 4$

$\therefore\ x = \pm 2$

but $x > 0$ $\therefore\ x = 2$

i.e., at (2, 4)

$$\begin{aligned} V &= \pi \int_0^2 \left(4^2 - (x^2)^2\right) dx \\ &= \pi \int_0^2 \left(16 - x^4\right) dx \\ &= \pi \left[16x - \frac{x^5}{5}\right]_0^2 \\ &= \pi \left(32 - \tfrac{32}{5} - 0\right) \\ &= \frac{128\pi}{5} \text{ units}^3 \end{aligned}$$

5 They meet where $\sin x = \cos x$

$\therefore\ \dfrac{\sin x}{\cos x} = 1$

$\therefore\ \tan x = 1$

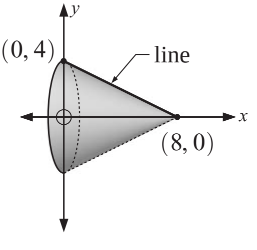
$\therefore\ x = \frac{\pi}{4}$

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\, dx \\ &= \pi \int_0^{\frac{\pi}{4}} \cos(2x)\, dx \\ &= \pi \left[\tfrac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} \\ &= \pi \left(\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sin 0\right) \\ &= \pi \left(\tfrac{1}{2}(1) - 0\right) \\ &= \tfrac{\pi}{2} \text{ units}^3 \end{aligned}$$

6 **a**

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi \times 4^2 \times 8 \\ &= \tfrac{1}{3}\pi \times 128 \\ &= \tfrac{128\pi}{3} \text{ units}^3 \end{aligned}$$

b



slope $= \frac{0-4}{8-0} = -\frac{1}{2}$

$\therefore$ line has equation

$y = -\frac{1}{2}x + 4$

$$\begin{aligned} \therefore\ V &= \pi \int_0^8 \left(-\tfrac{1}{2}x + 4\right)^2 dx \\ &= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx \\ &= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x\right]_0^8 \\ &= \pi \left\{\tfrac{128}{3} - 128 + 128 - 0\right\} \\ &= \tfrac{128\pi}{3} \text{ units}^3 \end{aligned}$$

Chapter 29

STATISTICAL DISTRIBUTIONS

EXERCISE 29A

1 **a** The quantity of fat in a lamb chop is a continuous random variable.
b The mark out of 50 for the Geography test is a discrete random variable.
c The weight of seventeen year-old students is a continuous random variable.
d The volume of water is a cup of coffee is a continuous random variable.
e The number of trout in a lake is a discrete random variable.
f The number of hairs on a cat is a discrete random variable.
g The length of hairs on a horse is a continuous random variable.
h The height of a sky-scraper is a continuous random variable.

2 **a** **i** The random variable is the height of water in the rain gauge.
ii $0 \leqslant x \leqslant 200$ mm
iii The variable is a continuous random variable.

b **i** The random variable is the stopping distance.
ii $0 \leqslant x \leqslant 30$ m
iii The variable is a continuous random variable.

c **i** The random variable could be the time taken for the switch to fail or it could be the number of times that the switch is turned on or off before it fails.
ii $0 \leqslant x \leqslant 10\,000$ hours if we are measuring time, or $1 \leqslant x \leqslant 2000$ if we are counting on/off.
iii The variable is a continuous random variable in the first case and a discrete random variable in the second.

3 **a** Since x is the number of weighing devices that are accurate, $x = 0, 1, 2, 3$ or 4.

b

		YYNN		
		YNYN		
	YYYN	YNNY	NNNY	
	YYNY	NNYY	NNYN	
	YNYY	NYNY	NYNN	
YYYY	NYYY	NYYN	YNNN	NNNN
$(x = 4)$	$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

c **i** If two are accurate then $x = 2$
ii If at least two are accurate then 2, 3 or 4 are accurate i.e., $x = 2, 3$ or 4

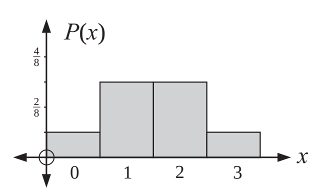
4 **a** If 3 coins are tossed then the number of heads x can be 0, 1, 2 or 3.

b Suppose H represents heads, T represents tails.

	HHT	TTH	
	HTH	THT	
HHH	THH	HTT	TTT
$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

c $P(x = 0) = \frac{1}{8}$ $P(x = 1) = \frac{3}{8}$
$P(x = 2) = \frac{3}{8}$ $P(x = 3) = \frac{1}{8}$

d



EXERCISE 29B

1 **a** Since this is a probability distribution $\sum P(i) = 1$

$$\therefore \quad a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$
$$\therefore \quad a + 0.4512 = 1$$
$$\therefore \quad a = 0.5488$$

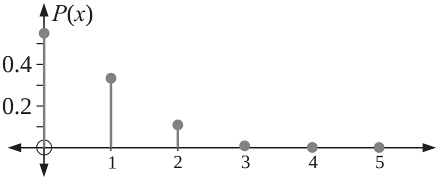
$P(0)$ is the probability that Jason does not hit a home run in a game.

b $P(2) = 0.1088$ (from table)

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.00$
$= 0.4512$

This represents the probability that Jason hits *at least one* home run in a game.

d



2 **a** The probabilities all lie in $0 \leqslant P(i) \leqslant 1$, $\therefore$ OK

Sum of probabilities $\sum P(i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$

$\therefore$ sum of probabilities $\neq 1$

$\therefore$ this is not a valid probability distribution.

b Notice that $P(5) = -0.2$, $\therefore$ not all of the probabilities lie in $0 \leqslant P(i) \leqslant 1$

$\therefore$ this is not a valid probability distribution.

3 **a** The random variable represents the number of hits that Sally has in each game.

b $0.7 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$ $\{\text{since } \sum p(i) = 1\}$

$$\therefore \quad k + 0.77 = 1$$
$$\therefore \quad k = 0.23$$

c **i** $P(x \geqslant 2)$
$= P(x = 2 \text{ or } x = 3 \text{ or } x = 4 \text{ or } x = 5)$
$= P(2) + P(3) + P(4) + P(5)$
$= 0.23 + 0.46 + 0.08 + 0.02$
$= 0.79$

ii $P(1 \leqslant x \leqslant 3)$
$= P(1) + P(2) + P(3)$
$= 0.14 + 0.23 + 0.46$
$= 0.83$

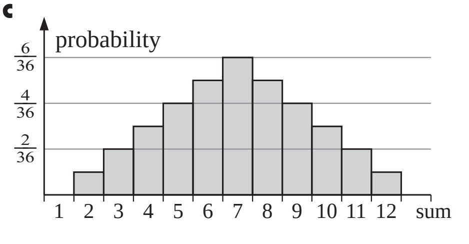
4 **a** Rolling a die twice, sample space:

roll 1							
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6

roll 2

b $P(0) = 0$ $P(1) = 0$ $P(2) = \frac{1}{36}$
$P(3) = \frac{2}{36}$ $P(4) = \frac{3}{36}$ $P(5) = \frac{4}{36}$
$P(6) = \frac{5}{36}$ $P(7) = \frac{6}{36}$ $P(8) = \frac{5}{36}$
$P(9) = \frac{4}{36}$ $P(10) = \frac{3}{36}$ $P(11) = \frac{2}{36}$
$P(12) = \frac{1}{36}$

c



5 **a** $P(x) = k(x+2), \quad x = 1, 2, 3$

$\therefore \quad P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$

and since this is a probability distribution, $\sum P(i) = 3k + 4k + 5k$

and $12k = 1 \qquad \{\text{as } \sum P(i) = 1\}$

$\therefore \quad k = \frac{1}{12}$

b $P(x) = \dfrac{k}{x+1} \qquad x = 0, 1, 2, 3$

$\therefore \quad P(0) = k, \qquad P(1) = \dfrac{k}{2}, \qquad P(2) = \dfrac{k}{3}, \qquad P(3) = \dfrac{k}{4}$

Now $k + \dfrac{k}{2} + \dfrac{k}{3} + \dfrac{k}{4} = 1 \qquad \{\text{as } \sum P(i) = 1\}$

$$\therefore \quad \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \quad \frac{25k}{12} = 1$$

$$\therefore \quad k = \tfrac{12}{25}$$

6 **a** $P(x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \qquad x = 0, 1, 2, 3, 4$

$P(0) = k\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = \dfrac{16k}{81} \qquad P(1) = k\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 = \dfrac{8k}{81} \qquad P(2) = k\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \dfrac{4k}{81}$

$P(3) = k\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 = \dfrac{2k}{81} \qquad P(4) = k\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = \dfrac{k}{81}$

b $\therefore \quad \dfrac{16k}{81} + \dfrac{8k}{81} + \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81} = 1 \qquad \{\text{as } \sum P(i) = 1\}$

$$\therefore \quad \frac{31k}{81} = 1$$

$$\therefore \quad k = \tfrac{81}{31} \doteqdot 2.6130$$

$$\begin{aligned} P(x \geqslant 2) &= P(2) + P(3) + P(4) \\ &= \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81} \\ &= \frac{7k}{81} \\ &= \frac{7 \times 2.6130}{81} \\ &\doteqdot 0.2258 \end{aligned}$$

7 **a** P(no faulty component)

$= \mathrm{P}(x = 0)$

$= P(0)$

$= C_0^{10}(0.04)^0(0.96)^{10-0}$

$= C_0^{10}(0.96)^{10}$

$= 0.96^{10}$

$= 0.6648$

b P(at least one faulty component)

$= 1 -$ P(none are faulty)

$= 1 - 0.6648$

$= 0.3352$

EXERCISE 29C

1 P(rain) $= 0.28 \qquad \therefore$ would expect rain on 0.28×365.25 days a year

i.e., $\doteqdot 102$ days.

2 **a** P(HHH)
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b For 200 tosses, we expect $200 \times \frac{1}{8}$ to be '3 heads'
i.e., 25 to be '3 heads'

3 **a** P(back) $\doteq \frac{125}{325}$ (i.e., $\doteq 0.385$) **b** Expectation is $\frac{125}{325} \times 50 \doteq 19$ 'backs'

4 **a** **i** P(wins \$10)
= P(rolls a 6)
$= \frac{1}{6}$

ii P(wins \$4)
= P(rolls 4 or 5)
$= \frac{2}{6}$ (or $\frac{1}{3}$)

iii P(wins \$1)
= P(rolls 1, 2 or 3)
$= \frac{3}{6}$ (or $\frac{1}{2}$)

b **i** Expectation
$= \frac{2}{6} \times \$4$
$\doteq \$1.33$

ii Expectation
$= \frac{3}{6} \times \$1$
$= \$0.50$

iii Expectation
$= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1$
$= \frac{1}{6}(\$21)$
$= \$3.50$

c It costs \$4 to play and the expected return is \$3.50.
$\therefore$ you expect to lose \$0.50 per game.

d So, over 100 games you expect to lose $100 \times \$0.50 = \$50$.

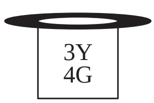
5 Expect to save $\frac{3}{10} \times 90 = 27$ goals

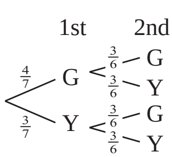
6 Expect to see snow falling on $\frac{3}{7} \times 5 \times 7$ days $= 15$ days

7 P(double) = P(1, 1 or 2, 2 or 3, 3 or 4, 4 or 5, 5 or 6, 6)
$= \frac{6}{36}$ {6 of the possible 36 outcomes}
$= \frac{1}{6}$

$\therefore$ when rolling 180 times we expect $180 \times \frac{1}{6} = 30$ doubles.

8





P(two greens)
= P(GG)
$= \frac{4}{7} \times \frac{3}{6}$
$= \frac{2}{7}$

$\therefore$ for 300 repetitions we expect $300 \times \frac{2}{7} \doteq 86$ times

9 **a** $165 + 87 + 48 = 300$ **i** P(votes A) $\doteq \frac{165}{300}$ **ii** P(votes B) $\doteq \frac{87}{300}$ **iii** P(votes C) $\doteq \frac{48}{300}$

b **i** Expect $7500 \times \frac{165}{300} = 4125$ to vote A **ii** Expect $7500 \times \frac{87}{300} = 2175$ to vote B
iii Expect $7500 \times \frac{48}{300} = 1200$ to vote C

10 **a** Expect to win $\frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \$6$
$= \frac{1}{6} \times \$21$
$= \$3.50$

b No, as on each occasion he would expect to lose \$0.50 (on average).

11

result	*win*
HH	\$10
HT or TH	\$3
TT	−\$5

a Expectation $= \frac{1}{4} \times \$10 + \frac{2}{4} \times \$3 + \frac{1}{4} \times (-\$5) = \$2.75$

b Expected win per game (payout) = \$2.75
$\therefore$ organiser would charge \$2.75 + \$1.00
= \$3.75 to play each game.

12

result	*win*
H	\$2
T	−\$1

For playing *once*,

expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$

$\therefore$ for 3 games, expect to win \$1.50.

13

result	*win*
1	\$1
2	\$2
3	\$3
4	\$5
5	\$10
6	\$25

a Expectation

$= \frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \$10 + \frac{1}{6} \times \$25$

$= \frac{1}{6} \times \$46$

$\doteqdot \$7.67$

b For 1 game at \$10 a game, expect to lose $\$10 - \$7\frac{2}{3} = \$2\frac{2}{3}$

So, for 100 games, expect to lose $\$2\frac{2}{3} \times 100 = \$233.33$

14 $P(R) = \frac{3}{6}$, $P(B) = \frac{2}{6}$, $P(W) = \frac{1}{6}$

$\therefore$ expectation is $\frac{3}{6} \times \$1 + \frac{2}{6} \times \$2 + \frac{1}{6} \times \$5 = \frac{1}{6} \times \$12 = \$2$

EXERCISE 29D

1

x_i	0	1	2	3	4	5	>5
$P(x_i)$	0.54	0.26	0.15	0.03	0.01	0.01	0.00

a $\mu = \sum x_i p_i$

$= 0 \times 0.54 + 1 \times 0.26 + \ldots\ldots + 5 \times 0.01$

$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$

$= 0.74$ i.e., over a long period the mean number of deaths per dozen crayfish is 0.74.

b $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(0 - 0.74)^2 \times 0.54 + (1 - 0.74)^2 \times 0.26 + \ldots\ldots + (5 - 0.74)^2 \times 0.01}$

$\doteqdot 0.9962$

2 $P(x) = \dfrac{x^2 + x}{20}$ for $x = 1, 2, 3$

x_i	1	2	3
$P(x_i) = p_i$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

$\mu = \sum x_i p_i$

$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6$

$= 2.5$

and $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(1 - 2.5)^2 \times 0.1 + (2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0}$

$\doteqdot 0.6708$

3 **a** $P(x) = C_x^3\,(0.4)^x (0.6)^{3-x}$ for $x = 0, 1, 2, 3$

$\therefore$ $P(0) = C_0^3\,(0.4)^0(0.6)^3 = (0.6)^3 = 0.216$

$P(1) = C_1^3\,(0.4)^1(0.6)^2 = 3(0.4)(0.6)^2 = 0.432$

$P(2) = C_2^3(0.4)^2(0.6)^1 = 3(0.16)(0.6) = 0.288$

$P(3) = C_3^3(0.4)^3(0.6)^0 = 1(0.4)^3 = 0.064$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

b $\mu = \sum x_i p_i = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2$

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{(0-1.2)^2(0.216) + (1-1.2)^2(0.432) + (2-1.2)^2 \times 0.288 + (3-1.2)^2 \times 0.064} \\ &\doteqdot 0.8485\end{aligned}$$

4

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ \therefore \quad \sigma^2 &= \sum (x_i - \mu)^2 p_i \\ &= (x_i - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \ldots\ldots + (x_n - \mu)^2 p_n \\ &= (x_1^2 - 2x_1\mu + \mu^2)p_1 + (x_2^2 - 2x_2\mu + \mu^2)p_2 + \ldots\ldots + (x_n^2 - 2x_n\mu + \mu^3)p_n \\ &= (x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \ldots\ldots + x_n^2 p_n) - 2\mu(x_1 p_1 + x_2 p_2 + \ldots\ldots + x_n p_n) \\ &\qquad + \mu^2(p_1 + p_2 + p_3 + \ldots\ldots + p_n)\end{aligned}$$

Now $p_1 + p_2 + \ldots\ldots + p_n = 1$

$$\begin{aligned}\therefore \quad \sigma^2 &= \sum x_i{}^2 p_i - 2\mu(\sum x_i p_i) + \mu^2(1) \\ &= \sum x_i{}^2 p_i - 2\mu(\mu) + \mu^2 \quad \{\text{since } \sum x_i^2 p_i = \mu\} \\ &= \sum x_i^2 p_i - \mu^2\end{aligned}$$

5 **a**

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

b

$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 1(0.1) + 2(0.2) + \ \ldots\ldots \ + 5(0.1) \\ &= 0.1 + 0.4 + 1.2 + 0.8 + 0.5 \\ &= 3\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{\sum x_i^2 p_i - \mu^2} \\ &= \sqrt{1^2(0.1) + 2^2(0.2) + \ldots\ldots + 5^2(0.1) - (3.0)^2} \\ &= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9} \\ &= \sqrt{1.2} \\ &\doteqdot 1.095\end{aligned}$$

c **i**

$$\begin{aligned}&\mathrm{P}(\mu - \sigma < x < \mu + \sigma) \\ &= \mathrm{P}(3 - 1.095 < x < 3 + 1.095) \\ &\doteqdot \mathrm{P}(x = 2,\ 3,\ 4) \\ &\doteqdot 0.2 + 0.4 + 0.2 \\ &\doteqdot 0.8\end{aligned}$$

ii

$$\begin{aligned}&\mathrm{P}(\mu - 2\sigma < x < \mu + 2\sigma) \\ &= \mathrm{P}(3 - 2.19 < x < 3 + 2.19) \\ &= \mathrm{P}(0.81 < x < 5.19) \\ &\doteqdot \mathrm{P}(x = 1,\ 2,\ 3,\ 4 \text{ or } 5) \\ &\doteqdot 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\ &\doteqdot 1.0\end{aligned}$$

6 Let x be the payout values, then $x = \$20\,000, \quad \$8000, \quad \text{or} \quad \$0$

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

The expectation is $\mu = \sum x_i p_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675)$
$= \$290$

i.e., the company expects to pay out \$290 on average in the long run

$\therefore$ the company should charge \$290 + \$100 = \$390

7

die 2 (columns), die 1 (rows)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b

$$\begin{aligned}\mu &= \sum m_i p_i \\ &= 1\left(\tfrac{1}{36}\right) + 2\left(\tfrac{3}{36}\right) + 3\left(\tfrac{5}{36}\right) + \ldots\ldots + 6\left(\tfrac{11}{36}\right) \\ &= \tfrac{1}{36} + \tfrac{6}{36} + \tfrac{15}{36} + \tfrac{28}{36} + \tfrac{45}{36} + \tfrac{66}{36} \\ &= \tfrac{161}{36} \\ &\doteqdot 4.472\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum {m_i}^2 p_i - \mu^2} \\ &= \sqrt{1^2\left(\tfrac{1}{36}\right) + 2^2\left(\tfrac{3}{36}\right) + \ldots\ldots + 6^2\left(\tfrac{11}{36}\right) - (4.4722)^2} \\ &= \sqrt{1.971\,65} \\ &\doteqdot 1.404\end{aligned}$$

8 One coin is tossed a maximum of three times, until either one tail or three heads occurs. Let x be the number of tosses needed. $\therefore$ $x_i = 1$, 2 or 3.

$\text{P}(x = 1) = \text{P(T)} = \frac{1}{2} = 0.50$

$\text{P}(x = 2) = \text{P(HT)} = (\frac{1}{2})^2 = \frac{1}{4} = 0.25$

$\text{P}(x = 3) = \text{P(HHT or HHH)} = (\frac{1}{2})^3 + (\frac{1}{2})^4 = \frac{2}{8} = 0.25$

$\therefore$ the probability distribution is

x_i	1	2	3
$P(x_i)$	0.50	0.25	0.25

and $\therefore$

$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 1(0.5) + 2(0.25) + 3(0.25) \\ &= 1.75\end{aligned}$$

EXERCISE 29E

1 **a** The binomial distribution applies, as tossing a coin has one of two possible outcomes (H or T) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.

2 Let X be the number of boys in the family. $\therefore$ $X = 0, 1, 2, 3, 4, 5, 6$ and X is Bin$(6, \frac{1}{2})$.

a $\text{P}(X = 6)$

$= \text{binompdf}\,(6, \frac{1}{2}, 6)$

$\doteqdot 0.0156$

b $\text{P}(X = 2)$

$= \text{binompdf}\,(6, \frac{1}{2}, 2)$

$\doteqdot 0.234$

c P(more than 4 girls)

$= \text{P}(X \leqslant 1)$

$= \text{binomcdf}\,(6, \frac{1}{2}, 1)$

$\doteqdot 0.109$

d P(more boys than girls)

$= \text{P}(X \geqslant 4)$

$= 1 - \text{P}(X \leqslant 3)$

$= 1 - \text{binomcdf}\,(6, \frac{1}{2}, 3)$

$= 0.344$

3 X is the random variable for the number working night-shift.

$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6, 7$ and X is Bin (7, 0.35).

a $P(X = 3)$
$= \text{binompdf}\,(7,\ 0.35,\ 3)$
$\doteqdot 0.268$

b $P(X < 4)$
$= P(X \leqslant 3)$
$= \text{binomcdf}\,(7,\ 0.35,\ 3)$
$\doteqdot 0.800$

c P(at least 4 work night-shift)
$= P(X \geqslant 4)$
$= 1 - P(X \leqslant 3)$
$\doteqdot 1 - 0.800$
$\doteqdot 0.200$

4 X is the random variable for the number of apples with a blemish.

$\therefore$ $X = 0, 1, 2, 3, 4, \text{.......}, 25$ and X is Bin (25, 0.05).

a $P(X = 2)$
$= \text{binompdf}\,(25,\ 0.05,\ 2)$
$\doteqdot 0.231$

b P(at least one has a blemish)
$= 1 -$ P(none of them have a blemish)
$= 1 - P(X = 0)$
$= 1 - \text{binompdf}\,(25,\ 0.05,\ 0)$
$\doteqdot 0.723$

5 X is the number of faulty items.

$\therefore$ $X = 0, 1, 2, 3, \text{......}, 12$ and X is Bin (12, 0.06).

a $P(X = 0)$
$= \text{binompdf}\,(12,\ 0.06,\ 0)$
$\doteqdot 0.476$

b P(at most one is faulty)
$= P(X \leqslant 1)$
$= \text{binomcdf}(12,\ 0.06,\ 1)$
$\doteqdot 0.840$

c P(at least 2 are faulty)
$= P(X \geqslant 2)$
$= 1 - P(X \leqslant 1)$
$\doteqdot 0.160$ {from **b**}

d P(less than 4 are faulty)
$= P(X < 4)$
$= 1 - P(X \leqslant 3)$
$= \text{binomcdf}\,(12,\ 0.06,\ 3)$
$= 0.996$

6 X is the random variable for the number of correct answers.

$\therefore$ $X = 0, 1, 2, 3, 4, \text{.....}, 20$ and X is Bin $(20,\ \frac{1}{2})$

a $P(X = 20)$
$= \text{binompdf}\,(20,\ \frac{1}{2},\ 20)$
$\doteqdot 9.54 \times 10^{-7}$

b P(half correct)
$= P(X = 10)$
$= \text{binompdf}(20,\ \frac{1}{2},\ 10)$
$\doteqdot 0.176$

c P(less than half correct)
$= P(X \leqslant 9)$
$= \text{binomcdf}\,(20,\ \frac{1}{2},\ 9)$
$\doteqdot 0.412$

d P(at least 15 correct)
$= P(X \geqslant 15)$
$= 1 - P(X \leqslant 14)$
$= 1 - \text{binomcdf}\,(20,\ \frac{1}{2},\ 14)$
$= 0.0207$

7 X is the random variable for the number of times in a week when the bus is on time.

$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6$ or 7 and X is Bin (7, 0.6) {late 2 in 5, on time 3 in 5}

a $\mathrm{P}(X = 7)$
$= \text{binompdf}(7, 0.6, 7)$
$\doteqdot 0.0280$

b P(on time only on Monday)
$= 0.6 \times (0.4)^6$
$= 0.002\,46$

c $\mathrm{P}(X = 6)$
$= \text{binompdf}(7, 0.6, 6)$
$\doteqdot 0.131$

d $\mathrm{P}(X \geqslant 4)$
$= 1 - \mathrm{P}(X \leqslant 3)$
$= 1 - \text{binomcdf}(7, 0.6, 3)$
$\doteqdot 0.710$

8 X is the random variable for the number with flu
$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and X is Bin (25, 0.3)

a $\mathrm{P}(X \geqslant 2)$
$= 1 - \mathrm{P}(X \leqslant 1)$
$= 1 - \text{binomcdf}(25, 0.3, 1)$
$\doteqdot 0.998$

b P(test cancelled)
$= \mathrm{P}(X \geqslant 6)$ $\{20\%$ of $25 = 5\}$
$= 1 - \mathrm{P}(X \leqslant 5)$
$= 1 - \text{binomcdf}(25, 0.3, 5)$
$\doteqdot 0.8065$

9 X is the number of goals thrown.
$\therefore$ $X = 0, 1, 2, 3, 4, ..., 20$ and X is Bin (20, 0.94)

a $\mathrm{P}(X = 20)$
$= \text{binompdf}(20, 0.94, 20)$
$\doteqdot 0.290$

b $\mathrm{P}(X \geqslant 18)$
$= 1 - \mathrm{P}(X \leqslant 17)$
$= 1 - \text{binomcdf}(20, 0.94, 17)$
$\doteqdot 0.885$

EXERCISE 29F

1 x is distributed Bin(6, p)

a If $p = 0.5$, x is distributed Bin(6, 0.5)

i $\mu = np$
$= 6 \times 0.5$
$= 3$

and $\sigma = \sqrt{npq}$
$= \sqrt{6 \times 0.5 \times 0.5}$
$\doteqdot 1.2247$

ii $\mathrm{P}(x = 0)$
$= C_0^6 (0.5)^0 (0.5)^6$
$= 0.0156$

$\mathrm{P}(x = 1)$
$= C_1^6 (0.5)^1 (0.5)^5$
$= 0.0938$

$\mathrm{P}(x = 2)$
$= C_2^6 (0.5)^2 (0.5)^4$
$= 0.2344$

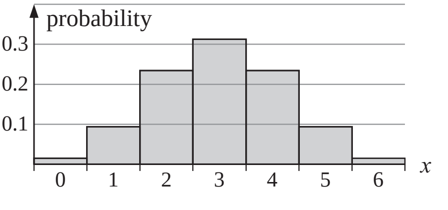
$\mathrm{P}(x = 3)$
$= C_3^6 (0.5)^3 (0.5)^3$
$= 0.3125$

$\mathrm{P}(x = 4)$
$= C_4^6 (0.5)^4 (0.5)^2$
$= 0.2344$

$\mathrm{P}(x = 5)$
$= C_5^6 (0.5)^5 (0.5)^1$
$= 0.0938$

$\mathrm{P}(x = 6)$
$= C_6^6 (0.5)^6 (0.5)^0$
$= 0.0156$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156



iii The distribution is bell-shaped.

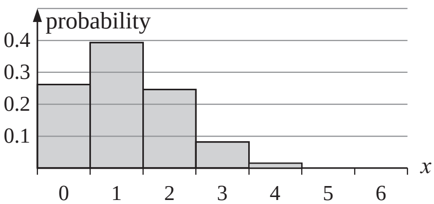
b If $p = 0.2$, x is distributed Bin(6, 0.2)

i $\mu = np = 6 \times 0.2 = 1.2$ and $\sigma = \sqrt{npq} = \sqrt{6 \times 0.2 \times 0.8} \doteqdot 0.980$

ii $\text{P}(x = 0) = C^6_0 (0.2)^0 (0.8)^6 = 0.2621$ $\text{P}(x = 1) = C^6_1 (0.2)^1 (0.8)^5 = 0.3932$ $\text{P}(x = 2) = C^6_2 (0.2)^2 (0.8)^4 = 0.2458$ $\text{P}(x = 3) = C^6_3 (0.2)^3 (0.8)^3 = 0.0819$

$\text{P}(x = 4) = C^6_4 (0.2)^4 (0.8)^2 = 0.0154$ $\text{P}(x = 5) = C^6_5 (0.2)^5 (0.8)^1 = 0.0015$ $\text{P}(x = 6) = C^6_6 (0.2)^6 (0.8)^0 = 0.0001$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



iii The distribution is skewed to the right, i.e., positively skewed.

c If $p = 0.8$, x is distributed Bin(6, 0.8)

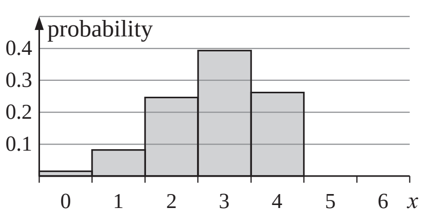
i $\mu = np = 6 \times 0.8 = 4.8$ $\sigma = \sqrt{npq} = \sqrt{6 \times 0.8 \times 0.2} \doteqdot 0.980$

ii $\text{P}(x = 0) = C^6_0 (0.8)^0 (0.2)^6 = 0.0001$ $\text{P}(x = 1) = C^6_1 (0.8)^1 (0.2)^5 = 0.0015$ $\text{P}(x = 2) = C^6_2 (0.8)^2 (0.2)^4 = 0.0154$ $\text{P}(x = 3) = C^6_3 (0.8)^3 (0.2)^3 = 0.0819$

$\text{P}(x = 4) = C^6_4 (0.8)^4 (0.2)^2 = 0.2458$ $\text{P}(x = 5) = C^6_5 (0.8)^5 (0.2)^1 = 0.3932$ $\text{P}(x = 6) = C^6_6 (0.8)^6 (0.2)^0 = 0.2621$

Notice that this is the reverse of **b**.

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii This distribution is the exact reflection of **b**. It is skewed to the left or negatively skewed.

2 Number of tosses, $n = 10$

X is the number of heads obtained. $\therefore$ x is distributed Bin(10, 0.5)

$\mu = np = 10 \times 0.5 = 5$ $\sigma = \sqrt{npq} = \sqrt{10 \times 0.5 \times 0.5} \doteqdot 1.58$

3 **a** x is distributed Bin(3, p)

$$\begin{aligned}\text{P}(x=0) &= C_0^3\, p^0 q^3 \\ &= C_0^3 p^0 (1-p)^3 \\ &= (1-p)^3\end{aligned} \qquad \begin{aligned}\text{P}(x=1) &= C_1^3\, p^1 q^2 \\ &= 3p(1-p)^2\end{aligned} \qquad \begin{aligned}\text{P}(x=2) &= C_2^3\, p^2 (q)^1 \\ &= 3p^2(1-p)\end{aligned}$$

$$\begin{aligned}\text{P}(x=3) &= C_3^3\, p^3 q^0 \\ &= p^3\end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

b
$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3(1-p) + 3p^3 \\ &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3 \\ &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3 \\ &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3 \\ &= 3p \quad \text{as required}\end{aligned}$$

c
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - \mu^2 \\ &= 0^2 \times (1-p)^3 \;+\; 1^2 \times 3p(1-p)^2 \;+\; 2^2 \times 3p^2(1-p) \;+\; 3^2 p^3 - (3p)^2 \\ &= 3p(1-p)^2 \;+\; 12p^2(1-p) \;+\; 9p^2(p-1) \\ &= (1-p)\left[3p(1-p) \;+\; 12p^2 \;-\; 9p^2\right] \\ &= (1-p)\left[3p - 3p^2 \;+\; 3p^2\right] \\ &= 3p(1-p) \\ &= 3pq \qquad \therefore \quad \sigma = \sqrt{3pq} \quad \text{as required}\end{aligned}$$

4 X is the number of defective bolts in the sample.

X is distributed Bin(30, 0.04) $\quad \mu = np = 30 \times 0.04 = 1.2 \quad$ and $\quad \sigma = \sqrt{npq} = \sqrt{30 \times 0.04 \times 0.96} \doteqdot 1.07$

5 X is the number of fatalities.

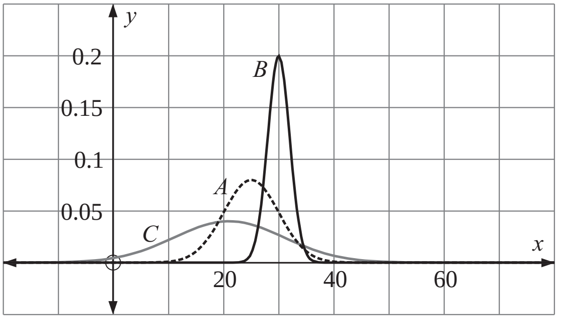
X is distributed Bin(243, 0.037) $\quad \mu = np = 243 \times 0.037 = 8.99 \quad$ and $\quad \sigma = \sqrt{npq} = \sqrt{243 \times 0.037 \times 0.963} \doteqdot 2.94$

6 X is the number of groups that do not arrive.

X is distributed Bin(5, 0.13) $\quad \mu = np = 5 \times 0.13 = 0.65 \quad$ and $\quad \sigma = \sqrt{npq} = \sqrt{5 \times 0.13 \times 0.87} = 0.752$

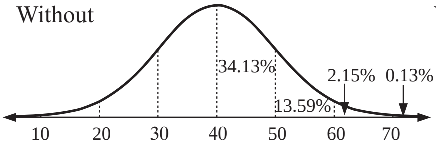
EXERCISE 29G.1

1

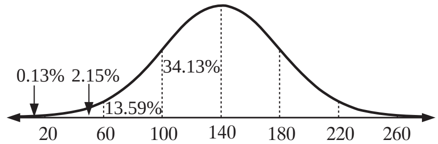


2 a/b/c/d The mean volume (or life time, weight, diameter etc) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 Without



With



a $\quad \text{P(without and} < 50)$
$\doteqdot 50\% + 34.13\%$
$\doteqdot 84.1\%$

b $\quad \text{P(with and} < 60)$
$\doteqdot 0.13\% + 2.15\%$
$\doteqdot 2.28\%$
$\doteqdot 2.3\%$

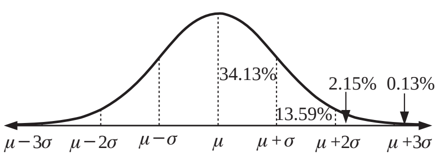
c $\quad \text{P(with and } 20 \leqslant x \leqslant 60)$
$\doteqdot 2.15\%$

$\text{P(without and } 20 \leqslant x \leqslant 60)$
$\doteqdot 2(34.13\% + 13.59\%)$
$\doteqdot 95.44\%$
$\doteqdot 95.4\%$

d $\quad \text{P(with and } x \geqslant 60)$
$\doteqdot 13.59\% + 34.13\% + 50\%$
$\doteqdot 97.7\%$

$\text{P(without and } x \geqslant 60)$
$\doteqdot 2.15\% + 0.13\%$
$\doteqdot 2.3\%$

4



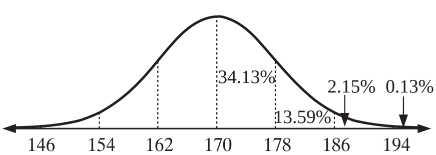
a $\quad P(\mu - \sigma < x < \mu + \sigma)$
$\doteqdot 2 \times 0.3413$
$\doteqdot 0.683$

b $\quad P(\mu < x < \mu + \sigma)$
$\doteqdot 0.341$

c $\quad P(\mu < x < \mu + 2\sigma)$
$\doteqdot 0.3413 + 0.1359$
$\doteqdot 0.477$

d $\quad P(\mu < x < \mu + 3\sigma)$
$\doteqdot 0.3413 + 0.1359 + 0.0215$
$\doteqdot 0.499$

5



a $\quad P(162 < x < 170)$
$\doteqdot 34.1\%$

b $\quad P(170 < x < 186)$
$\doteqdot 34.13\% + 13.59\%$
$\doteqdot 47.7\%$

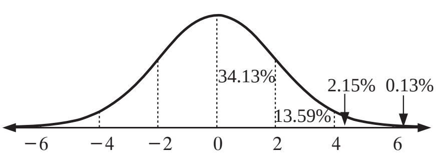
c $\quad P(178 < x < 186)$
$\doteqdot 13.59\%$
$\doteqdot 0.136$

d $\quad P(x < 162)$
$\doteqdot 1 - (0.5 + 0.3413)$
$\doteqdot 0.159$

e $\quad P(x < 154)$
$\doteqdot 0.0215 + 0.0013$
$\doteqdot 0.0228$

f $\quad P(x > 162)$
$\doteqdot 1 - 0.159$ {using **d**}
$\doteqdot 0.841$

6

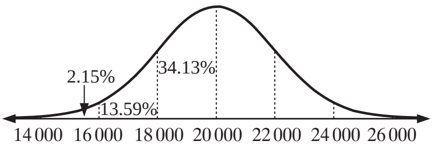


a $\quad P(0 < x < 4)$
$\doteqdot 0.3413 + 0.1359$
$\doteqdot 0.4772$
$\therefore$ expect $800 \times 0.4772 \doteqdot 382$

b $\quad P(-6 < x < 0)$
$\doteqdot 0.5 - 0.0013$
$\doteqdot 0.4987$
$\therefore$ expect $800 \times 0.4987 \doteqdot 399$

c $\quad P(-4 < x < 6)$
$\doteqdot 2 \times 0.3413 + 2 \times 0.1359 + 0.0215$
$\doteqdot 0.9759$
$\therefore$ expect $800 \times 0.9759 \doteqdot 781$

7



a $\text{P}(x < 18\,000)$
$\doteq 1 - 0.5 - 0.3413$
$\doteq 0.1587$
$\therefore$ expect $260 \times 0.1587 = 41$ days

b $\text{P}(x > 16\,000)$
$\doteq 0.1359 + 0.3413 + 0.5$
$\doteq 0.9772$
$\therefore$ expect $260 \times 0.9772 = 254$ days

c $\text{P}(18\,000 \leqslant x \leqslant 24\,000)$
$\doteq 0.3413 \times 2 + 0.1359$
$\doteq 0.8185$
$\therefore$ expect $0.8185 \times 260 = 213$ days

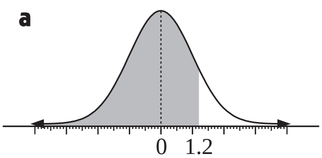
EXERCISE 29G.2 Using Technology

1 **a** 0.341 **b** 0.383 **c** 0.106

2 **a** 0.341 **b** 0.264 **c** 0.212 **d** 0.945 **e** 0.579 **f** 0.383

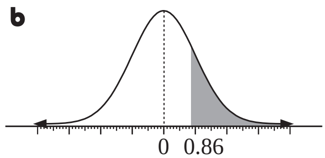
EXERCISE 29H.1

1 **a**



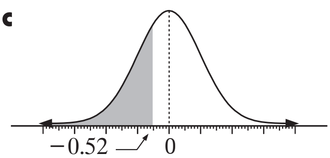
$\text{P}(z \leqslant 1.2)$
$= 0.8849$
$\doteq 0.885$

b



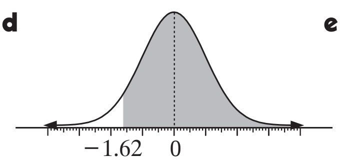
$\text{P}(z \geqslant 0.86)$
$= 1 - \text{P}(z \leqslant 0.86)$
$= 1 - 0.8051$
$\doteq 0.195$

c



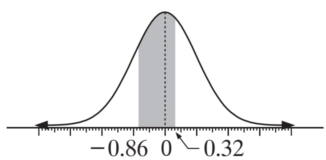
$\text{P}(z \leqslant -0.52)$
$\doteq 0.3015$

d



$\text{P}(z \geqslant -1.62)$
$= 1 - \text{P}(z \leqslant -1.62)$
$\doteq 1 - 0.0526$
$\doteq 0.947$

e



$\text{P}(-0.86 < z < 0.32)$
$= \text{P}(z < 0.32) - \text{P}(z < -0.86)$
$= 0.6255 - 0.1949$
$\doteq 0.431$

2 **a** $\text{P}(z \geqslant 0.837)$
$= 1 - \text{P}(z \leqslant 0.837)$
$\doteq 1 -$ normalcdf $(-\text{E}99, 0.837)$
$\doteq 0.201$

b $\text{P}(z \leqslant 0.0614)$
$=$ normalcdf $(-\text{E}99, 0.0614)$
$\doteq 0.524$

c $\text{P}(z \geqslant -0.876)$
$= 1 - \text{P}(z \leqslant -0.876)$
$\doteq 0.809$

d $\text{P}(-0.3862 \leqslant z \leqslant 0.2506)$
$=$ normalcdf $(-0.3862, 0.2506)$
$\doteq 0.249$

e $\text{P}(-2.367 \leqslant z \leqslant -0.6503)$
$=$ normalcdf $(-2.367, -0.6503)$
$\doteq 0.249$

3 **a** $\text{P}(-0.5 < z < 0.5)$
$=$ normalcdf $(-0.5, 0.5)$
$\doteq 0.383$

b $\text{P}(-1.960 < z < 1.960)$
$=$ normalcdf $(-1.96, 1.96)$
$\doteq 0.950$

4 **a** $\mathrm{P}(z \leqslant a) = 0.95$

$\therefore \quad a \doteqdot 1.645$

{searching in tables}

b $\mathrm{P}(z \geqslant a) = 0.90$

$\therefore \quad 1 - \mathrm{P}(z \leqslant a) = 0.90$

$\therefore \quad \mathrm{P}(z \leqslant a) = 0.1$

$\therefore \quad a \doteqdot -1.28 - \frac{3}{18}(0.01)$

$\therefore \quad a \doteqdot -1.282$

EXERCISE 29H.2

1 X is normal with mean 70, standard deviation 4.

a $\mathrm{P}(x \geqslant 74)$

= normalcdf (74, E99, 70, 4)

$\doteqdot 0.159$

b $\mathrm{P}(x \leqslant 68)$

= normalcdf (−E99, 68, 70, 4)

$\doteqdot 0.309$

c $\mathrm{P}(60.6 \leqslant x \leqslant 68.4)$

= normalcdf (60.6, 68.4, 70, 4)

$\doteqdot 0.335$

2 X is normal with mean 58.3 and standard deviation 8.96.

a $\mathrm{P}(x \geqslant 61.8)$

= normalcdf (61.8, E99, 58.3, 8.96)

$\doteqdot 0.348$

b $\mathrm{P}(x \leqslant 54.2)$

= normalcdf (−E99, 54.2, 58.3, 8.96)

$\doteqdot 0.324$

c $\mathrm{P}(50.67 \leqslant x \leqslant 68.92)$

= normalcdf (50.67, 68.92, 58.3, 8.96)

$\doteqdot 0.685$

3 L is normal with mean 50.2 mm and standard deviation 0.93 mm.

a $\mathrm{P}(l \geqslant 50)$

= normalcdf (50, E99, 50.2, 0.93)

$\doteqdot 0.585$

b $\mathrm{P}(l \leqslant 51)$

= normalcdf (−E99, 51, 50.2, 0.93)

$= 0.805$

c $\mathrm{P}(49 \leqslant l \leqslant 50.5)$

= normalcdf (49, 50.5, 50.2, 0.93)

$\doteqdot 0.528$

EXERCISE 29H.3

1 **a** $\mathrm{P}(z \leqslant k) = 0.81$

$\therefore \quad k \doteqdot 0.87 + \frac{22}{28}(0.01)$

$\therefore \quad k \doteqdot 0.878$

b $\mathrm{P}(z \leqslant k) = 0.58$

$\therefore \quad k \doteqdot 0.20 + \frac{7}{39}(0.01)$

$\therefore \quad k \doteqdot 0.202$

c $\mathrm{P}(z \leqslant k) = 0.17$

$\therefore \quad k \doteqdot -0.96 + \frac{15}{26}(0.01)$

$\therefore \quad k \doteqdot -0.954$

2 **a** $\mathrm{P}(z \leqslant k) = 0.384$

$\therefore \quad k \doteqdot$ invNorm (0.384)

$\therefore \quad k \doteqdot -0.295$

b $\mathrm{P}(z \leqslant k) = 0.878$

$\therefore \quad k \doteqdot$ invNorm (0.878)

$\therefore \quad k \doteqdot 1.165$

c $\mathrm{P}(z \leqslant k) = 0.1384$

$\therefore \quad k \doteqdot$ invNorm (0.1384)

$\therefore \quad k \doteqdot -1.088$

3 **a** $\quad \mathrm{P}(-k \leqslant z \leqslant k)$

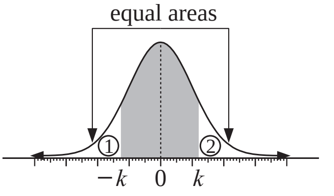
$= \mathrm{P}(z \leqslant k) - \mathrm{P}(z \leqslant -k)$

$= \mathrm{P}(z \leqslant k) - \mathrm{P}(z \geqslant k)$ {as area ① = area ②}

$= \mathrm{P}(z \leqslant k) - [1 - \mathrm{P}(z \leqslant k)]$

$= \mathrm{P}(z \leqslant k) - 1 + \mathrm{P}(z \leqslant k)$

$= 2\mathrm{P}(z \leqslant k) - 1$



b **i**
$$\begin{aligned} \mathrm{P}(-k \leqslant z \leqslant k) &= 0.238 \\ \therefore \quad 2\mathrm{P}(z \leqslant k) - 1 &= 0.238 \\ \therefore \quad 2\mathrm{P}(z \leqslant k) &= 1.238 \\ \therefore \quad \mathrm{P}(z \leqslant k) &= 0.619 \\ \therefore \quad k &= \text{invNorm } (0.619) \\ \therefore \quad k &\doteqdot 0.303 \end{aligned}$$

ii
$$\begin{aligned} \mathrm{P}(-k \leqslant z \leqslant k) &= 0.7004 \\ \therefore \quad 2\mathrm{P}(z \leqslant k) - 1 &= 0.7004 \\ \therefore \quad 2\mathrm{P}(z \leqslant k) &= 1.7004 \\ \therefore \quad \mathrm{P}(z \leqslant k) &= 0.8502 \\ \therefore \quad k &= \text{invNorm } (0.8502) \\ \therefore \quad k &\doteqdot 1.037 \end{aligned}$$

EXERCISE 29I

1 Let X be the length (in cm) of a bolt.
Then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.

$\mathrm{P}(19.7 < x < 20)$ = normalcdf (19.7, 20, 19.8, 0.3)
$\doteqdot 0.378$

2 Let X be the money collected (in \$)
Then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

a $\mathrm{P}(30.00 < x < 50.00)$
= normalcdf (30, 50, 40, 6)
$\doteqdot 0.904$
$\doteqdot 90.4\%$

b $\mathrm{P}(x \geqslant 50)$
= normalcdf (50, E99, 40, 6)
$\doteqdot 0.0478$
$\doteqdot 4.78\%$

3 Let X be the result of the Physics test.
Then X is normally distributed with $\mu = 46$ and $\sigma = 25$.

We need to find k such that
$$\begin{aligned} \mathrm{P}(x \geqslant k) &= 0.07 \\ \text{i.e.,} \quad 1 - \mathrm{P}(x \leqslant k) &= 0.07 \\ \text{i.e.,} \quad \mathrm{P}(x \leqslant k) &= 0.93 \\ \therefore \quad k &= \text{invNorm } (0.93, 46, 25) \\ \therefore \quad k &= 82.894...... \\ \therefore \quad k &\doteqdot 83 \quad \{k \text{ assumed to be an integer}\} \end{aligned}$$

i.e., lowest score would be 83 to get an A.

4 Let X be the length of an eel (in cm)
Then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$

a $\mathrm{P}(x \geqslant 50)$
= normalcdf (50, E99, 41, $\sqrt{11}$)
$\doteqdot 0.003\,33$

b $\mathrm{P}(40 \leqslant x \leqslant 50)$
= normalcdf (40, 50, 41, $\sqrt{11}$)
$\doteqdot 0.615$
$\doteqdot 61.5\%$

c $\mathrm{P}(x \geqslant 45)$
= normalcdf (45, E99, 41, $\sqrt{11}$)
$\doteqdot 0.114$ $\quad \therefore$ we expect $200 \times 0.114 \doteqdot 23$ eels

5 X is the height of a palm tree (in cm).
X is normally distributed with $\mu = 181$, $\sigma = 4$.

a $\mathrm{P}(x \geqslant 175)$
= normalcdf (175, E99, 181, 4)
$\doteqdot 0.933$

b $\mathrm{P}(177 \leqslant x \leqslant 180)$
= normalcdf (177, 180, 181, 4)
$\doteqdot 0.243$

6 **a** Let the mean be μ and standard deviation be σ.

$$\begin{aligned}
\text{Then}\quad \mathrm{P}(x \geqslant 80) &= 0.1 \\
\therefore\quad \mathrm{P}(x \leqslant 80) &= 0.9 \\
\therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} \leqslant \frac{80-\mu}{\sigma}\right) &= 0.9 \\
\therefore\quad \mathrm{P}\left(z \leqslant \frac{80-\mu}{\sigma}\right) &= 0.9 \\
\therefore\quad \frac{80-\mu}{\sigma} &= \text{invNorm }(0.9) \\
\therefore\quad 80-\mu &\doteqdot 1.2816\sigma \quad \text{...... (1)}
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad \mathrm{P}(x \leqslant 30) &= 0.15 \\
\therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) &= 0.15 \\
\therefore\quad \mathrm{P}\left(z \leqslant \frac{30-\mu}{\sigma}\right) &= 0.15 \\
\therefore\quad \frac{30-\mu}{\sigma} &= \text{invNorm }(0.15) \\
\therefore\quad 30-\mu &\doteqdot -1.0364\sigma \quad \text{...... (2)}
\end{aligned}$$

From (1) and (2)

$$\begin{aligned}
(80-\mu)-(30-\mu) &\doteqdot 1.2816\sigma + 1.0364\sigma \\
50 &\doteqdot 2.318\sigma \\
\therefore\quad \sigma &\doteqdot \frac{50}{2.318} \doteqdot 21.57
\end{aligned}$$

$$\begin{aligned}
\text{and in (1)}\quad 80-\mu &\doteqdot 1.2816 \times 21.57 \doteqdot 27.6 \\
\therefore\quad \mu &\doteqdot 52.4 \\
\text{i.e.,}\quad \mu &\doteqdot 52.4 \quad \text{and} \quad \sigma \doteqdot 21.6
\end{aligned}$$

b X is the result of the Maths Exam.
X is normally distributed with mean μ and standard deviation σ.
Then $\mathrm{P}(x \geqslant 80) = 0.1$ and $\mathrm{P}(x \leqslant 30) = 0.15$
So, from **a** $\mu \doteqdot 52.4$ and $\sigma \doteqdot 21.6$

$\mathrm{P}(x > 50)$
$=$ normalcdf (50, E99, 52.4, 21.6)
$\doteqdot 0.544$
$\doteqdot 54.4\%$

or $\mathrm{P}(x \geqslant 51)$ {for integer marks only}
$=$ normalcdf (51, E99, 52.4, 21.6)
$\doteqdot 0.526$
$\doteqdot 52.6\%$

This answer assumes 'part marks' can be given.

7 **a** Let the mean be μ and standard deviation be σ and X be the diameter (in cm).

$$\begin{aligned}
\therefore\quad \mathrm{P}(x < 1.94) &= 0.02 \quad \text{and} \\
\therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} < \frac{1.94-\mu}{\sigma}\right) &= 0.02 \\
\therefore\quad \mathrm{P}\left(z < \frac{1.94-\mu}{\sigma}\right) &= 0.02 \\
\therefore\quad \frac{1.94-\mu}{\sigma} &= \text{invNorm }(0.02) \\
\therefore\quad 1.94-\mu &\doteqdot -2.054\sigma \quad \text{...... (1)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(x > 2.06) &= 0.03 \\
\therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} > \frac{2.06-\mu}{\sigma}\right) &= 0.03 \\
\therefore\quad \mathrm{P}\left(z > \frac{2.06-\mu}{\sigma}\right) &= 0.03 \\
\text{i.e.,}\quad \mathrm{P}\left(z < \frac{2.06-\mu}{\sigma}\right) &= 0.97 \\
\therefore\quad \frac{2.06-\mu}{\sigma} &= \text{invNorm }(0.97) \\
\therefore\quad 2.06-\mu &\doteqdot 1.881\sigma \quad \text{..... (2)}
\end{aligned}$$

From (1) and (2)

$$\begin{aligned}
(2.06-\mu)-(1.94-\mu) &= 1.881\sigma + 2.054\sigma \\
\therefore\quad 3.935\sigma &= 0.12 \\
\therefore\quad \sigma &\doteqdot 0.0305
\end{aligned}$$

$$\begin{aligned}
\text{and in (1)}\quad 1.94-\mu &= -2.054 \times 0.0305 \doteqdot -0.0626 \\
\therefore\quad \mu &\doteqdot 2.00 \\
\text{i.e.,}\quad \mu &\doteqdot 2.00 \quad \text{and} \quad \sigma \doteqdot 0.0305
\end{aligned}$$

b This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$

$$\begin{aligned}\therefore\quad & \text{P(less than 2 will operate)}\\ &= \text{P}(x \leqslant 1)\\ &= \text{binomcdf}\,(20,\ 0.05,\ 1)\\ &\doteqdot 0.736\end{aligned}$$

8 S is distributed normally with mean 60 and standard deviation 10.

$$\begin{aligned}\text{P}(s \geqslant 70) &= \text{normalcdf}\,(70,\ \text{E}99,\ 60,\ 10)\\ &\doteqdot 0.158\\ &\doteqdot 16\%\end{aligned}$$

G is distributed normally with mean 50 and standard deviation 12.

$$\begin{aligned}\text{P}(g \geqslant 66) &= \text{normalcdf}\,(66,\ \text{E}99,\ 50,\ 12)\\ &\doteqdot 0.0912\\ &\doteqdot 9\%\end{aligned}$$

i.e., only 9% achieved a higher grade in Geography while almost 16% achieved a higher grade in Science. The student achieved a grade higher than 91% of the class in Geography and 84% of the class in Science.

REVIEW SET 29A

1 **a** $P(x) = \dfrac{a}{x^2+1}$ for $a = 0, 1, 2, 3$

x_i	0	1	2	3
$P(x_i)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

$$\begin{aligned}\text{Now}\quad a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} &= 1 \qquad \{\text{as } \textstyle\sum P(x_i) = 1\}\\ \therefore\quad 10a + 5a + 2a + a &= 10 \qquad \{\times \text{ each term by } 10\}\\ \therefore\quad 18a &= 10\\ \therefore\quad a &= \tfrac{5}{9}\end{aligned}$$

b
$$\begin{aligned}\text{P}(x \geqslant 1) &= \text{P}(x = 1,\ x = 2 \text{ or } x = 3)\\ &= \text{P}(x = 1) + \text{P}(x = 2) + \text{P}(x = 3)\\ &= \tfrac{5}{18} + \tfrac{1}{9} + \tfrac{5}{90}\\ &= \tfrac{4}{9}\end{aligned}$$

or
$$\begin{aligned}\text{P}(x \geqslant 1) &= 1 - \text{P}(x < 1)\\ &= 1 - \text{P}(x = 0)\\ &= 1 - \tfrac{5}{9}\\ &= \tfrac{4}{9}\end{aligned}$$

2 **a**
$$\begin{aligned}P(x) &= C^4_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{4-x}\\ \therefore\quad P(0) &= C^4_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 = 0.0625\\ P(1) &= C^4_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^3 = 0.25\end{aligned}$$

$$\begin{aligned}P(2) &= C^4_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^2 = 0.375\\ P(3) &= C^4_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^1 = 0.25\\ P(4) &= C^4_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^0 = 0.0625\end{aligned}$$

i.e.,

x_i	0	1	2	3	4
$P(x_i)$	0.0625	0.25	0.375	0.25	0.0625

b
$$\begin{aligned}\mu &= \textstyle\sum x_i P(x_i)\\ &= 0 \times 0.0625 + 1 \times 0.25 + 2 \times 0.375 + 3 \times 0.25 + 4 \times 0.0625\\ &= 2\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\textstyle\sum (x_i - \mu)^2 P(x_i)}\\ &= \sqrt{(-2)^2(0.0625) + (-1)^2(0.25) + 0^2(0.375) + 1^2(0.25) + 2^2(0.0625)}\\ &= 1\end{aligned}$$

3 X is the number of defectives. Then X is Bin (10, 0.18). $X = 0, 1, 2, 3, \ldots, 10$.

a $P(X = 1)$
$= \text{binompdf}(10, 0.18, 1)$
$\doteqdot 0.302$

b $P(X = 2)$
$= \text{binompdf}(10, 0.18, 2)$
$\doteqdot 0.298$

c $P(X \geqslant 2) = 1 - P(X \leqslant 1)$
$= 1 - \text{binomcdf}(10, 0.18, 1)$
$\doteqdot 0.561$

4 Let X be the number of defective toothbrushes.
X is distributed Bin(120, 0.04), $n = 120$, $p = 0.04$, $q = 0.96$,

a $\mu = np$
$= 120 \times 0.04$
$= 4.8$

b $\sigma = \sqrt{npq}$
$= \sqrt{120 \times 0.4 \times 0.6}$
$\doteqdot 2.15$

5

Result	*Pays*
1, 3, 5	\$2
2	\$3
4	\$6
6	\$9

a Expected return $= \frac{3}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$9$
$= \frac{1}{6}(\$24)$
$= \$4$

b For a \$5 amount to play the game the club expects a \$1 return/game
$\therefore$ for 75 people, the return expected is \$75.

6 **a**

Result	*Pays*
1	\$2
2	\$4
3	\$6
4	\$8
5	\$10
6	\$12

Expectation
$= \frac{1}{6} \times \$2 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$8 + \frac{1}{6} \times \$10 + \frac{1}{6} \times \$12$
$= \frac{1}{6} \times \$42$
$= \$7$

b No, as \$1 is expected to be lost per game in the long run.

7 If X is the arm length random variable (in cm) then X is normally distributed with $\mu = 64$ and $\sigma = 4$.

a i $P(60 < x < 72)$
$= \text{normalcdf}(60, 72, 64, 4)$
$\doteqdot 0.8186$
$\doteqdot 81.9\%$

ii $P(X > 60)$
$= \text{normalcdf}(60, \text{E}99, 64, 4)$
$\doteqdot 0.841$
$\doteqdot 84.1\%$

b $P(56 < x < 68) = \text{normalcdf}(56, 68, 64, 4)$
$\doteqdot 0.819$

8 X is the rod length (in mm)
X is normally distributed with mean μ and $\sigma = 3$

Now $P(x < 25) = 0.02$

$$\therefore\ P\left(\frac{x-\mu}{3} < \frac{25-\mu}{3}\right) = 0.02$$

$$\therefore\ P\left(z < \frac{25-\mu}{3}\right) = 0.02$$

$$\therefore\ \frac{25-\mu}{3} = \text{invNorm}(0.02)$$

$$\therefore\ \frac{25-\mu}{3} \doteqdot -2.0537$$

$$\therefore\ 25-\mu \doteqdot -6.161$$

$$\therefore\ \mu \doteqdot 31.2$$

REVIEW SET 29B

1 **a** $P(x_i) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$P(0) = k\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3 = \dfrac{k}{64}$ $\quad P(2) = k\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 = \dfrac{9k}{64}$

$P(1) = k\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2 = \dfrac{3k}{64}$ $\quad P(3) = k\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \dfrac{27k}{64}$

x_i	0	1	2	3
$P(x_i)$	$\frac{k}{64}$	$\frac{3k}{64}$	$\frac{9k}{64}$	$\frac{27k}{64}$

Now $\dfrac{k}{64} + \dfrac{3k}{64} + \dfrac{9k}{64} + \dfrac{27k}{64} = 1$ $\quad$ {as $\sum p(x_i) = 1$}

$\therefore \dfrac{40k}{64} = 1$

$\therefore k = 1.6$

b $\text{P}(x \geqslant 1) = 1 - \text{P}(x = 0)$

$= 1 - \dfrac{k}{64}$

$= 1 - \dfrac{1.6}{64}$

$= 0.975$

2 Let X be the payout values then $x = \$45\,000$, $\$30\,000$, $\$10\,000$ or $\$0$

x_i	0	10 000	30 000	45 000
$P(x_i)$	0.988 18	0.0088	0.0023	0.000 72

$\therefore$ expected payout over the long run

$= \sum x_i p_i$

$= 0(0.988\,18) + 10\,000(0.0088) + 30\,000(0.0023) + 45\,000(0.000\,72)$

$= \$189.40$

$\therefore$ to cover an expected return of \$250 they need to charge $\$189.40 + \$250 = \$439.40$

3 X is the number of hits, then $X = 0, 1, 2, 3, 4$ and X is Bin (4, 0.96)

a $\text{P}(X = 4)$
$=$ binompdf (4, 0.96, 4)
$\doteqdot 0.849$

b $\text{P}(X = 0)$
$=$ binompdf (4, 0.96, 0)
$\doteqdot 0.000$

c $\text{P}(X \geqslant 3)$
$= 1 - \text{P}(X \leqslant 2)$
$= 1 -$ binomcdf (4, 0.96, 2)
$\doteqdot 0.991$

d $\text{P}(X = 1)$
$=$ binompdf (4, 0.96, 1)
$\doteqdot 0.000\,246$

4 X is distributed Bin(4, p)

$\therefore$ $\text{P}(x = 0) = C_0^4\,p^0(1-p)^4$
$= (1-p)^4$

$\text{P}(x = 1) = C_1^4 p^1(1-p)^3$
$= 4p(1-p)^3$

$\text{P}(x = 2) = C_2^4\,p^2(1-p)^2$
$= 6p^2(1-p)^2$

$\text{P}(x = 3) = C_3^4\,P^3(1-p)^1$
$= 4p^3(1-p)$

$\text{P}(x = 4) = C_4^4 p^4(1-p)^0$
$= p^4$

x_i	0	1	2	3	4
$P(x_i)$	$(1-p)^4$	$4p(1-p)^3$	$6p^2(1-p)^2$	$4p^3(1-p)$	p^4

$$\begin{aligned}\mu &= \sum x_i p_i\\ &= 0(1-p)^4 + 1\times 4p(1-p)^3 + 2\times 6p^2(1-p)^2 + 3\times 4p^3(1-p) + 4\times p^4\\ &= 4p(1-p)^3 + 12p^2(1-p)^2 + 12p^3(1-p) + 4p^4\\ &= 4p(1-3p+3p^2-p^3) + 12p^2(1-2p+p^2) + 12p^3 - 12p^4 + 4p^4\\ &= 4p - \cancel{12p^2} + \cancel{12p^3} - \cancel{4p^4} + \cancel{12p^2} - \cancel{24p^3} + \cancel{12p^4} + \cancel{12p^3} - \cancel{12p^4} + \cancel{4p^4}\\ &= 4p, \quad \text{as required}\end{aligned}$$

Now
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - (\mu)^2\\ &= \left[0\times(1-p)^4 + 1^2\times 4p(1-p)^3 + 2^2\times 6p^2(1-p)^2 + 3^2\times 4p^3(1-p) + 4^2p^4\right]\\ &\quad -(4p)^2\\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) + 16p^4 - 16p^2\\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1-p^2)\\ &= 4p(1-p)^3 + 24p^2(1-p)^2 + 36p^3(1-p) - 16p^2(1+p)(1-p)\\ &= (1-p)\left[4p(1-p)^2 + 24p^2(1-p) + 36p^3 - 16p^2(1+p)\right]\\ &= q\left[4p - 8p^2 + 4p^3 + 24p^2 - 24p^3 + 36p^3 - 16p^2 - 16p^3\right]\\ &= q\times 4p\\ &= 4pq\end{aligned}$$

$\therefore \quad \sigma = \sqrt{4pq}$, as required.

5 X is the contents of the container (in mL).
X is normally distributed with $\mu = 377$ and $\sigma = 4.2$

a **i** $\text{P}(X < 368.6)$
$= \text{normalcdf}\ (-\text{E}99, 368.6, 377, 4.2)$
$\doteqdot 0.0228$
$\doteqdot 2.28\%$

ii $\text{P}(372.8 < X < 389.6)$
$= \text{normalcdf}\ (372.8, 389.6, 377, 4.2)$
$\doteqdot 0.840$
$\doteqdot 84.0\%$

b $\text{P}(364.4 < X < 381.2)$
$= \text{normalcdf}\ (364.4, 381.2, 377, 4.2)$
$\doteqdot 0.840$

6 X is the life of a battery (in weeks)
X is normally distributed with $\mu = 33.2$, $\sigma = 2.8$

a $\text{P}(X \geqslant 35)$
$= \text{normalcdf}\ (35, \text{E}99, 33.2, 2.8)$
$\doteqdot 0.260$

b We need to find k such that $\text{P}(X \geqslant k) = 0.08$
i.e., $\text{P}(X \leqslant k) = 0.92$
$\therefore \quad k = \text{invNorm}\ (0.92, 33.2, 2.8)$
$\therefore \quad k \doteqdot 37.134$

So, the manufacturer can expect that no more than 8% will fail for a maximum of 37 weeks and 1 day, i.e., 260 days.

REVIEW SET 29C

1

x_i	0	1	2	3	4
$P(x_i)$	0.10	0.30	0.45	0.10	k

a If this is a probability distribution then $\sum p(x_i) = 1$

$$\therefore \quad 1 = 0.1 + 0.3 + 0.45 + 0.1 + k$$
$$\therefore \quad k = 1 - 0.95$$
$$\therefore \quad k = 0.05$$

b
$$\begin{aligned}\mu &= \sum x_i p_i \\ &= 0(0.1) + 1(0.3) + 2(0.45) + 3(0.1) + 4(0.05) \\ &= 0 + 0.3 + 0.9 + 0.3 + 0.2 \\ &= 1.7\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - (\mu)^2 \\ &= 0^2(0.1) + 1^2(0.3) + 2^2(0.45) + 3^2(0.1) + 4^2(0.05) - (1.7)^2 \\ &= 0.3 + 1.8 + 0.9 + 0.8 - 2.89 \\ &= 0.91 \qquad \therefore \quad \sigma = \sqrt{0.91} \doteqdot 0.954\end{aligned}$$

2 X is the number of trees which survive the first year.

$\therefore \quad X = 0, 1, 2, 3, 4, 5$ and X is Bin (5, 0.4)

a $\text{P}(X = 1)$
$= \text{binompdf}\ (5, 0.4, 1)$
$\doteqdot 0.259$

b $\text{P}(X \leqslant 1)$
$= \text{binomcdf}\ (5, 0.4, 1)$
$\doteqdot 0.337$

c $\text{P}(X \geqslant 1)$
$= 1 - \text{P}(X = 0)$
$= 1 - \text{binompdf}\ (5, 0.4, 0)$
$\doteqdot 0.922$

3 Let X denote the number of cases of netballers needing knee surgery.

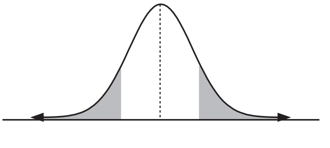
x is distributed Bin(487, 0.0132), $n = 487$, $p = 0.0132$, $q = 0.9868$

$$\therefore \quad \mu = np = 487 \times 0.0132 \doteqdot 6.43$$

$$\sigma = \sqrt{npq} = \sqrt{487 \times 0.0132 \times 0.9868} \doteqdot 2.52$$

4 Let X denote the mass of a Coffin Bay Oyster. X is distributed normally with a mean of 38.6 and a standard deviation of 6.3.

a
$$\text{P}(38.6 - a \leqslant x \leqslant 38.6 + a) = 0.6826$$
$$\text{P}\left(\frac{38.6 - a - 38.6}{6.3} \leqslant \frac{x - 38.6}{6.3} \leqslant \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$$
$$\text{P}\left(-\frac{a}{6.3} \leqslant z \leqslant \frac{a}{6.3}\right) = 0.6826$$
$$\therefore \text{ by symmetry, } \quad \text{P}\left(z \leqslant -\frac{a}{6.3}\right) = \frac{1 - 0.6826}{2} = 0.1587$$
$$\therefore \quad -\frac{a}{6.3} = \text{invNorm}\ (0.1587)$$
$$\therefore \quad -\frac{a}{6.3} \doteqdot -0.9998 \quad \text{and} \quad \therefore \quad a \doteqdot 6.3 \text{ gms}$$



b

$$\begin{aligned} \mathrm{P}(x \geqslant b) &= 0.8413 \\ \therefore\quad \mathrm{P}(x \leqslant b) &= 0.1587 \\ \mathrm{P}\left(\frac{x-38.6}{6.3} \leqslant \frac{b-38.6}{6.3}\right) &= 0.1587 \\ \text{i.e.,}\quad \mathrm{P}\left(z \leqslant \frac{b-38.6}{6.3}\right) &= 0.1587 \\ \therefore\quad \frac{b-38.6}{6.3} &\doteqdot \text{invNorm}\ (0.1587) \\ \therefore\quad b-38.6 &\doteqdot 6.3 \times -0.9998 \\ \therefore\quad b &\doteqdot -6.298 + 38.6 \\ \therefore\quad b &\doteqdot 32.3 \text{ gms} \end{aligned}$$

5 Let X denote the life of the staplers. X is distributed normally with a mean of 3.42 and standard deviation 0.4.

$$\begin{aligned} \mathrm{P}(x \leqslant 3) &= \text{normalcdf}\ (-\text{E}99,\ 3,\ 3.42,\ 0.4) \\ &\doteqdot 0.147 \end{aligned}$$

i.e., out of 2000 in the batch we would expect to have to replace $2000 \times 0.147 \doteqdot 294$

$\therefore$ profit $= \$15 \times (2000 - 294) = \$15 \times 1706 = \$25\,590$

6 $f(x) = ax^2(2-x)$ for $0 < x < 2$

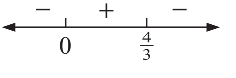
a Since $f(x)$ is a probability distribution function the area under the curve is 1

$$\begin{aligned} \text{i.e.,}\quad \int_0^2 ax^2(2-x)\,dx &= 1 \\ \therefore\quad a\int_0^2 \left(2x^2 - x^3\right)\,dx &= 1 \\ \therefore\quad a\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 &= 1 \\ \therefore\quad a\left(\tfrac{16}{3} - \tfrac{16}{4}\right) - 0 &= 1 \\ \therefore\quad a\left(\tfrac{4}{3}\right) &= 1 \\ \therefore\quad a &= \tfrac{3}{4} \end{aligned}$$

b The mode is the most frequently occurring score, i.e., the value of x when $f(x)$ is a maximum.

$$\begin{aligned} f(x) &= \tfrac{3}{4}\left(2x^2 - x^3\right) \\ f'(x) &= \tfrac{3}{4}\left(4x - 3x^2\right) \\ &= \tfrac{3}{4}x(4-3x) \end{aligned}$$

which has sign diagram:



$\therefore$ is a maximum when $x = \frac{4}{3}$

$\therefore$ the mode is $\frac{4}{3}$.

c If the median is m, then,

$$\begin{aligned} \int_0^m \tfrac{3}{4}x^2(2-x)\,dx &= \tfrac{1}{2} \\ \tfrac{3}{4}\int_0^m \left(2x^2 - x^3\right)\,dx &= \tfrac{1}{2} \\ \therefore\quad \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^m &= \tfrac{2}{3} \\ \therefore\quad \frac{2m^3}{3} - \frac{m^4}{4} &= \tfrac{2}{3} \\ \therefore\quad m &\doteqdot 1.2285 \end{aligned}$$

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i.e., the median is approximately 1.23.

d $$\begin{aligned} P(0.6 < x < 1.2) &= \int_{0.6}^{1.2} \tfrac{3}{4}x^2(2-x)\,dx \\ &= 0.3915 \end{aligned}$$

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